Homotopy Theoretic Techniques in Arrangements

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Configuration Spaces: Geometry, Combinatorics and Topology
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Outline

Outline

1. Recap

2. Cohomology of Homotopy Colimits
   - Constructive Approach
   - Examples

3. The Fundamental Group

Last Week

- Homotopy colimit is a functorial construction.
- Homotopy colimits are homotopy invariant.
- There is a natural map $hocolim\mathcal{D} \to colim\mathcal{D}$ which is not always a homotopy equivalence.
- There is more than one way to construct the homotopy colimit (up to homotopy equivalence).

Homotopy Colimit

Definition

Let P be a ranked poset and $\mathcal{D}\colon P \to Top$ be a diagram of spaces then the homotopy colimit of $\mathcal{D}$ is defined as

$$hocolim_P \mathcal{D} := \coprod_{p \in P} (\Delta(\mathcal{F}_{\leq p}) \times \mathcal{D}(p))/\sim$$

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the equivalence relation is defined as follows. For all $q \geq p$ consider the maps:

$$\alpha : \begin{cases} \Delta(P_{\leq p}) \times \mathcal{D}(q) & \to & \Delta(P_{\leq p}) \times \mathcal{D}(p), \\ (p, x) & \mapsto & (p, \mathcal{D}(q \geq p)(x)), \end{cases}$$

$$\beta : \begin{cases} \Delta(P_{\leq p}) \times \mathcal{D}(q) & \to & \Delta(P_{\leq q}) \times \mathcal{D}(q), \\ (p, x) & \mapsto & (\Delta(q \geq p)(p), x). \end{cases}$$

Then '$\sim$' is the transitive closure of $\alpha(p,x) \sim \beta(p,x)$.

back

Subspace of Joins

Lemma

Let $\mathcal{D}\colon P \to Top$ be a diagram over a finite poset P. Let $J(\mathcal{D}) := *_{p\in P}\mathcal{D}(p)$ be the join of all spaces in this diagram and let

$$\Delta(\mathcal{D}) := \{\sum_{i=0}^{m} t_i x_i \in J(\mathcal{D}) \mid x_i \in \mathcal{D}(p_i), p_0 \leq \cdots \leq p_m, x_{j+1} \mapsto x_j\}$$

Then,

$$\Delta(\mathcal{D}) \cong hocolim_P \mathcal{D}$$

Why *hocolim* ?

- Salvetti complex is a homotopy colimit.

Why *hocolim* ?

- Salvetti complex is a homotopy colimit.
- Homotopy colimits come equipped with lot of fancy tools from category theory.
- The Bousfield-Kan spectral sequence can be used to understand (co)homology of *hocolim* in terms of the spaces involved.
- The fundamental group(oid) can also be expressed in terms of the fundamental group(oid)s of the input spaces (Seifert-van Kampen-Brown theorem).

Outline

The Bousfield-Kan Spectral Sequence

Let $\mathcal{D}\colon P \to Top$ be a diagram of spaces, defined over an arbitrary poset P. Then there is a spectral sequence converging to $H^*(hocolim_P\mathcal{D})$, its second page is,

$$E_2^{r,s} = \varprojlim{}^r H^s(\mathcal{D}, \mathbb{Z})$$

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Where,

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- lim is the limit functor $Ab^{P^{op}} \to Ab$.

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What is lim ?

- Universal cone on a diagram.
- Right adjoint to the diagonal functor $\Delta\colon C \to C^J$.
- Products, equalizers and pullbacks are examples of lim.
- lim is left exact.

Cochains with Coefficients in a Functor

Let $F\colon P \to Ab$ be a functor;

$$\begin{aligned} C^k(P,F) &:= \{\phi\colon N_kP \to \coprod_{x\in P} F(x) \mid \phi(x_0 < \cdots < x_k) \in F(x_k)\} \\ &= \prod_{x_0<\cdots<x_k} F(x_k) \end{aligned}$$

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$$= \prod_{x_0<\cdots<x_k} F(x_k)$$

for $0 \leq j \leq k+1$ define the maps $\delta_k^j\colon C^k \to C^{k+1}$ as

$$(\delta_k^j \phi)(x_0 < \cdots < x_{k+1}) := \phi(x_0 < \cdots < \hat{x}_j < \cdots < x_{k+1})$$

