Applications of Hodge theory to cohomology of local systems

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5. Log-Canonical thresholds and Shokurov Conjecture

References:

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Goals:

Obtain information on fundamental groups of quasiprojective varieties, their homotopy types, special classes of quasi-projective varieties such as complements to hypersurfaces or arrangements of hyperplanes.

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Study invariants of fundamental groups and homotopy types.
This talk: **Alexander type invariants twisted by a non abelian representation and their Hodge theoretical counterparts**.

Quick survey of untwisted Alexander polynomials of algebraic curves in $\mathbf{C}^2$:

Let C be possibly reducible curve in $\mathbf{C}^2$ (e.g. an arrangement of lines). Assume C is transversal to the line at infinity. Let $G = \pi_1(\mathbf{C}^2 - C)$. If r is the number of irreducible components then

$$G/G' = \mathbf{Z}^r$$

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G has prefered surjection $lk : G \to \mathbf{Z}$. Let $K = Kerlk$ and let

$$\widetilde{\mathbb{C}^2 - C} \to \mathbb{C}^2 - C$$

be the corresponding covering space. Generator $T \in Im(lk)$ of the Galois group of covering acts on $\widetilde{\mathbb{C}^2 - C}$

Definition

(Global) Alexander polynomial of C is

$$det(T_* - tI, H_1(\widetilde{\mathbb{C}^2 - C}, \mathbb{C})) = det(T_* - t, ker(lk)/(kerlk)' \otimes \mathbb{C})$$

For irreducible curve last polynomial is

$$det(T_* - tI, G'/G'' \otimes \mathbb{C})$$

For each singular point one has **local** Alexander polynomial: $\Delta(C, P)(t)$

Example

(of local Alexander polynomials) For cusp:

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Example

For oridinary triple point (intersection of three lines):

$$\Delta(t) = (t-1)^2(t^2+t+1)$$

Theorem

*(***Dependence of local type: Divisibility***) (L-1982)*
$\Delta_C(t)$ divides product of local Alexander polynomials of all singularities. For irredcudible curve $\Delta_C(1) \neq 0$.

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Example

The Alexander polynomial of an arrangement of lines with triple and double points only $\Delta = (t-1)^{r-1}(t^2+t+1)^s$ for some s.

Dependence of global information: *(L -1983)*

$$\Delta_C(t) = \Pi_\alpha[(t - exp(2\pi i\alpha)(t - exp(-2\pi i\alpha)]^{s(\alpha,C)}$$

where α runs through all roots of local Alexander polynomials and $s(\alpha, C)$ is the dimension of linear system of curves passing through the singularities of C and defined using α and the data of singularities of C.

For an irreducible curve with nodes and cusps:

$$\Delta_C(t) = 1 \textit{ if } d \not\equiv 0(6), \quad \textit{otherwise } \Delta_C(t) = (t^2 - t + 1)^s$$

where s is the difference betwen the actual and expected dimensions of curves of degree $d - 3 - \frac{d}{6}$ *passing through cusps*

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For curves with triple points (arrangments of r lines)

$$\Delta_C(t) = (t-1)^{r-1} \text{ if } r \not\equiv 0(3) \text{ and otherwise}$$

$$\Delta_C(t) = (t-1)^{r-1}(t^2 + t + 1)^{\bar{s}}$$

where $\bar{s}$ *is difference betwen actual and expected dimensions of curves of degree* $d - 3 - \frac{d}{3}$ *passing through the triple points.*

Remarks

There is bound on the degree of the Alexander polynomial in terms of the degree of the curve (with J.I. Cogolludo):
For curves and nodes and cusps only:

$$dim G'/G'' = deg \Delta_C(t) \leq \frac{5}{3}d - 2$$

Definition

Twisted Alexander polynomial Defined for a CW-complex X with the following data:

$$\epsilon : \pi_1(X) \to \mathbf{Z}$$

and unitary representation:

$$\rho : \pi_1(X) \to U(V)$$

($U(V)$ is the unitary group of a hermitian space V). Cohomology of chain complex:

$$... \to C_i(\tilde{X}) \otimes_{Ker(\epsilon)} V \to$$

are $\mathbb{C}[\pi_1(X)/Ker\epsilon]$-modules. Here the action of $\pi_1(X)$ is given by $g(c \otimes v) = cg^{-1} \otimes gv$. Then

$$\Delta(X, \epsilon, \rho) = \Pi[ordTorH_k(\tilde{X}, \rho)]^{(-1)^{k+1}}$$

where $ord \oplus \mathbb{C}[t, t^{-1}]/(a_i) = \Pi_i a_i$.

