Configuration Spaces: Algebra, Combinatorics, Topology
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Brunnian braids and representations of arrangement groups
(joint work w/ D. Cohen & R. Randell)

Outline:
I. Introduction - homomorphisms of arrangement groups
II. Brunnian braids and Stanford's Thm
III. Generalization to arrangement groups
IV. Injectivity criterion; examples
V. Embeddings in right-angled Artin groups
VI. Decomposable arrangements; residual nilpotence

I. $\mathcal{A}$: affine arrangement in $\mathbb{C}^\ell$; $M = M_{\mathcal{A}} = \mathbb{C}^\ell - \cup\mathcal{A}$;
$G = G_{\mathcal{A}} = \pi_1(M)$. Observe, if $\mathcal{B} \subseteq \mathcal{A}$, $M_{\mathcal{A}} \hookrightarrow M_{\mathcal{B}}$ induces $G_{\mathcal{A}} \to G_{\mathcal{B}}$
For $S \subseteq \mathcal{A}$ set $G_S = \pi_1(\mathbb{C}^\ell - \cup S)$,

$$\rho_S : G \longrightarrow G_S \quad \text{induced by inclusion.}$$

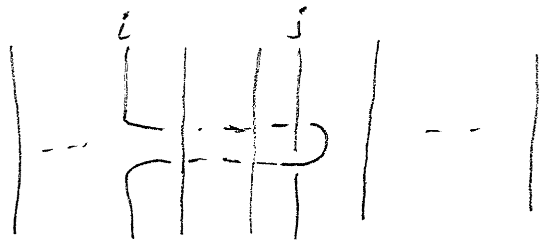
$\mathcal{S}$: family of subarrangements of $\mathcal{A}$

$$\rho = \prod_{S \in \mathcal{S}} \rho_S : G \longrightarrow \prod_{S \in \mathcal{S}} G_S$$

<u>Goal</u>
Choose $\mathcal{S}$ so that $G_S$ is "nice" for $S \in \mathcal{S}$, and $\rho$ is injective. (e.g. $\mathcal{S}$ = set of supersolvable (fiber-type) subarrangements.)

II. Example : $\mathcal{A}_\ell = \{H_{ij} \mid 1 \le i < j \le \ell\}$ : braid arrangement in $\mathbb{C}^\ell$

$G = \langle a_{ij} \mid 1 \le i < j \le \ell \rangle$




$a_{ij}$

For $I \subseteq \{1, \dots, \ell\}$,
the projection $\mathbb{C}^\ell \to \mathbb{C}^I$
restricts to (up to homotopy) the inclusion

$$M \hookrightarrow M_I = \{\underline{x} \in \mathbb{C}^\ell \mid x_i \neq x_j \ \forall\, i,j \in I\}$$
$$= \text{complement of } S = \{H_{ij} \mid i,j \in I\}$$

$$p_I : G \longrightarrow G_I \quad \text{kills } \{a_{ij} \mid i \notin I \text{ or } j \notin I\}$$

($p_I \leftrightarrow$ delete strands outside of $I$).

For $\mathcal{S}$ a family of subsets of $\{1, \dots, \ell\}$, get

$$p = \prod_{I \in \mathcal{S}} p_I : G \longrightarrow \prod_{I \in \mathcal{S}} G_I$$

Special case : $\mathcal{S} = \{I \mid |I| = \ell - 1\}$.

Then $\ker(p) = \mathrm{Brunn}_\ell$ = gp. of Brunnian braids on $\ell$ strands

Brunnian = becomes trivial upon deletion of any strand

Ex/ $[a_{12}, a_{23}]$

T. Stanford gave a description of Brunnian braids in terms of iterated commutators of braid generators ("monic commutators")

<u>Thm</u> (Stanford) The kernel of $\rho = \prod_{I \in \mathcal{A}} \rho_I$ is generated by iterated commutators of $a_{ij}$'s having at least one entry involving an index outside of $I$, for each $I \in \mathcal{A}$.

