Homotopy Theoretic Techniques in Arrangements

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Configuration Spaces: Geometry, Combinatorics and Topology
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Outline

1. Introduction
2. Colimits
3. Homotopy Colimits
4. Examples
   - Examples from Arrangements
5. Applications
6. The Toolkit
7. Nutshell

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Motivation

Definition

Let V be a finite dimensional vector space (over $\mathbb{R}$ or $\mathbb{C}$), an arrangement of hyperplanes is a finite collection $\mathcal{A} = \{H_1 \ldots, H_n\}$ of codimension 1 affine subspaces of V.

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An arrangement of subspaces is a finite collection $\mathcal{A} = \{H_1, \ldots, H_n\}$ of subspaces in a topological space V which is closed under intersection.

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One of the themes in arrangement theory is to understand the interaction between the combinatorics of the intersections and the topology of the complement, $V \setminus \bigcup_{i=1}^{n} H_i$.

What is a Homotopy Colimit ?

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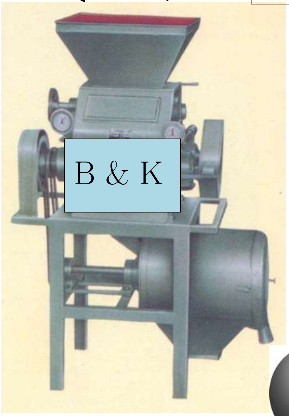
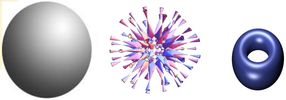
Topological spaces and maps between them



Space with homotopically smart gluing



Homotopy Colimits are also Familiar Spaces

- Simplicial Complexes
- Spheres
- Complex Projective Spaces, $\mathbb{C}P^n$
- Orbit Spaces, X/G
- Classifying Spaces of Groups
- Toric Varieties

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Why do we need homotopy colimits ?

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Categorical Language

Definition (Small Category)

A category C is small if both $ob(C)$ and $hom(C)$ are sets. For example,

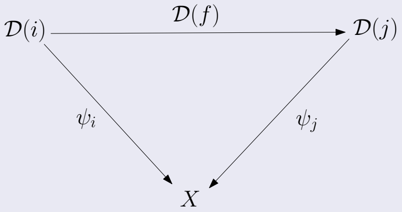
- A poset $(P, \leq)$, elements as objects and morphisms given by $\leq$.
- A group G, with a single object and morphisms indexed by $g \in G$.
- $\{\bullet \leftarrow \bullet \rightarrow \bullet\}$ and $\{\bullet \rightarrow \bullet \rightarrow \bullet\}$

Definition (Diagram of Spaces)

A diagram of spaces over a small category C is a covariant functor $\mathcal{D}\colon C \rightarrow Top$ into the category of topological spaces.

Definition (Co-cone)

A co-cone of a diagram $\mathcal{D}$ is a topological space X with a family of maps $\psi_i \colon \mathcal{D}(i) \to X$ such that for every $f \colon i \to j$ the following triangle commutes.



The Definition

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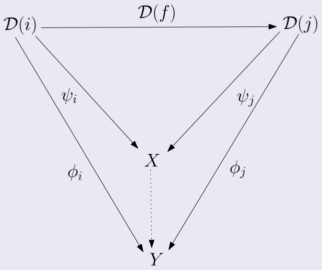
The colimit of a diagram $\mathcal{D}$ (*colim*$\mathcal{D}$) is the *universal co-cone*



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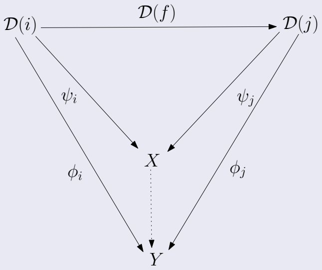

Constructive definition

$$colim\mathcal{D} = \coprod_{i \in Ob(C)} \mathcal{D}(i)/\sim$$

for $x \in \mathcal{D}(i), y \in \mathcal{D}(j), x \sim y$ iff $\mathcal{D}_g(x) = y$ for some $g \in Mor(i, j)$.