Remarks

This definition is due to Lin and later studied by Wada and others. Useful for distingushing knots and links which cannot be done by usual Alexander polynomial. In the case of complements to plane algenraic curves twisted Alexander polynomials were studied by Cogolludo-Florens (also Suciu-Cohen).

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Remarks

In the case when X is a complements to plane curve (or arrangment of lines) $H_2(\tilde{X}, \rho)$ is free and hence $\Delta = \frac{OrdTorH_1}{OrdTorH_0}$

Theorem

Let C be a plane curve transversl to the line at infinity. Let γ *be the conjugacy class of boundary of a small disk transversal to* C. *Let* F *be the field generated by the eigenvalues of* $\rho(\gamma)$ *Then the roots of twisted Alexander polynomial belong to a cyclotomic extension of* F. *(of degree bounded by the degrees of root of local Alexander polynomials)*

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Instead the proof uses the Hodge theory of local systems

Twisted moduli space of local systems *Let X be a smooth quasi-projective variety. Assume for simplicity that X admits a smooth compactification $\bar{X}$ such that $H^1(\bar{X}, \mathbb{C}^*) = 0$.*

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A local system is unitary if this representation is unitary

Local systems of rank one i.e. the characters of π_1 are parametrized by $(\mathbb{C}^*)^{b_1}$ if $H_1(X, \mathbb{Z}) = \mathbb{Z}^{b_1}$ and in general the moduli space of rank one local systems is a finite union of ${\mathbb{C}^*}^{rkH_1(X,\mathbb{Z})}$.
Unitary local systems of rank one are parametrized by (a union of copies of) $U(1)^{b_1}$.

Definition

Cohomology of local systems: let $\tilde{X}$ .be the universal cover of X then $H_i(X, \rho)$ is the cohomology of chain complex:

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This space contains subsets of jumping cohomology.

Theorem

Structure of jumping loci for twisted cohomology *The subsets*

$$S^n_{\rho,l} = \{\chi \in Char^u \pi_1(X) | \dim H^n(X, L_{\rho\otimes\chi}) \geq l\}$$

are unions of finite collections of translated subgroups. If ρ *has an abelian image and if* $\bar{\rho}_1, .., \bar{\rho}_N \in Char^u(\pi_1(X))$ *are the irreducible components of* ρ*, then translations can be made by points generating a subgroup of* $Char^u(\pi_1(X))$ *containing the subgroup generated by* $\rho_1, ..., \rho_N$ *as a subgroup of finite order.*

Corollary

1. If ρ is trivial then this implies that the cosets have finite order.

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2. If $b_1 = 1$ then the characters corresponding to local system with non vanishing cohomology have as their values the root of unity.

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Definition

The twisted by ρ l-th characteristic variety (in degree n) (denoted as Ch^l_n) is the support of the module

$$\Lambda^l H_n(\tilde{X}_{ab}, V_{ab})$$

i.e., the subset of $Spec\mathbb{C}[H_1(X,\mathbb{Z})]$ consisting of prime ideals $\wp$ in $\mathbb{C}[H_1(X,\mathbb{Z})]$ such that localization of $\Lambda^l H_n(\tilde{X}_{ab}, V)$ at $\wp$ is non zero.

