Generalized Moment-Angle Complexes

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joint work with Tony Bahri, Martin Bendersky, and Sam Gitler

The setting and the problems:

This report addresses

- joint work with Tony Bahri, Martin Bendersky, and Sam Gitler in addition to
- an engineering question in joint work with Dan Koditschek, and Clark Haynes.
- Much of this report gives a picture of 'how and where' some mathematical structures fit together with an outline of basic proofs and basic properties.

The setting and the problems continued:

- Let (X, A) denote a pair of spaces.
- Let K denote an abstract simplicial complex with m vertices.
- The main subject of these lectures are properties of so-called generalized

  generalized moment-angle complexes

  denoted

  $$Z(K; (X, A))$$

  which are subspaces of the product X^m.

Ingredients:

- Let $(\underline{X}, \underline{A}) = \{(X_i, A_i, x_i)_{i=1}^m\}$ denote a set of triples of CW–complexes with base-point x_i in A_i.
- Let K denote an abstract simplicial complex with m vertices labeled by the set

  $$[m] = \{1, 2, \ldots, m\}.$$

  Thus, a $(k-1)$-simplex σ of K is given by an ordered sequence

  $$\sigma = (i_1, \cdots, i_k)$$

  with $1 \leq i_1 < \cdots < i_k \leq m$ such that if $\tau \subset \sigma$, then τ is required to be a simplex of K. In particular the empty set ϕ is a subset of σ and so it is in K. Define define the length of I by the formula $|I| = k$.

Definition of generalized moment-angle complexes:

- The *generalized moment-angle complex or polyhedral product functor* determined by $(\underline{X}, \underline{A})$ and K denoted

$$Z(K; (\underline{X}, \underline{A}))$$

is defined as follows: For every σ in K, let

$$D(\sigma) = \prod_{i=1}^{m} Y_i, \quad \text{where} \quad Y_i = \begin{cases} X_i & \text{if} \quad i \in \sigma \\ A_i & \text{if} \quad i \in [m] - \sigma \end{cases}$$

with $D(\emptyset) = A_1 \times \cdots \times A_m$.

- The generalized moment-angle complex is

$$Z(K; (\underline{X}, \underline{A})) = \bigcup_{\sigma \in K} D(\sigma) = \text{colim } D(\sigma).$$

Remarks:

- In the special case where $X_i = X$ and $A_i = A$ for all $1 \leq i \leq m$, it is convenient to denote the generalized moment-angle complex by $Z(K; (X, A))$ to coincide with the notation in work of Graham Denham, and Alex Suciu who inspired much of the work here.

Examples:

- Let K denote the 2-point complex $\{1,2\}$ with $(X,A)=(D^1,S^0)$. Then

$$Z(K;(D^1,S^0))=(D^1\times S^0)\cup(S^0\times D^1)=S^1.$$

- Let K denote the 2-point complex $\{1,2\}$ with $(X,A)=(D^n,S^{n-1})$. Then

$$Z(K;(D^n,S^{n-1}))=(D^n\times S^{n-1})\cup(S^{n-1}\times D^n)=S^{2n-1}.$$

More examples:

- More generally, $Z(K;(D^2,S^1))$ has the homotopy type of the complement of unions of certain coordinate planes in $\mathbb{C}^m$ corresponding to 'coordinate subspace arrangements' as described next.
- Given a simplicial complex K with m vertices, and a simplex $\omega \in \Delta[m-1]$, define

$$L_\omega = \{(z_1,\cdots,z_m) \in \mathbb{C}^m |\ z_{i_1} = \cdots = z_{i_k} = 0\}$$

for

$$(i_1,\cdots,i_k) \in \omega.$$

Complements of coordinate planes:

Definition:

$$U(K) = \cup_{\omega \notin K} \mathbb{C}^m - L_\omega$$

The following is an elegant result due to Taras Panov.

Proposition: The natural inclusion

$$Z(K; (D^2, S^1)) \to U(K)$$

is

1. a homotopy equivalence (more precisely, a strong deformation), and
2. $(S^1)^m$-equivariant.