Examples of Colimits

1. $colim(\{\} \to Top) = \emptyset$ (the initial object in *Top*).
2. $colim(\{X, Y\}) = X \coprod Y$ (coproduct in *Top*).
3. $colim(X \xrightarrow{f} Y) = Y$.
4. $colim(* \rightrightarrows [0, 1]) = S^1$ (coequalizer in *Top*).
5. $colim(Z \xleftarrow{f} X \xrightarrow{g} Y) = Z \bigcup_X Y$ (pushout in *Top*).
6. $colim(G \xrightarrow{X} Top) = X/G$.

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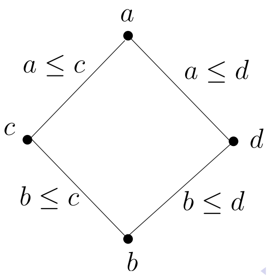
Definition (Geometric Realization of a Poset - $\Delta(P)$)

Let $(P, \leq)$ be a poset, order complex of P is a simplicial complex whose vertex set corresponds to the elements of P and k-simplices correspond to k-chains in P. The space $\Delta(P)$ is the geometric realization of the order complex of P.

$$P = \{a \leq c, a \leq d, b \leq c, b \leq d\}$$

$\Delta(P) =$



$\simeq S^1$

The Homotopy Colimit

Constructive Definition

The space $hocolim(\mathcal{D})$ can be constructed by taking -

- for each object c_0 of C, a copy of $\mathcal{D}(c_0)$;
- for each chain $c_0 \to \cdots \to c_n$ of composable arrows in C, a copy of $\mathcal{D}(c_0) \times \Delta^n$;

and making identifications as follows -

- collapse $\mathcal{D}(c_0) \times \Delta^n$ to something smaller if it arises from a chain containing an identity arrow;
- identify $\mathcal{D}(c_0) \times \partial\Delta^n \subset \mathcal{D}(c_0) \times \Delta^n$ with the appropriate subspace of $hocolim(\mathcal{D})$ arising from chains of smaller lengths;
- finally identify $\mathcal{D}(c_0) \times \Delta^0$ with $\mathcal{D}(c_1) \times \Delta^0$ via the induced map $\mathcal{D}(c_0 \to c_1)$.

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- There are more than one ways to construct the homotopy colimit (up to homotopy equivalence).
- In general it is not the *colimit* in the homotopy category.
- Homotopy colimit is the total left derived functor of *colimit*.

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then identifying

- $(X \times \{0\})_f \sim X$ via 1_X and $(X \times \{1\})_f \sim Z$ via f
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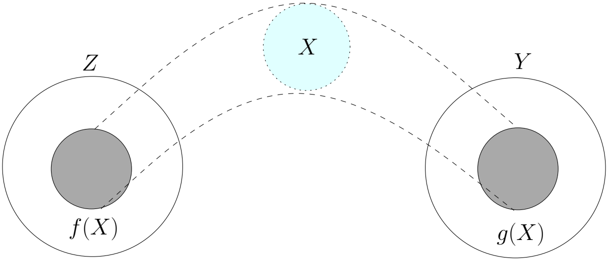

Examples of Homotopy Pushouts

- $hocolim(*\leftarrow X \rightarrow *) = \Sigma X$ (suspension of X).
- $hocolim(*\leftarrow S^{n-1} \rightarrow *) = S^n$
- $hocolim(Z \stackrel{f}{\leftarrow} X \rightarrow *)$ (mapping cone of f).
- $hocolim(Z \twoheadleftarrow (Z \times Y) \twoheadrightarrow Y) = Y * Z$ (join construction).

Classifying Space of a Category

Consider the (trivial) diagram

$$\mathcal{D}(i) = *, \ \ \forall i \in Ob(C)$$

then *hocolim*$\mathcal{D}$ is called as the *classifying space of* C, denoted BC.

Classifying Space of a Category

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- If P is a poset then classifying space of P is $\Delta(P)$.
- If G is a group considered as a small category then classifying space of G is BG.
- If C is a groupoid then $BC \simeq \coprod BC(\{x\}, \{x\})$, where $x \in Ob(C)$ and $C(\{x\}, \{x\})$ is the vertex group of the isomorphism class of x.

Nerve Diagram

Let $U := \{X_i\}_{i\in I}$ be an open cover of a space X. Let P_U be the poset of all non-empty intersections, ordered by reverse inclusion. Define a diagram of spaces $\mathcal{D}_U$ as follows -

- $\mathcal{D}_U(J) = \cap_{i\in J} X_i$, for $J \subseteq I$;
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Nerve Lemma

Every topological space is homotopy equivalent to classifying space of some category.