Remarks

One has obvious inclusions:

$$...Ch^{l+1}n \subseteq Ch^{l}_{n} \subseteq Ch^{l-1}_{n}...$$

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If codimension of this set is one, one can define **multivariable** *twisted Alexander polynomial (can work also in some cases when the codim is also zero)*

Theorem

Relation between roots of Alexander polynomial and charactersitic varieties. *Let X be a CW complex such that $H_1(X,\mathbb{Z})=\mathbb{Z}$, $\rho:\pi_1(X)\to U(V)$ is a unitary local system and $Ch^l_n\subseteq Spec\mathbb{C}[\mathbb{Z}]=\mathbb{C}^*$ is the collection of characteristic varieties associated with (X,ρ) in above definition. For $\xi\in\mathbb{C}^*$ let $l_n(\xi)=\max\{l\,|\xi\in Ch^l_n(X,\rho)\}$ and $b_n=\min\{l_n(\xi)|\xi\in Spec\mathbb{C}[\mathbb{Z}]\}$. Then $\Delta_n(\xi)=0$ if and only if $l_n(\xi)>b_n$.*

Theorem

Relationship between characterisitc varieties and jumping loci *If $\pi_i(X)=0$ for $2\le i<k$ or if $k=1$ then the dimension of the homology group $H_k(X,V\otimes L_\chi)$ corresponding to a character $\chi\in Char\pi_1(X)$ and a local system ρ is given by:*

$$\max\{i|\chi\in Ch^i_k(X,\rho)\}$$

Example

Varieties with range of vanishing of homotopy groups

1.Complements to germs of isolated non-normal crossings.

$f_1(x_1, ..., x_{n+1}) \cdot .... \cdot f_r(x_1, ..., x_{n+1}) = 0$

Example

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Example

2.Complement to a divisor having ample component and isolated non normal crossings (k is the dimension of divisor).

Example

3.More generally k can be the dimension of the set of of points of D in which D is not a normal crossing divisor.

Hodge theory and local systems *Let $\rho : \pi_1(X) \to U_N$ be N-dimensional unitary representation. There exist a locally trivial bundle $\mathcal{V}$ of rank N on $\bar{X}$ such that and meromorphic connection*

$$\nabla : \mathcal{V} \to \Omega^1_{\bar{X}}(logD) \otimes \mathcal{V}$$

for which restriction on X is flat with holonomy $\rho : \pi_1(X) \to U_n$

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Extension $\mathcal{V}$ is **NOT** *unique but can be specified by requiring that the eigenvalues ξ of residues of connection along each component satisfy:*

$$0 \leq Re\xi < 1$$

(such canonical extension is called Deligne extension of bundle of connection)

Theorem

(Deligne-Timmerscheidt) **Hodge filtration** *Let X be a smooth quasi-projective variety and $\mathcal{V}$ be a unitary local system. Let $\bar{\mathcal{V}}$ be Deligne extension of logarithmic with holonomy ρ. Then there is spectral sequence:*

$$H^p(\Omega^q(logD) \otimes \bar{\mathcal{V}}) \Rightarrow H^{p+q}(X, \rho)$$

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Remarks

1.Deligne's situation: ρ is trivial, one-dimensional representation.

2.Also true in the case X is a complement in a small ball to a germ of a divisor with several irreducible componenets.

Theorem

Twisted jumping polytopes:
Let X be a quasiprojective manifold without non-trivial rank one local systems on a non singular compactification, $\rho : \pi_1(X) \to U_N$ be a N-dimensional unitary representation , $Char \pi_1(X)$ be the torus of characters of the fundamental group and $Char^u \pi_1(X)$ be the subgroup of unitary characters. Let $\mathcal{U}$ be the fundamental domain of $\pi_1(Char^u \pi_1(X))$ acting on the universal cover $\widetilde{Char^u \pi_1}(X)$ of the torus $Char^u \pi_1(X)$ and $\exp : \mathcal{U} \to Char^u \pi_1(X)$ be the universal covering map Then

$$S^{n,p}_{\rho,l} = \{\chi \in Char^u \pi_1(X) | \dim Gr^p_F H^n(V_\rho \otimes L_\chi) \geq l\}$$

is a finite union of polytopes in $\mathcal{U}$ i.e. the subsets, each of which is the set of solutions for a finite set of inequalities $L \geq 0$ where L is a linear function **with integer coefiicients if ρ is trvial**.