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$$(\delta_k^j\phi)(x_0 < \cdots < x_{k+1}) := \phi(x_0 < \cdots < \hat{x}_j < \cdots < x_{k+1})$$

differential $d^k\colon C^k \to C^{k+1}$ is

$$d^k := \sum_{j=0}^{k}(-1)^j\delta_k^j + (-1)^{k+1}F(x_k < x_{k+1}) \circ \delta_k^{k+1}$$

Right Derived Functors of lim

$$\varprojlim^r F := H^r((C^*, d^*))$$

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Properties -

- $\varprojlim^0 F \cong \lim F$
- For a short exact sequence $0 \to F \to G \to H \to 0$ there is a LES

$$0 \to \lim F \to \lim G \to \lim H \to \lim^1 F \to \cdots$$

- $\{\varprojlim^n\}$ form a cohomological δ-functor.
- If P is a poset of rank k then $\varprojlim^r = 0$ for $r \geq k+1$.
- If M is a G-module considered as a diagram then

$$\varprojlim^r M = H^r(G, M)$$

- $\varprojlim^r$ are the cohomotopy groups of cosimplicial replacement of F.

Cohomology of Pushouts

Consider the pushout;

$$F : (Z \stackrel{f}{\leftarrow} X \stackrel{g}{\rightarrow} Y)$$

then applying $H^s(-, \mathbb{Z})$

$$H^s(F, \mathbb{Z}) : H^s(Z) \stackrel{f^*}{\rightarrow} H^s(X) \stackrel{g^*}{\leftarrow} H^s(Y)$$

(which is a pullback of abelian groups)

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(which is a pullback of abelian groups)

$$\begin{aligned} E_2^{0,s} &\cong Ker(H^s(Z) \oplus H^s(Y) \to H^s(X)) = Ker < f^*, -g^* > \\ E_2^{1,s} &\cong Coker(H^s(Z) \oplus H^s(Y) \to H^s(X)) = Coker < f^*, -g^* > \end{aligned}$$

Sheaves over Posets

Consider a poset P with 'ideal' topology induced by the order.
Given a diagram of spaces $\mathcal{D}\colon P \to Top$, consider the following sheaf on P, for $r \geq 0$:

- $\mathcal{H}^r\mathcal{D}\colon P \to Ab$,
- $\mathcal{H}^r\mathcal{D}(p) := H^r(\mathcal{D}(p))$,
- $\mathcal{H}^r\mathcal{D}(p \leq q) := H^r(\mathcal{D}(p)) \to H^r(\mathcal{D}(q))$

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Theorem

There is a spectral sequence converging to $H^*(hocolim_P\mathcal{D})$ given by

$$E_2^{r,s} = H^s(P, \mathcal{H}^r\mathcal{D})$$

Filtration on *hocolim*

Assume that $\mathcal{F}$ is a poset of cells of a regular cell complex of dimension n, i.e.

- $\mathcal{F}$ is a ranked poset,
- $\Delta(\mathcal{F}_{<x}) \simeq S^{r-1}$, where $0 < r = rk(x) \leq n$.

Let $\mathcal{D}\colon \mathcal{F} \to \mathit{Top}$ be a diagram of spaces with $X := \mathit{hocolim}_{\mathcal{F}}\mathcal{D}$

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$$X_r := \coprod_{rk(x) \leq r} (\Delta(\mathcal{F}_{\leq x}) \times \mathcal{D}(x))/ \simeq$$

there is an increasing filtration

$$\emptyset = X_{-1} \subset X_0 \subset \cdots \subset X_n = X$$

Leray's Theorem

Theorem

Let X be a topological space with an increasing filtration $\{X_i\}_{i=-1}^{n}$ *of subspaces. Then there is a (first quadrant) cohomology spectral sequence converging to* $H^*(X,\mathbb{Z})$*. The first page of this spectral sequence is*

$$E_1^{r,s} = H^{r+s}(X_r, X_{r-1}; \mathbb{Z})$$

and the differentials

$$d_1^{r,s} \colon H^{r+s}(X_r, X_{r-1}) \to H^{r+1+s}(X_{r+1}, X_r)$$

are the connecting homomorphisms of the long exact sequence in cohomology arising from the triple (X_{r+1}, X_r, X_{r-1}).