Note: This is an infinite set, even for $\ell = 3$.

## III. Generalization to arrangement groups

Let $G$ be a group with finite generating set $Y$.

For $S \subseteq Y$ let $G_S = G / \langle\langle Y - S \rangle\rangle$ ← normal closure

and $\rho_S : G \to G_S$.

<u>Def'ns</u> $S \subseteq Y$ is <u>retractive</u> iff the composite

$$\langle S \rangle \hookrightarrow G \longrightarrow G_S \text{ is an isomorphism.}$$

A family $\mathcal{A}$ of subsets of $Y$ is a <u>retractive family</u> iff every intersection of elements of $\mathcal{A}$ is retractive.

A <u>monic commutator</u> in $G$ relative to $Y$ is defined recursively by

(1) $y^{\pm 1}$ is a monic commutator $\forall\ y \in Y$.

(2) if $a$ and $b$ are monic commutators then $[a, b]$ is a monic commutator.

The <u>support</u> of a monic commutator $g \in G$ is

$$\operatorname{supp}(g) = \bigcap \{ S \subseteq Y \mid g \in \langle S \rangle \}.$$

We say a monic commutator $g$ has support <u>transverse</u> to $\mathcal{S}$ iff $\mathrm{supp}(g) \not\subseteq S \quad \forall\, S \in \mathcal{S}$.

<u>Thm</u> (CFR) Suppose $\mathcal{S}$ is a retractive family. Then the kernel of $\rho = \prod_{S \in \mathcal{S}} \rho_S : G \longrightarrow \prod_{S \in \mathcal{S}} G_S$ is generated by monic commutators whose support is transverse to $\mathcal{S}$.

pf: same as Stanford

Note: The hypothesis applies to families of parabolic subgroups of Coxeter or Artin groups.

<u>Retractive families for arrangement groups</u>

Let $\mathcal{A}$ be an affine arrangement and $G = G(\mathcal{A})$.

Let $L(\mathcal{A})$ be the intersection poset of $\mathcal{A}$. For $X \in L(\mathcal{A})$

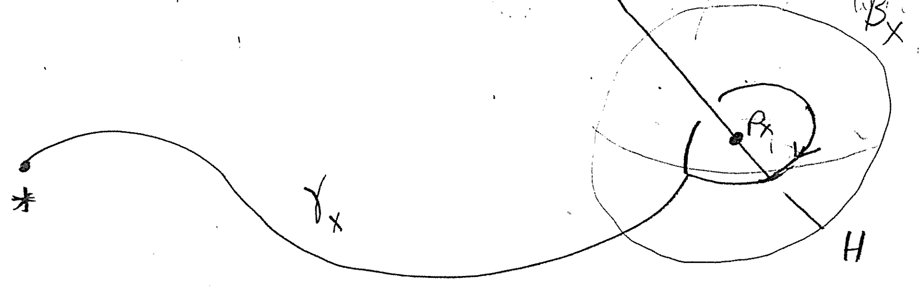
let $\mathcal{A}_X = \{ H \in \mathcal{A} \mid X \subseteq H \}$ ; $M_X = \mathbb{C}^\ell - \cup \mathcal{A}_X$ ;

$p_X$ = a generic point on $X$ ;

$B_X$ = a small ball centered at $p_X$, satisfying $B_X \cap M_X \subseteq M$.

Note: $B_X \cap M \hookrightarrow M \hookrightarrow M_X$ is a homotopy equivalence.

Let $a_{X,H} \in G = G(\mathcal{A})$ be as pictured:



Def A standard generating set for $G$ is a subset $Y$ of $\{ a_{X,H} \mid X \in L(\mathcal{A}),\ H \in \mathcal{A}_X \}$ containing exactly one element of the form $a_{X,H}$ for each $H \in \mathcal{A}$.

(A standard generating set generates $G$.).