Example

Consider local case: $X = B - (f(x_1, ..., x_{n+1} = 0))$ where f has an isolated singularity. $\pi_1(X) = \mathbb{Z}$ *if* $n > 1$. Let $\Delta(t)$ be characteristic polynomial of monodromy or equivalently “Alexander polynomial” of the link of $f = 0$. Then $\xi \in \mathbb{C}^*$ defines a L_ξ local system sending $1 \to \xi$.

$$H^n(L_\xi) \neq 0 \textit{ iff } \Delta(\xi) = 0$$

$$Gr_F^p H^n(L_\xi) \neq 0 \iff \xi = exp(2\pi i\alpha)$$

where $\alpha \in \mathbf{Q}, \alpha \in [a, a+1]$ is element of spectrum of singularity $f = 0$ and a is determined by p, n.

Example

Given a collection of germs $f_1(x_1,...,x_{n+1}),...,f_r(x_1,...,x_{n+1})$ of reduced local equations of divisors $D_i = V((f_i))$ at a point $P \in X = \mathbf{C}^{n+1}$, one associates with each $\phi \in \mathcal{O}_P$ the top degree form:

$$\omega_\phi(j_1,...,j_r|m_1,....,m_r) =$$

$$f_1^{\frac{j_1-m_1+1}{m_1}} \cdot \ldots \cdot f_r^{\frac{j_r-m_r+1}{m_r}} \phi(x_1,...,x_{n+1})dx_1 \wedge \ldots \wedge dx_{n+1}$$

on unramified covering $X_{m_1,...,m_r}$ of $X - \sum D_i$ with Galois group $\oplus \mathbf{Z}/m_i\mathbf{Z}$. One can think that $X_{m_1,..,m_r}$ is subset of set of solutions:

$$z_1^{m_1} = f_1(x_1,...,x_{n+1})...$$

$$z_r^{m_r} = f_r(x_1,....,x_{n+1})$$

- A form ω_ϕ extends to a holomorphic form on a resolution of singularities of a compactification $\bar{X}_{m_1,...,m_r}$ of $X_{m_1,...,m_r}$ iff $(x_1,...,x_r)=(\frac{j_1+1}{m_1},....,\frac{j_r+1}{m_r})\in \mathbf{R}^r$ satisfies a system of linear inequalities i.e. belongs to a polytope $\mathcal{P}(\phi|f_1,...,f_r)$.

- This system can be obtained in terms of a log-resolution $\pi : Y \to X$ of principal ideals $(f_1), ..., (f_r)$ as above using resolution of $\bar{X}_{m_1,...,m_r}$ which is the resolution of quotient singularities of the normalization of $\bar{X}_{m_1,...,m_r} \times_X Y$. This leads to the following explicit collection of inequalities defining $\mathcal{P}(\phi|f_1, ..., f_r)$

$$\sum_{i=1}^{r} \alpha_{i,j}(1 - x_i) \leq \kappa_j + 1 + e_j(\phi) \text{ for } 1 \leq j \leq N.$$

is the multiplicity of $\pi^*(\phi)$ along E_j).

Remarks

1. For a polytope $\mathcal{P}(\phi_0|f_1,...,f_r)$, the germs ϕ such that the corresponding form extends as a holomorphic form for all $(\frac{j_1+1}{m_1},...,\frac{j_r+1}{m_r}) \in \mathcal{P}(\phi_0|f_1,...,f_r)$ form an ideal $\mathcal{A}(\mathcal{P}(\phi_0|f_1,...,f_r))$ in the local ring of P (ideal of quasi-adjunction).