Examples continued:

- In addition, $(S^1)^m = T^m$ acts naturally on the product $(D^2)^m$ and on $Z(K;(D^2,S^1))$. The Davis-Januszkiewicz space is the associated Borel construction

$$\mathcal{DJ}(K) = ET^m \times_{T^m} Z(K;(D^2,S^1)).$$

- A special case of a beautiful theorem of Denham-Suciu as well as Davis-Januskiewicz gives that $\mathcal{DJ}(K)$ is homeomorphic to

$$Z(K;(\mathbb{CP}^\infty,*)).$$

Examples continued:

- The generalized moment-angle complex $Z(K;(S^1,*))$ is a $K(\pi,1)$ where π is a right-angled Artin group (as described by Ruth Charney in her lecture).
- These spaces are examples of spaces listing positions of robotic 'legs' as illustrated next.

A moving example:

- The following is a moving example of $Z(K;(S^1,*))$.

Yet more examples

- Configuration spaces of certain singular spaces are sometimes homotopy equivalent to generalized moment-angle complexes as discovered by Sun Qiang in his thesis. Let X denote a topological space. Define

$$\mathrm{Conf}(X, k) = \{(x_1, \ldots, x_k) \in X^k |\ x_i \neq x_j \text{ if } i \neq j\}.$$

Yet more examples continued:

- Let
$$X_n = \mathbb{R}^n \vee \mathbb{R}^n.$$
Then Qiang shows that the configuration space
$$\mathrm{colim}_{n\to\infty}\mathrm{Conf}(\mathbb{R}^n \vee \mathbb{R}^n, k)$$
is homotopy equivalent to a certain choice of $Z(K;(X,A))$.
- Here,
$$K = \Delta[k-1]_0,$$
the 0-skeleton of the $(k-1)$-simplex, and
$$(X,A) = (\mathbb{RP}^\infty, *).$$

- These results at the nascent stages represent ways to measure

  'bottlenecks'

  such as in traffic flow, and configuration spaces of singular spaces.

Initial structure theorems (from the eyes of homotopy theory)

- The purpose of the next few sections is to provide structure for the generalized moment-angle complex after suspending the space.
- The motivation is partially homological as well as computational and arises from a geometric decomposition.
- The basic decompositions arise from the suspension of a space X is given by
$$\Sigma(X) = C_+(X) \cup C_-(X)$$
where $C_+(X)$ is the "upper cone" and $C_-(X)$ is an "lower cone" glued together along X.

Technical point regarding base-points

- Notice that $\Sigma(X)$ does not have a natural base-point. This is remedied by using the 'reduced suspension'

$$C_+(X) \cup C_-(X)/([0,1] \times *_X).$$

Wedge products, and smash products

- Let $(X, *_X)$ and $(Y, *_Y)$ be pointed CW complexes.
- The wedge product (or 'wedge' in the vernacular)

$$X \vee Y$$

is the subspace of the product $X \times Y$ given by

$$X \vee Y = (X \times *_Y) \cup (*_X \times Y).$$

Wedge products, and smash products continued

- The smash product (or 'smash' in the vernacular)

$$X \wedge Y$$

is the quotient space of the product $X \times Y$ given by

$$X \wedge Y = (X \times Y)/(X \vee Y).$$

Examples of smash products:

- The smash product
$$S^p \wedge S^q$$
is homotopy equivalent to
$$S^{p+q}.$$

- The smash product
$$S^1 \wedge X$$
is homotopy equivalent to the suspension
$$\Sigma(X).$$

Elementary properties of suspensions, wedge products, and smash products

- Let X be a connected CW complex, and

$$H_i(X)$$

the i-th homology group of X. Then there are natural isomorphisms:

$$\Delta : H_{i+1}\Sigma(X) \to H_i(X)$$

for all $i > 0$.