Homotopy Orbit Spaces

Let G be a group and X be a G-space, which is also the diagram $X\colon G \to Top$. The *homotopy orbit space* of the action of G on X, denoted X_{hG}, is the space

$$\begin{aligned} X_{hG} &:= hocolim_G X \\ &= (X \times EG)/G \end{aligned}$$

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- X_{hG} is a homotopically correct version of the orbit space, also known as the *Borel construction*.
- If X is a free G-space then $X_{hG} \simeq colim_G X = X/G$.
- If the G action is trivial then $X_{hG} \simeq X \times BG$.

Joins and Projective Spaces

Let P_n be the poset of nonempty faces of an n-simplex ordered by inclusion, let $\{X_i\}_{i=0}^n$ be a collection of spaces.
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- $\mathcal{D}(A) = \prod_{i \in A} X_i$, for $A \in P_n$;
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then

$$hocolim_{P_n}\mathcal{E} \cong \mathbb{C}P^n$$

Toric Diagrams

For Σ, a complete and rational fan in $\mathbb{R}^n$, let P_Σ be the poset of cones ordered by reverse inclusion. Define a diagram as follows -

- $\mathcal{D}(\sigma) = T^{n-dim(\sigma)}$, for $\sigma \in \Sigma$;
- $\mathcal{D}(\sigma \to \tau) \colon T^{n-dim(\sigma)} \to T^{n-dim(\tau)}$ be the projection.

Then

$$hocolim\mathcal{D} \cong X_\Sigma$$

X_Σ is the *toric variety* associated with the given fan.

Subspace Arrangement

Definition (Arrangement of Subspaces)

is a finite collection $\mathcal{A} = \{A_1, \ldots, A_m\}$ of affine subspaces in $\mathbb{R}^n$ s.t.

- $\mathcal{A}$ is closed under intersection, and
- for $A, B \in \mathcal{A}$ and $A \subseteq B$ the inclusion is a cofibration.

Let P be the poset of all non-empty intersections of subspaces in $\mathcal{A}$ ordered by reversed inclusion and the link $L := \bigcup_{i=1}^{n} A_i$.

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Theorem (Ziegler & Živaljević - 91)

$$L \simeq hocolim_P \mathcal{D}$$

Compactified Affine Arrangements

Definition

Consider a subspace arrangement $\mathcal{A}$ in $\mathbb{R}^n$, let $\hat{\mathcal{A}}$ denote arrangment of one point compactification of A_i's in S^n. Let $\hat{L}$ denote the corresponding link and P denote the corresponding intersection poset.

Define a diagram over P,

$$\begin{aligned} \hat{\mathcal{D}}(p) &= S^{d(p)} \ (\, d(p) := dim(\mathcal{D}(p))) \\ \hat{D}(p \to q) &= \text{constant} \end{aligned}$$

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Spherical Arrangements

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Let $\mathcal{A}$ be a central arrangements of subspaces in $\mathbb{R}^n$. The spherical arrangement is a finite collection $\mathcal{A}_S = \{S_1 \ldots, S_m\}$ of spheres $S_i = A_i \cap S^{n-1}$ in S^{n-1}. Let P denote the intersection poset and L_S denote the link.

Define a P-diagram of spaces as -

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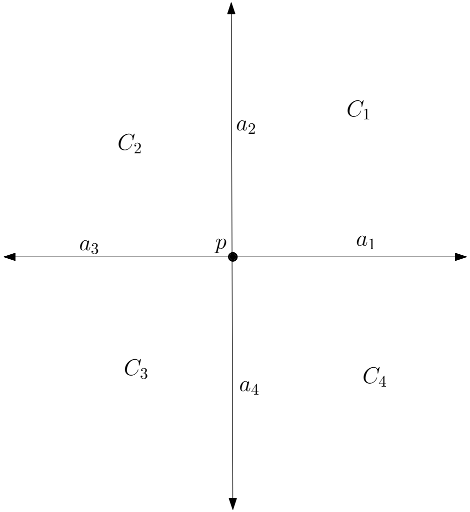
$$L_S \simeq hocolim_P \mathcal{D}_S$$