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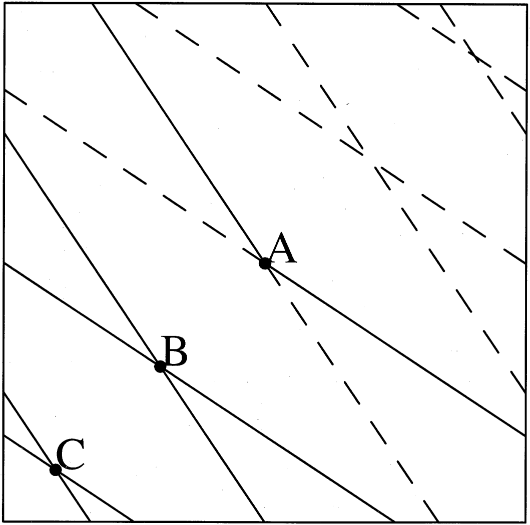
2. One has

$$\pi_1(B-(f_1\cdot...\cdot f_r=0))=\mathbb{Z}^r$$

$$Char^u_{B-(f_1\cdot...\cdot f_r=0)}=U(1)^r$$

Polytope $\mathcal{P}(\phi_0|f_1,...,f_r)$ is one of jumping polytops, specifically jumping for $Gr^0_F H^n(L_\chi)$

3. The ideal of quasi-adjunction $(j_1, ..., j_r, |m_1, ..., m_r)$ *coincides with the multiplier ideal of the divisor* $\sum \lambda_i(f_i)$ *where* $\lambda_i = 1 - \frac{j_i+1}{m_i}$*. (In the case* $r = 1$*, the collection of polytopes of quasi-adjunction becomes the collection of jumping coefficients of multiplier ideals. In this case, if the singularity of f is isolated, this collection coincides with the subset of spectrum of singularity in interval* $[0, 1]$*).*



Polytopes of quasiadjunction for
$(x^2+y^3)(x^3+y^2)$

The map $exp : [0,1]^r \to U(1)^r$ *takes solid lines* $2u+3v = 1/2, 3/2, 5/2$ *and* $3u+2v = 1/2, 3/2, 5/2$ *into union of segments. Zariski closure is the zero set of the multivariable Alexander polynomial:*

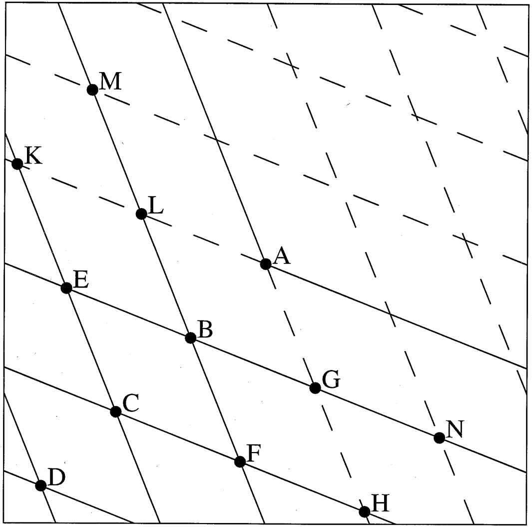
$$(t_1^2 t_2^3 + 1)(t_1^3 t_2^2 + 1) = 0$$

Multivariable Alexander polynomial has expression in terms of resolution of singularity $f_1(x,y)...f_r(x,y)$ *with exeptional curves* $E_1, ...E_k$

$$\Pi(t_1^{m_{1,j}} \cdot ....t_r^{m_{r,j}} - 1)^{\chi(E_i^0)}$$

where $m_{i,j}$ *is the multiplicity of pull back of* f_i *along* E_j, E_i^0 *is subset of point in* E_i *which are non-singular on exceptional set.*

In this example there are 4 exceptional curves with non zero euler characteristic.



Polytopes of quasiadjunction for
$(x^2+y^5)(x^5+y^2)$

The map exp $: [0,1]^r \to U(1)^r$ *takes solid lines into union of segments. Zariski closure is the zero set of the Alexander polynomial:*

$$(t_1^2 t_2^5 + 1)(t_1^5 t_2^2 + 1) = 0$$

Theorem

Relation between local and global polytopes quasi-adjunction *Let D be divisor with isolated non normal crossings in* $\mathbf{C}^{n+1}$ *(transversal to hyperplane at infinity) Let r be the number of components of D. Let* B_P *be a small ball about* $P \in \mathbb{C}^{n+1}$. *For each non-normal crossings P one has surjection:*

$$Char\pi_1(\mathbb{C}^{n+1} - D) \to Char\pi_1(B_P - B_P \cap D)$$

Let $b_P = rk\pi_1(B_P - B_P \cap D), b = rk\pi_1(\mathbb{C}^{n+1} - D)$ *Let* $\pi_P : [0,1]^{b_P} \to [0,1]^b$ *induced map of universal covers of subgroups of unitary characters. Then each global polytope of quasi-adjunction defines a non empty local polytope of quasi-adjunction. Vice-versa: Collection of local polytopes of quasi-adjunction (i.e. local jumping polytopes for each non-normal crossing point) yields a global polytope iff certain linear system of divisors correspodning to chosen polytopes is superabundant.*