For $X \in L$ let $S_X = \{ a_{X,H} \mid H \in \mathcal{A}_X \}$.

Lemma Suppose $Y$ is a standard generating set for $G$ containing $S_X$. Then $S_X$ is retractive relative to $Y$.

Thm Let $\mathcal{S}_0 \subseteq L(\mathcal{A})$. Let $Y$ be a standard generating set of $G$ containing $S_X$, for each $X \in \mathcal{S}_0$, then $\mathcal{S}$ is a retractive family relative to $Y$.

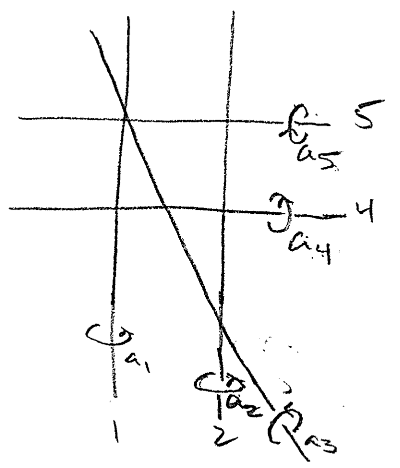
When the hypothesis of the previous thm is satisfied we say $Y$ is adapted to $\mathcal{S}_0$.

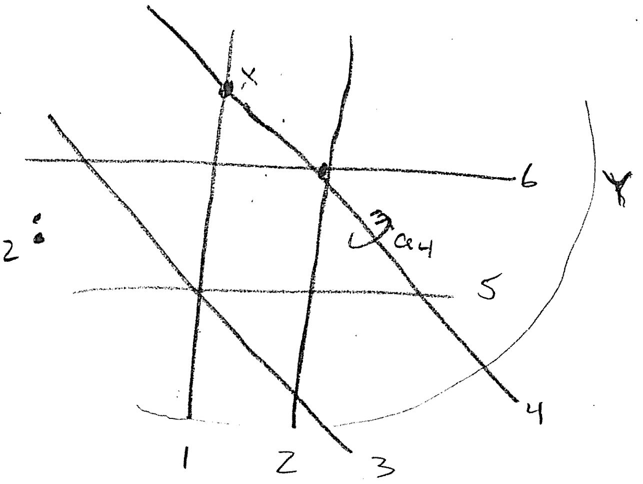
Let $\mathcal{S}_\infty$ be a family of sets of mutually parallel hyperplanes of $\mathcal{A}$.

Thm Suppose $Y$ is a standard generating set of $G$ adapted to $\mathcal{S}_0 \subseteq L(\mathcal{A})$. Then $\mathcal{S} = \mathcal{S}_0 \cup \mathcal{S}_\infty$ is a retractive family relative to $Y$.

pf: If $S \in \mathcal{S}_\infty$ then $G_S$ is a free group, hence is Hopfian.

Exy

$X_3$: 


$X_2$: 


$\mathcal{S} = \{12, 135, 45\}$ is retractive rel $\mathcal{Y}$.

$\mathcal{S} = \{12, 135, 246, 34, 56\}$ is retractive rel $\mathcal{Y}$.

<u>Thm</u> Suppose $\mathcal{Y}$ is a standard generating set adapted to $\mathcal{S}_0 \subseteq L(\mathcal{A})$, $\mathcal{S}_\infty$ is a family of parallel classes in $\mathcal{A}$, and $\mathcal{S} = \mathcal{S}_0 \cup \mathcal{S}_\infty$. Suppose $\mathcal{S}$ is an anti-chain and $\bigcup \mathcal{S} = \mathcal{A}$. Then $\rho$ is injective iff

(i) $[a_H, a_K] = 1$ if $\{H, K\}$ is transverse to $\mathcal{S}$,

and (ii) $[a_H, [G_S, G_S]] = 1 \quad \forall\ S \in \mathcal{S},\ H \notin S$.