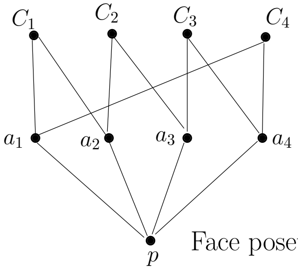
Real Arrangements of Hyperplanes

Definition

- A real arrangement of hyperplanes is a finite collection $\mathcal{A} = \{H_1, \ldots, H_m\}$ of affine hyperplanes in $\mathbb{R}^n$.
- Set of codimension 0 cells is called *chambers*, denoted by $\mathcal{C}(\mathcal{A})$.
- The face poset $\mathcal{F}$ is the poset of cells ordered by reverse containment.
- The complexified complement $M(\mathcal{A})$ is the complement of the union of complexified hyperplanes in $\mathbb{C}^n$.
- For a face F and a chamber C, let $F \circ C$ denote the unique chamber containing F that lies on the same side as C with respect to F.

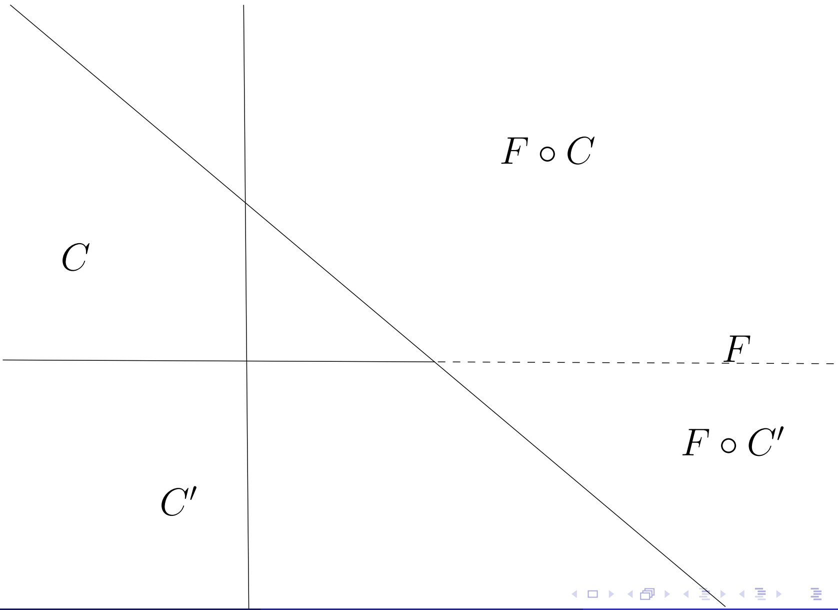
Example of an Arrangement





Face poset

Face Action on Chambers



Combinatorial Model for $M(\mathcal{A})$

Definition

Let $\mathcal{A}$ be an arrangement of hyperplanes in $\mathbb{R}^n$ and consider the set

$$\mathcal{S} := \{[F, C] \in \mathcal{F} \times \mathcal{C}(\mathcal{A}) \mid C < F\}$$

with the partial order given by

$$[F_1, C_1] \prec [F_2, C_2] \Longleftrightarrow F_1 < F_2 \; \& \; C_1 = F_1 \circ C_2.$$

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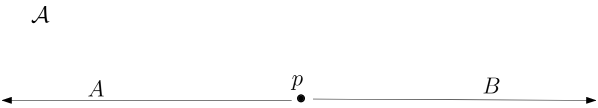
Theorem (Salvetti - 87)