Example

Arrangements of lines with triple points in $\mathbb{C}^2$.
Consider an arrangement of r lines with triple and double poitns only. Polytopes belong to the cube I^r ($I = [0,1]$). Local polytopes of $l_1 l_2 l_3 = 0$ is $x_1 + x_2 + x_3 = 1$ (zero set of Alexander polynomial is $t_1 t_2 t_3 = 1$). Global polytopes correspond to collections S of triple points and are sets of solutions to system of equations

$$x_{i_1} + x_{i_2} + x_{i_3} = 1$$

where triples (i_1, i_2, i_3) are indices of lines forming a tripe point in S. Such polytope is actual polytopes iff $dimH^1(\mathbf{P}^2, \mathcal{I}_S(r - 3 - r/3)) > 0$ (sheaf of germs of sections of $\mathcal{O}_{\mathbf{P}^2}$ which belong to maximal ideals of points which are triple points in collection S.)

Theorem

Roots of twisted Alexander polynomial belong to a cyclotomic extension of the field F generated by the eigenvalues of $\rho(\gamma)$

Idea of proof: Need to find holonomy only along one component and components of exceptional curves. One shows that if matrix of connection along $f = 0$ *is* $A\frac{df}{f}$ *and pullback of f has multiplicity m along E then residues of connection on* $V \otimes L_\chi$ *along E satisfy* $m(\xi_i + \alpha) = n$ *where* $\xi_{i,j}$ *are eigenvalues of A. Hence jumping element* α_i *for which one has jumping in cohomology is* $exp(-2\pi\sqrt{-1}\xi_i)exp(2\pi i\frac{m}{n})$ *for some* ξ_i *i.e. is product of a root of unity and eigenvalue of the holonomy of V.*

Distribution of the polytopes of singularities of fixed dimension is quite interesting. There are preferred polytopes among local jumping polytopes (or polytopes of quasi-adjunction)-i.e.polytopes of log-canonical thresholds which satisfy ascending chains conditions.(joint with M.Mustata) This extends recent work on Shokurov conjecture.

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It is numerical invariant of pair (X, D). X *a possibly singular variety and* D *a possibly singular divisor.*

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Discrepancies: X is normal, integral scheme, D is a $\mathbf{R}$-divisor such that $K_X + D$ is $\mathbf{R}$-Cartier, $f : Y \to X$ birational morphism, Y is normal. Let

$$K_Y = f^*(K_X + D) + \sum a(E, X, D)E \quad (a(E, X, D) \in \mathbf{R})$$

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Pair (X, D) is log-canonical if the discrepancy:

$$inf_E\{a(E, X, D)|E \textit{ is exceptional with non empty center on } X\} \geq -1$$

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Remarks

For pair a $(\mathbf{C}^n, f(x_1, ..., x_n) = 0)$ where f has isolated singularity at the origin this is the complex singularity exponent: upper bound for s such that $|f|^{-s}$ is integrable near the origin.

- Example: log canonical threshold for x^n is $\frac{1}{n}$.

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$$LCT = \frac{1}{n} + \frac{1}{m}$$

and for $x_1^{b_1} + ...x_r^{b_r} = 0$ one has:

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- For a divisor with an isolated singularity: $1 - lct$ is the largest element of the spectrum in interval [0,1] (Budur: case of non isolated singularities)

Conjecture

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Relation between collections of lct in different dimensions: Set of accumulation points in dimension n is the set of log-canonical thresholds in dimension $n-1$ *different from 1.*

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Relation between collections of lct in different dimensions: Set of accumulation points in dimension n is the set of log-canonical thresholds in dimension $n-1$ different from 1.