Elementary properties of suspensions, wedge products, and smash products continued

- If X and Y are pointed CW-complexes, there are natural (pointed) homotopy equivalences

$$\Sigma(X \vee Y \vee (X \wedge Y)) \to \Sigma(X \times Y).$$

- Thus there is a homotopy equivalance

$$S^{p+1} \vee S^{q+1} \vee S^{p+q+1} \to \Sigma(S^p \times S^q).$$

- The spaces $X \vee Y \vee (X \wedge Y)$, and $X \times Y$ are usually not homotopy equivalent as they usually have different cup product structures in cohomology.

Smash moment-angle complexes and their applications

The purpose of the next few slides is to describe the structure of the suspension of moment-angle complexes

$$Z(K; (\underline{X}, \underline{A}))$$

in terms of 'smash moment-angle complexes' to be made precise next. This information is then applied to obtain information about

- homology,
- cohomology for various cohomology theories in addition to singular cohomology, and
- the cup product structure for the cohomology ring of the moment angle complex.

‘Smash moment-angle complexes’

- Recall that moment-angle complexes are subspaces of product spaces.
- Passing to the ‘world’ of pointed spaces (in which all maps are required to preserve base-points), there are natural analogues called

  ‘smash moment-angle complexes’

  where all products in the earlier definition are replaced by smash products.

‘Smash moment-angle complexes’ continued

- Given $(\underline{X}, \underline{A}, \underline{*})$, and a simplicial complex K with m vertices, recall that

$$Z(K; (\underline{X}, \underline{A}))$$

is a subspace of the product

$$X_1 \times X_2 \times \cdots \times X_m.$$

- Define the ‘smash moment-angle complex’

$$\widehat{Z}(K; (\underline{X}, \underline{A}))$$

to be the image of $Z(K; (\underline{X}, \underline{A}))$ in the smash product

$$X_1 \wedge X_2 \wedge \cdots \wedge X_m.$$

Example of a decomposition:

A classical theorem is stated first.

- **Theorem 1**
  Given pointed CW-complexes $X_1, \cdots, X_m$, there are natural homotopy equivalences

$$H : \Sigma(X_1 \times X_2 \times \cdots \times X_m) \to \Sigma(\bigvee_{1 \le i_1 < i_2 \cdots < i_k \le m} X_{i_1} \wedge \cdots \wedge X_{i_k}).$$

Sketch proof of this example:

- Given a sequence
$$I = (i_1, \cdots, i_k)$$
with $1 \leq i_1 < \cdots < i_k \leq m$, define he natural projection map
$$\pi_I : X_1 \times X_2 \times \cdots \times X_m \to X^I$$
onto the product X^I specified by I.
- Thus there are induced maps
$$\pi_I : X_1 \times X_2 \times \cdots \times X_m \to \widehat{X}^I.$$
where $\widehat{X}^I$ is the smash product of the X_{i_j}.

Sketch proof of this example continued:

- The homotopy classes of pointed maps out of suspensions can be added. Thus, do so.
- The induced map gives the equivalence stated in Theorem 1.

Language:

- Let K denote a simplicial complex with m vertices. Given a sequence

  $$I = (i_1, \cdots, i_k)$$

  with $1 \leq i_1 < \cdots < i_k \leq m$, define $K_I \subseteq K$ to be the

  full sub-complex

  of K consisting of all simplices of K which have all of their vertices in I, that is $K_I = \{\sigma \cap I | \sigma \in K\}$.

A second decomposition:

- **Theorem 2**
  Let K be an abstract simplicial complex with m vertices. Given $(\underline{X}, \underline{A}) = \{(X_i, A_i)\}_{i=1}^{m}$ where (X_i, A_i, x_i) are pointed triples of CW-complexes, the homotopy equivalence of Theorem 1 induces a natural pointed homotopy equivalence

$$H : \Sigma(Z(K; (\underline{X}, \underline{A}))) \to \Sigma(\bigvee_{I \subseteq [m]} \widehat{Z}(K_I; (\underline{X_I}, \underline{A_I}))).$$

Remarks:

- A sketch of a proof together with applications to Davis-Januskiewicz spaces, right-angled Artin groups as well as other moment-angle complexes will be given in the next few lectures.