Successive Quotients

For $0 \leq r \leq n$, let $Q_r = X_r / X_{r-1}$ then,

$$\begin{aligned} Q_r &= \bigvee_{rk(x)=r} [\Sigma(\Delta(\mathcal{F}_{<x}) \times \mathcal{D}(x))]/[\{*\} \times \mathcal{D}(x)] \\ &= \bigvee_{rk(x)=r} [S^r \times \mathcal{D}(x)]/[\{*\} \times \mathcal{D}(x)] \end{aligned}$$

▸ definition

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Corollary

For any $0 \leq r \leq n$ and $s \geq 0$ we have,

$$H^{r+s}(Q_r, \mathbb{Z}) \cong \bigoplus_{rk(x)=r} H^s(\mathcal{D}(x), \mathbb{Z})$$

The Differentials

Let $r + s = k$ then $d_1 : H^k(Q_r) \to H^{k+1}(Q_{r+1})$
Plan : follow a cocyle in $H^n(Q_r)$ through various maps that appear while constructing the connecting homomorphism.

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- wlog $[\alpha] \in H^k(\mathcal{D}(x)),\ rk(x) = r$ then as a cochain

$$\alpha = \phi \otimes \psi \in C^r(S^r) \otimes C^s(\mathcal{D}(x))$$

- necessary condition that $\phi \otimes \psi$ is a cocycle is that ψ is a cocycle in $C^s(\mathcal{D}(x))$
- apply the coboundary map of $C^k(X_{r+1}/X_{r-1})$ to $\phi \otimes \psi$

$$\delta\phi \otimes \psi \in C^{r+1}(S^{r+1}) \otimes C^s(\mathcal{D}(x))$$

The Differential

- then use the map

$$1 \otimes f_{yx}^{*} \colon C^{r+1}(S^{r+1}) \otimes C^{s}(\mathcal{D}(x)) \to C^{r+1}(S^{r+1}) \otimes C^{s}(\mathcal{D}(y))$$

  to land in $C^{k+1}(Q_{r+1})$
- $\delta\phi \otimes f_{yx}^{*}(\psi)$ is a cocycle in $C^{k+1}(Q_{r+1})$
- $\delta\phi(y) = [x : y]$ where $\mathrm{rank}(y) = r + 1$
- The cocycle now looks like

$$\delta\phi \otimes f_{yx}^{*}(\psi) = \sum_{rk(y)=r+1} [x : y]\phi \otimes f_{yx}^{*}(\psi)$$

The Consequence

Theorem

Let $\mathcal{D}\colon \mathcal{F} \to$ Top be a diagram of spaces and suppose that $\mathcal{F}$ is a face poset of a regular CW complex. Then there is a spectral sequence converging to hocolim$_{\mathcal{F}}\mathcal{D}$ with the E_1 page given by

$$E_1^{r,s} = \bigoplus_{rk(x)=r} H^s(\mathcal{D}(x))$$

and the differentials

$$d_1^{r,s}(\alpha_x) = \sum_{rk(y)=r+1} [x : y]\, f_{yx}^*(\alpha_x),$$

where

- *$[x : y]$ is the incidence index of x in y*
- *α_x is a cohomology class in $H^s(\mathcal{D}(x))$*
- *f_{yx}^* is induced on cohomology by $f_{yx}\colon \mathcal{D}(y) \to \mathcal{D}(x)$.*

Arbitrary Posets

Let P be an arbitrary poset and $\mathcal{D}$ a P-diagram and let $Sd(P)$ denote the poset of chains in P ordered by inclusion

Lemma

Consider the poset map $g\colon Sd(P) \to P$ given by $g(\tau) = max\ \tau$, for a chain τ. Given a diagram $\mathcal{D}\colon P \to Top$, define a new diagram $g^*\mathcal{D}$ as follows

$$(g^*\mathcal{D})(\tau) := \mathcal{D}(max\ \tau)$$

$$(g^*\mathcal{D})(\tau > \tau') := \mathcal{D}(max\ \tau \geq max\ \tau')\ \ \forall \tau, \tau' \in Sd(P)$$

Then the map g induces a homotopy equivalence

$$hocolim(g^*\mathcal{D}) \simeq hocolim(\mathcal{D})$$

The General Case

Theorem

Let $\mathcal{D}\colon P \to Top$ be a diagram of spaces over any poset P. Then there is a spectral sequence converging to $H^*(hocolim\mathcal{D})$ with the E_1 page given by,