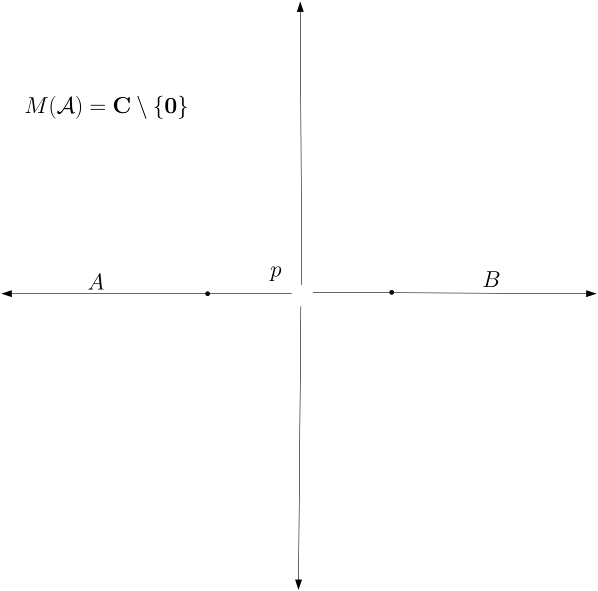
If $\mathcal{A}$ is a hyperplane arrangement in $\mathbb{R}^n$ and $Sal(\mathcal{A})$ then the associated Salvetti complex $Sal(\mathcal{A})$ embeds into the complexified complement and is also a strong deformation retract of $M(\mathcal{A})$.

The Salvetti Complex



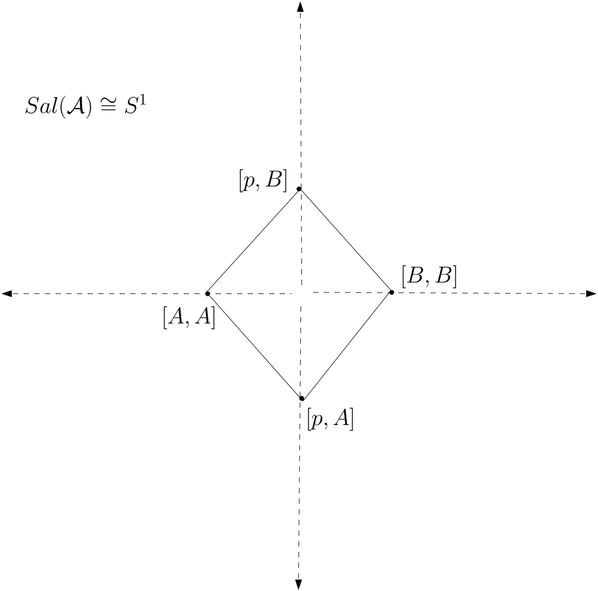
The Salvetti Complex

$M(\mathcal{A}) = \mathbf{C} \setminus \{\mathbf{0}\}$



The Salvetti Complex

$Sal(\mathcal{A}) \cong S^1$



$M(\mathcal{A})$ as a Homotopy Colimit

Given a real arrangement of hyperplanes $\mathcal{A}$, define a diagram of spaces $\mathcal{D}$ over the face poset $\mathcal{F}$ by

$$\mathcal{D}(F) := \{C \in \mathcal{C}(\mathcal{A}) \mid C < F\}$$

and the maps

$$\begin{aligned} \mathcal{D}(F_1 \to F_2): \quad & \mathcal{D}(F_1) \to \mathcal{D}(F_2) \\ & C \mapsto F_2 \circ C \end{aligned}$$

$M(\mathcal{A})$ as a Homotopy Colimit

Given a real arrangement of hyperplanes $\mathcal{A}$, define a diagram of spaces $\mathcal{D}$ over the face poset $\mathcal{F}$ by

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Then,

Theorem (Delucchi - 06)

$$hocolim_{\mathcal{F}}\mathcal{D} \cong Sal(\mathcal{A})$$

$M(\mathcal{A})$ as a Homotopy Colimit, continued

Given a real arrangement of hyperplanes $\mathcal{A}$, define a diagram of spaces $\mathcal{E}$ over the face poset $\mathcal{F}^{op}$ by

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2. Colimits
3. Homotopy Colimits
4. Examples
   - Examples from Arrangements
5. Applications
6. The Toolkit
7. Nutshell

Why Care ?

- Homotopy colimits come equipped with lot of fancy tools from category theory.
- The Bousfield-Kan spectral sequence can be used to understand (co)homology of *hocolim* in terms of the spaces involved.
- The fundamental group(oid) can also be expressed in terms of the fundamental group(oid)s of the input spaces (Seifert-van Kampen-Brown theorem).
- They provide input spaces for important theorems in homotopy theory.