Partial orderings:

- Let K denote a simplicial complex.
- There is a partially ordered set (**poset**) $\bar{K}$ associated to any simplicial complex K as follows. A point σ in $\bar{K}$ corresponds to a simplex $\sigma \in K$ with order given by *reverse* inclusion of simplices.
- Thus $\sigma_1 \leq \sigma_2$ in $\bar{K}$ if and only if $\sigma_2 \subseteq \sigma_1$ in K.
- The empty simplex $\emptyset$ is the unique maximal element of $\bar{K}$.
  Let P be a poset with $p \in P$.
- There are further posets given by

$$P_{\leq p} = \{q \in P | q \leq p\}$$

as well as

$$P_{<p} = \{q \in P | q < p\}.$$

Thus

$$\bar{K}_{<\sigma} = \{\tau \in \bar{K} | \tau < \sigma\} = \{\tau \in K | \tau \supset \sigma\}.$$

The order complex and further decompositions:

- Given a poset P, there is an associated simplicial complex $\Delta(P)$ called the order complex of P which is defined as follows.
- Given a poset P, the *order complex* $\Delta(P)$ is the simplicial complex with vertices given by the set of points of P and k-simplices given by the ordered $(k+1)$-tuples $(p_1, p_2, \ldots, p_{k+1})$ in P with $p_1 < p_2 < \cdots < p_{k+1}$.

A further decomposition:

To state the next theorem, recall that the symbol $*$ denotes the join of two spaces.

- **Theorem 3**

  Let K be an abstract simplicial complex with m vertices, and let
  $$(\underline{X},\underline{A}) = \{(X_i, A_i, x_i)\}_{i=1}^{m}$$
  denote m choices of connected pairs of CW-complexes with the inclusion $A_i \subset X_i$ null-homotopic for all i. Then there is a homotopy equivalence
  $$\Sigma(Z(K;(\underline{X},\underline{A}))) \to \Sigma \bigvee_{I} \bigvee_{\sigma \in K_I} |\Delta((\overline{K}_I))_{<\sigma}| * \widehat{D}(\sigma).$$

Examples:

- **Corollary**

  Let (X_i, A_i, x_i) denote the triple $(D^{n+1}, S^n, *)$ for all i. Then there are homotopy equivalences

$$\Sigma(Z(K;(D^{n+1}, S^n))) \to \bigvee_{I \notin K} \Sigma^{2+n|I|}|K_I|.$$

Examples continued:

- **Theorem 4**

  Let K be an abstract simplicial complex with m vertices and $(\underline{X}, \underline{A})$ have the property that all the A_i are contractible. Then there is a homotopy equivalence

$$\Sigma\big(Z\big(K; (\underline{X}, \underline{A})\big)\big) \to \Sigma\big(\bigvee_{I \in K} \widehat{X}^I\big).$$

Examples continued:

- **Theorem 5**

  Let K be an abstract simplicial complex with m vertices and $(\underline{X}, \underline{A})$ have the property that the the X_i are contractible for all i. Then there is a homotopy equivalence

$$\Sigma Z(K; (\underline{X}, \underline{A})) \to \Sigma(\bigvee_{I \notin K} |K_I| * \widehat{A}^I).$$

Examples continued:

- **Theorem 6**

  Let K be an abstract simplicial complex with m vertices and $(\underline{X}, \underline{A})$ have the property that all the A_i are contractible. Then there is a homotopy equivalence

$$\widehat{Z}(K; (\underline{X}, \underline{A})) = \begin{cases} * & \text{if } K \text{ is not the simplex } \Delta[m-1], \text{ and} \\ \widehat{X}^{[m]} & \text{if } K \text{ is the simplex } \Delta[m-1] \end{cases}$$

  where

$$\widehat{X}^{[m]} = X_1 \wedge \cdots \wedge X_m$$

  the m-fold smash product.