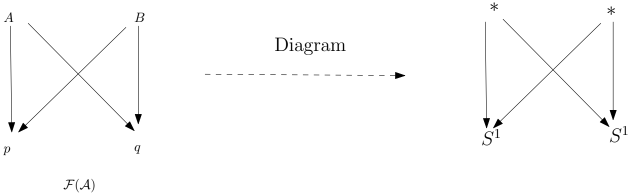
$$E_1^{r,s} := \bigoplus_{x_0 < \cdots < x_r} H^s(\mathcal{D}(x_r)), \qquad x_i \in P \quad \forall i$$

and the differentials $d_1^{r,s} : E_1^{r,s} \to E_1^{r+1,s}$,

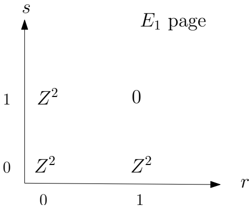
$$\begin{aligned}(d_1^{r,s}\alpha)(x_0 < \cdots < x_{r+1}) &= \sum_{i=0}^{r}(-1)^i \alpha(x_0 < \cdots < \hat{x}_i < \cdots < x_{r+1}) \\ &+ (-1)^{r+1} f^*_{x_{r+1}x_r}(\alpha(x_0 < \cdots < x_r))\end{aligned}$$

Where $\alpha(x_0 < \cdots < x_r)$ is a cohomology class in $H^s(\mathcal{D}(x_r))$.

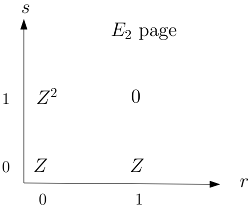
Example 1



A
B
p
q
$\mathcal{F}(\mathcal{A})$
Diagram
*
*
S^1
S^1

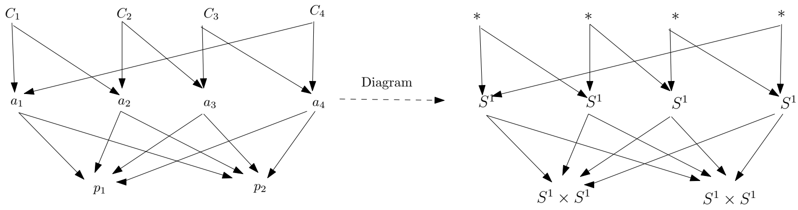


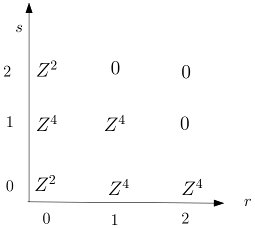
E_1 page
s
r
1
0
0
1
Z^2
0
Z^2
Z^2

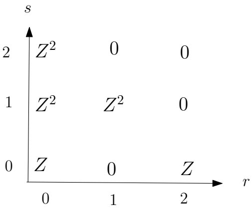


E_2 page
s
r
1
0
0
1
Z^2
0
Z
Z

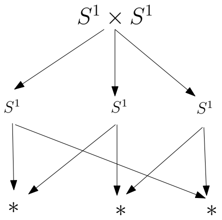
Example 2


C_1 C_2 C_3 C_4
a_1 a_2 a_3 a_4
p_1 p_2
Diagram
$*$ $*$ $*$ $*$
S^1 S^1 S^1 S^1
$S^1 \times S^1$ $S^1 \times S^1$


s
2 Z^2 0 0
1 Z^4 Z^4 0
0 Z^2 Z^4 Z^4
0 1 2
r


s
2 Z^2 0 0
1 Z^2 Z^2 0
0 Z 0 Z
0 1 2
r

Example 3



s \ r	0	1	2
2	0	0	Z
1	0	Z^3	Z^2
0	Z^3	Z^3	Z

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2	0	0	Z
1	0	Z	0
0	Z	0	0

Outline

A Million Dollar Question

Is there a functor $F\colon \{I-\text{ diagram of Groups }\} \to \text{ Groups }$ such that if $\mathcal{D}$ is a I-diagram of connected and pointed spaces, then

$$F(\pi_1(\mathcal{D})) \cong \pi_1(hocolim_I \mathcal{D})$$

where $\pi_1(\mathcal{D})$ is the induced diagram of fundamental groups.

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Seifert-van Kampen Theorem

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A counter example

Consider S^2 as a $\mathbb{Z}/2$-diagram (via antipodal action)

$$\begin{aligned} \pi_1(S^2) &\cong \{*\} \\ \pi_1(colim_{\mathbb{Z}/2} S^2) &\cong \pi_1(\mathbb{R}P^2) \cong \mathbb{Z}/2 \end{aligned}$$

The Land of Groupoids

Definition

A groupoid is a small category in which every morphism is invertible.