Exy In example $X_2$ above:

$[a_1, a_4^{a_6}] = 1$ by Randell presentation, but in fact one can show $[a_1, a_4] = 1$. Et cetera.

$$[a_2, [G_{135}, G_{135}]] \subseteq \langle [a_2, [a_3, a_5]] \rangle = 1$$

because $G_{135} \cong \langle a_1 a_3 a_5 \rangle \times \langle a_3, a_5 \rangle$. Et cetera.

II. Embedding $G$ in a product of free groups.

Suppose now $\mathcal{A}$ is a central arrangement, with projectivization $\overline{\mathcal{A}}$. Let $\overline{G} = \pi_1(\mathbb{CP}^{\ell-1} - \cup \overline{\mathcal{A}})$; and $\overline{G_S} = \pi_1(\mathbb{CP}^{\ell-1} - \cup \overline{S})$. Assume $\overline{a}_1 \cdots \overline{a}_n = 1$ in $\overline{G}$.

$\overline{G} \cong \langle a_1, \dots, a_{n-1} \rangle := G_0 \subseteq G$.

Assume $\mathcal{S}_0$ consists of rank-two flats. Then $\overline{G_S}$ is free of rank $|S| - 1$.

Assume $\overline{a}_1 \cdots \overline{a}_n = 1$.

$$
\begin{tikzcd}
G_0 \arrow[d, hook] \arrow[drr, "\rho|_{G_0}"] \arrow[dd, bend right, "\cong"'] & & \\
G \arrow[rr, "\rho"] \arrow[d] & & \prod_{S \in \mathcal{S}} G_S = A \arrow[d] \\
\overline{G} \arrow[rr, "\overline{\rho}"] & & \prod_{S \in \mathcal{S}} \overline{G_S} = \overline{A}
\end{tikzcd}
$$

<u>Thm</u> Suppose $H_n \notin S$ for some $S \in \mathcal{S}$. Then $\rho$ is injective iff $\rho|_{G_0}$ is injective, iff $\overline{\rho}$ is injective.

<u>Thm</u> $\operatorname{im}(\overline{\rho})$ is normal in $\overline{A}$, and $\overline{A}/\operatorname{im}(\overline{\rho})$ is free abelian of rank

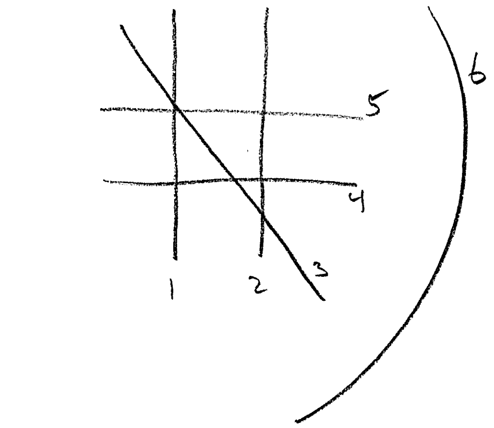
$$\sum_{S \in \mathcal{S}} |S| - |\mathcal{A}| - |\mathcal{S}| + c$$ where $c$ is

the number of components in the incidence graph of

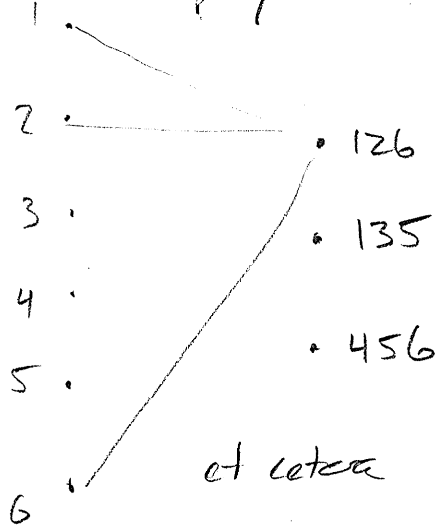
$\mathcal{A}$ with $\mathcal{A}$.