Some history: 1

- The spaces $Z(K;(D^2,S^1))$ are at the confluence of work of many people. A short introduction to a small sample of some of this work is given next.

Some history: 2

- Generalized moment-angle complexes, have been studied by topologists since the 1960's thesis of G. Porter. In the 1970's E. B. Vinberg and in the late 1980's S. Lopez de Medrano developed some of their features.

Some history: 3

- In seminal work during the early 1990's, M. Davis and J. Januszkiewicz introduced quasi-toric manifolds, a topological generalization of projective toric varieties which were being studied intensively by algebraic geometers. They observed that every quasi-toric manifold is the quotient of a moment-angle complex $Z(K; (D^2, S^1))$ by the free action of a real torus.
- Namely, a quasi-toric manifold M is given by the quotient

$$M = Z(K; (D^2, S^1))/T^d$$

where

$$T^d \subset T^m$$

is a sub-torus of T^m, and where T^d acts freely on $Z(K; (D^2, S^1))$.

Some history: 4

- Let R denote the ring given by $\mathbb{Z}$, $\mathbb{Q}$, or a finite field. Given K, there is an associated ring known as the Stanley-Reisner ring of K, defined below, and denoted $R[K]$ here. The ring $R[K]$ is a quotient of a finitely generated polynomial ring $P(K) = R[v_1, \ldots, v_m]$ with generators v_i for each vertex of K and relations given by

$$v_{i_1} \cdots v_{i_k} = 0$$

  for every simplex $\sigma = (i_1, \cdots, i_k)$ in K.
- M. Hochster, in purely algebraic work, calculated the Tor-modules $\mathrm{Tor}_{P(K)}(R[K], R)$ in terms of the full subcomplexes of K. In this work Hochster also produced an algebraic decomposition of these Tor-modules.

Some history: 5

- Subsequently, and independently, Goresky-MacPherson studied the cohomology of complements of subspace arrangements $U(\mathcal{A})$ and related decompositions of their cohomology. These spaces included complements of certain coordinate subspace arrangements. A more direct proof was subsequently given by Ziegler-Zivaljević.

Some history: 6

- Later, as well as independently, Davis-Januszkiewicz introduced manifolds now called quasi-toric varieties, a topological generalization of projective toric varieties. They proved that a certain choice of Borel construction for the space $Z(K;(D^2,S^1))$ which they define precisely had cohomology ring given by $R[K]$ for $R=\mathbb{Z}$, the Stanley-Reisner ring of K. These spaces are now known as the Davis-Januszkiewicz spaces.

Some history: 7

- Buchstaber-Panov synthesized these different developments by proving that the spaces $Z(K;(D^2,S^1))$ are strong deformation retracts of complements of certain coordinate subspace arrangements $U(\mathcal{A})$ appearing earlier in work of Goresky-MacPherson. They also proved that the cohomology algebra of $Z(K;(D^2,S^1))$ is isomorphic to $\mathrm{Tor}_{P(K)}(\mathbb{Z}[K],\mathbb{Z})$ which had been considered earlier by Hochster.

Some history: 8

- There has been further, extensive work on moment-angle complexes. A few samples are Notbohm-Ray, Grbic-Theriault, Strickland, Baskakov, Buchstaber-Panov, Panov, Baskakov-Buchstaber-Panov, Buchstaber-Panov-Ray, M. Franz, Panov-Ray-Vogt, and Kamiyama-Tsukuda.
- Further elegant closely related results are due to De Concini-Procesi, Danilov, Hu, Jewell, Jewell-Orlik-Shapiro. Extensions to generalized moment-angle complexes had been defined earlier in work of Anick.
- Applications to robotics are the focus of work by Cohen-Haynes-Koditschek.

Some history: 9

- The direction of this current joint work is guided by the development in elegant work of Denham-Suciu.

Thank you very much.

- Please remember to hand in the homework !