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A List of Fancy Tools

- Projection lemma
- Homotopy lemma
- Wedge lemma
- Quillen's theorem A
- Simplicial model lemma

Projection Lemma

Lemma

Let P be a finite poset and let $\mathcal{D}\colon P \to Top$ be a diagram of spaces such that for each $p \in P$ the induced map $colim_{P_{<p}}\mathcal{D} \to \mathcal{D}(p)$ is a closed cofibration. Then the natural projection map

$$hocolim_P\mathcal{D} \to colim_P\mathcal{D}$$

is a homotopy equivalence.

Homotopy Lemma

Lemma

Let $\mathcal{D}$ and $\mathcal{E}$ be two P-diagrams s.t. there is a map $\alpha_p : \mathcal{D}(p) \to \mathcal{E}(p)$ for every $p \in P$. If α_p is a (weak) homotopy equivalence for every p then the induced map

$$\hat{\alpha} : hocolim_P \mathcal{D} \to hocolim_P \mathcal{E}$$

is also a (weak) homotopy equivalence.

Wedge Lemma

Lemma (Ziegler & Živaljević - 91)

Let $\mathcal{D}$ be a P-diagram so that there exists points $c_p \in \mathcal{D}(p)$ $\forall p \in P$ such that Image $\{(\mathcal{D}(q \to p))\} = \{c_p\}$ for all $q \to p$, then -

$$hocolim_P \mathcal{D} \simeq \Delta(P) \vee \bigvee_{p \in P} (\mathcal{D}(p) * \Delta(P_{<p})).$$

Homotopy Types of Links

Theorem (Ziegler & Živaljević - 91)

Let $\mathcal{A}$ be an arrangement of subspaces with L as its link, then -

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If $\mathcal{A}$ is a spherical arrangement then the homotopy type of the link is

$$L \simeq \bigvee_{p \in P} (\Delta(P_{<p}) * S^{d(p)-1})$$

Quillen's Theorem A

Theorem

If $f: P \to Q$ *is a poset map s.t.* $\Delta(f^{-1}(Q_{\geq q}))$ *is contractible* $\forall q$ *then,*

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Corollary (Order Homotopy Theorem)

If $f : P \to P$ *is a poset map such that* $f(p) \leq p \; \forall p$*, then*

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Corollary (Order Homotopy Theorem)

If $f : P \to P$ is a poset map such that $f(p) \leq p$ $\forall p$, then

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Corollary

Let $f : P \to Q$ be a map satisfying the hypothesis of theorem A. If $\mathcal{D}$ is any Q-diagram and $f^*\mathcal{D}$ the corresponding (pull back) P-diagram then,

$$hocolim_P(f^*\mathcal{D}) \overset{\simeq}{\to} hocolim_Q\mathcal{D}$$

A Generalization

Theorem (Björner, Wachs, Welker - 04)

Let $f : P \to Q$ be a poset map such that $\Delta(Q)$ is connected and for all $q \in Q$ the fiber $\Delta(f^{-1}(Q_{\leq q}))$ is $l(f^{-1}(Q_{<q}))$-connected. Then

$$\Delta(P) \simeq \Delta(Q) \vee \bigvee_{q \in Q} \Delta(f^{-1}(Q_{\leq q})) * \Delta(Q_{>q})$$

Simplicial Model Lemma

Definition (Poset Limit of a Diagram ($Plim\mathcal{D}$))

Let $\mathcal{D}\colon P \to pos$ be a diagram of posets. Then $Plim\mathcal{D}$ is again a poset whose objects are

$$\coprod_{p\in P} \{p\} \times \mathcal{D}(p)$$

and the order relations are

$$(p_1, q_1) \leq (p_2, q_2) \iff \begin{cases} p_1 \leq p_2 \text{ and} \\ q_1 \leq \mathcal{D}_{p_2 \to p_1}(q_2) \end{cases}$$

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Lemma

Let $\mathcal{D}$ be a P-diagram of posets, define a new diagram $\Delta(\mathcal{D})\colon P \to Top$ by assigning $\Delta(\mathcal{D}(p))$ to each p. Then we have;

$$hocolim_P \Delta(\mathcal{D}) \simeq \Delta(Plim\mathcal{D})$$

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- Complement of a c.r. arrangement is a gluing construction,
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- need a better result explaining the nature of the gluing, and
- how about complex arrangements ?

Trailer

Next episode

Trailer

Next episode

- (Co)homology of the homotopy colimits.
- Fundamental group(oid) of the homotopy colimits.

Thank You !