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Definition

Let Y be a CW complex and A be a choice of 0-skeleton for Y. Then the fundamental groupoid of (Y, A) denoted by $\tilde{\pi}_1(Y, A)$ is defined as the small category whose object set is A and whose morphisms are the homotopy classes of paths between any two of these zero cells.

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Example

$$\tilde{\pi}_1(S^2, \{x, -x\}) \cong [\bullet \rightleftharpoons \bullet] =: J$$

Seifert-van Kampen-Brown Theorem

Theorem (E. Dror Farjoun '04)

For any P-diagram of spaces $\mathcal{D}$ there is a natural equivalence of groupoids:

$$\tilde{\pi}_1(hocolim_P \mathcal{D}) \xrightarrow{\cong} hocolim_P(\tilde{\pi}_1 \mathcal{D})$$

If the homotopy colimit is a connected space, this gives a corresponding isomorphism of groups.
(Further, if the diagram of groupoids is cofibrant then the right hand side can be replaced by the strict colimit of groupoids.)

Example

Example

Consider the antipodal action of $\mathbb{Z}/2$ on S^2 then the LHS is-

$$\tilde{\pi}_1(\mathbb{R}P^2) \cong \mathbb{Z}/2$$

and the right hand side

$$\begin{aligned} hocolim_{\mathbb{Z}/2}\tilde{\pi}_1(S^2, \{x, -x\}) &= hocolim_{\mathbb{Z}/2}J \\ &= colimJ \\ &= J/(\mathbb{Z}/2) \\ &\cong \mathbb{Z}/2 \end{aligned}$$

homotopy colimit is neccessary

Let $G = \mathbb{Z}/2$ and it acts on $\mathbb{I} = [0,1]$ by $x \mapsto 1-x$. Then,

$$\tilde{\pi}_1(\mathbb{I}, \{0,1\}) = J$$

If we take the colimit of this diagram -

$$\mathbb{I}/G \simeq *$$

however on the right hand side -

$$(\tilde{\pi}_1\mathbb{I})/G = J/G \cong \mathbb{Z}/2$$

the two sides do not agree. Now taking homotopy colimit of the diagram:

$$hocolim_G(\mathbb{I}) = \mathbb{R}P^\infty$$

Grothendieck Construction

Let $F\colon I \to Cat$ be a functor from I to the category of small categories.

Definition

Grothendieck construction$(Gr(F))$, is a category

- whose objects are the pairs (i,x) where $i \in Ob(I)$ and $x \in Ob(F(i))$.
- A morphism $(i,x) \to (j,y)$ is a pair (f,g) where $f \in Hom_I(i,j)$ and $g \in Hom_{F(j)}(F(f)(x),y)$.

Arrows compose according to the rule

$$(f,g) \circ (f',g') = (f'',g'')$$

where

- $f'' = f \circ f'$
- $g'' = g \circ F(f)(g')$

hocolim for groupoids

Definition

Let $\mathcal{G} : P \to Gpd$ be a diagram of groupoids the homotopy colimit of $\mathcal{G}$ is defined as follows

$$hocolim_P\, \mathcal{G} := \tilde{\pi}_1(B\, Gr(\mathcal{G}))$$

This definition is a consequence of Thomason's theorem.

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This definition is a consequence of Thomason's theorem.

Example

Let P be a finite, connected poset and $\mathcal{D}\colon P \to Top$ be the constant diagram, hence the corresponding diagrams of fundamental groupoids $\tilde{\pi}_1\mathcal{D}$ is also constant. Then,

$$\begin{aligned} hocolim_P \tilde{\pi}_1\mathcal{D} &= \tilde{\pi}_1\Delta(P) \\ &= \pi_1(\Delta(P), *) \end{aligned}$$

Diagrams of Connected Spaces

Corollary

Let P be a small category which is connected and has an initial object. Let $\mathcal{D}$ be a P-diagram of connected and pointed spaces. Then,

$$\pi_1(hocolim_P \mathcal{D}) \cong colim_P \pi_1 \mathcal{D}$$

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Corollary

Let P be a small category which is connected and $\mathcal{E} : P \to \{Groups\}$ be a diagram of groups, there is a pushout diagram of groupoids:

$$\begin{array}{ccc} \tilde{\pi}_1(BP) & \longrightarrow & hocolim_P(\mathcal{E}) \\ \downarrow & & \downarrow \\ 1 & \longrightarrow & colim_P\ (\mathcal{E}) \end{array}$$

If BP is connected, then, up to equivalence, this is a pushout of groups.

Thank You !

The town of Manarola (Cinque Terra).