<u>Cor</u> If $\bar{p}$ is injective then $\bar{G}$ is of type $F_{m-1}$ but not $F_m$, where $m = |\mathcal{A}|$.

<u>Examples</u>



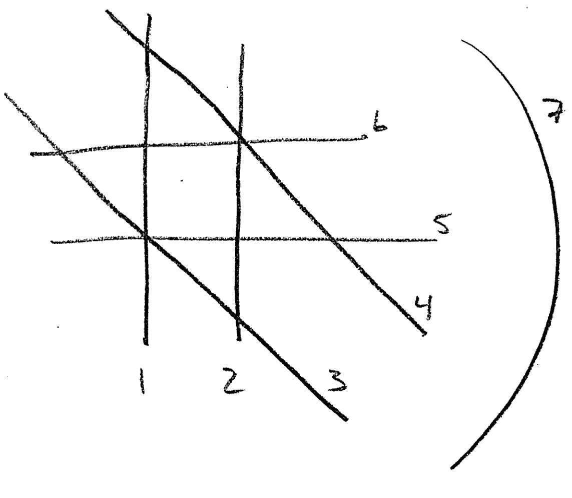

$\bar{p}$ is injective.




$c = 1$

et cetera

$3 \cdot 3 - 6 - 3 + 1 = 1$, $\bar{A}/\mathrm{im}(\bar{p}) \cong \mathbb{Z}$,
$\bar{G}$ is $F_2$ but not $F_3$.




$c = 1$, $m = 5$, $n = 7$

$5 \cdot 3 - 7 - 5 + 1 = 4$

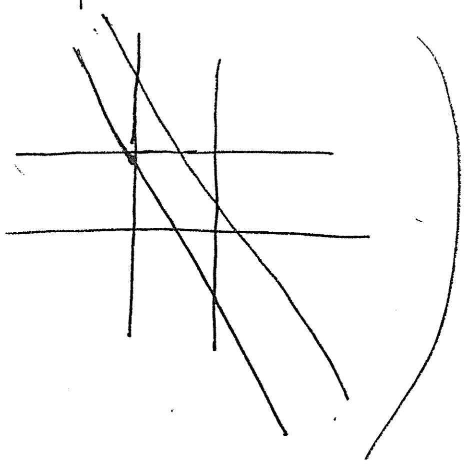
$\bar{p}$ is injective,

$\bar{A} \cong \prod^{5} F_2$ ;

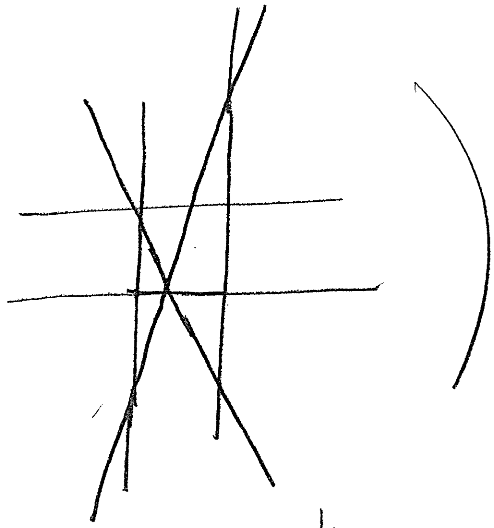
$\bar{A}/\mathrm{im}(\bar{p}) \cong \mathbb{Z}^4$

$\bar{G}$ is $F_4$ but not $F_5$.

A couple of new (old) examples



$\bar{p}$ is injective,
$c=1,\ m=4,\ n=7$



$\bar{p}$ is injective,
$c=1,\ m=4,\ n=7.$

$$\bar{A} = \prod^{4} F_2$$

$$\bar{A}/\mathrm{im}(\bar{p}) \cong \mathbb{Z}^2 .$$

$$(4\cdot 3 - 7 - 4 + 1 = 2)$$

$\bar{G}$ is $F_3$ but not $F_4$.