Remarks

a) Ein-deFernex-Mustata: OK if ambient variety is smooth (or on varieties with bounded singularities)
b)Since 1 is not an accumulation point of an increasing sequence, there is $\epsilon_n > 0$ such that $1-\epsilon_n$ does not contain log-canonical thresholds of singularities of dimension n e.g. $\epsilon_1 = \frac{1}{2}, \epsilon_2 = \frac{1}{6}, \epsilon_3 = \frac{1}{42}$.

Definition

Polytope of log-canonical thresholds:

$$LCT(\mathfrak{a}_1, ..., \mathfrak{a}_r) = \{\lambda = (\lambda_1, ..., \lambda_r \in \mathbf{R}^+ | (X, a_1^{\lambda_1} ... a_r^{\lambda_r}) \text{ is log} - \text{canonical}\}$$

- Explicit description in terms of resolutions: Let $\pi : Y \to X$ be a log resolution of $\mathfrak{a}_1 \cdot ... \cdot \mathfrak{a}_r$ i.e. exist a simple normal crossings divisor $\sum E_i$ such that

$$\mathfrak{a}_i \mathcal{O}_Y = \mathcal{O}_Y(-\sum_{j=1}^{j=N} \alpha_{ij} E_j)$$

Let also:

$$K_{Y/X} = \sum_{j=1}^{j=N} \kappa_j E_j$$

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- $$LCT(\mathfrak{a}_1, ..., \mathfrak{a}_r) = \textit{solutions to system}$$

$$\sum_i \alpha_{i,j} \lambda_i \leq \kappa_i + 1$$

Remarks

LCT polytope of pair $\mathbb{C}^{n+1}, f_1 \cdot .... \cdot f_r$ is a polytopes of quasi-adjunction.

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In particular for lct is one of elements of the spectrum

Definition

Distance between a point and a set:
If $K \subset \mathbb{R}^r$ is an arbitrary compact set, for every $x \in \mathbb{R}^r$ we put $d(x, K) = \min_{y \in K} d(x, y)$, where $d(x, y)$ denotes the Euclidean distance between x and y.

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Distance between two sets:
The Hausdorff distance between two compact sets K_1 and K_2 is defined by

$$\delta(K_1, K_2) := \max\{\max_{x \in K_1} d(x, K_2), \max_{x \in K_2} d(x, K_1)\}.$$

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Set of all non empty compact subsets in $\mathbb{R}^r$ is a complete metric space.

Set of all subsets of a compact set is compact.

Theorem

If $P_m = LCT(\mathfrak{a}_1^{(m)}, \dots, \mathfrak{a}_r^{(m)})$ for $m \geq 1$, where the $\mathfrak{a}_i^{(m)}$ are proper nonzero ideals in $k[[x_1, \dots, x_n]]$, and if the P_m converge in the Hausdorff metric to a compact set $Q \subseteq \mathbf{R}^r$, then Q is again an LCT-polytope. More precisely, if I is the set of those $i \leq r$ such that $Q \not\subseteq (x_i = 0)$, then we can find proper nonzero ideals $\mathfrak{a}_1, \dots, \mathfrak{a}_s$ in $K[[x_1, \dots, x_n]]$, with $s = \#I$ and K an algebraically closed field extension of k, such that $Q = j_I(LCT(\mathfrak{a}_1, \dots, \mathfrak{a}_s))$, where $j_I \colon \mathbf{R}^s \hookrightarrow \mathbf{R}^r$ is the inclusion corresponding to the coordinates in I.

Theorem

If $(P_m)_{m\geq 1}$ and $Q = lim_{m\to\infty} P_m$ are as in the above theorem. Then there is m_0 such that $Q = \bigcap_{m\geq m_0} P_m$.

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Corollary

If $P_m = LCT(\mathfrak{a}_1^{(m)}, \ldots, \mathfrak{a}_r^{(m)})$ for $m \geq 1$, where the $\mathfrak{a}_i^{(m)}$ are proper nonzero ideals in $k[[x_1, \ldots, x_n]]$, and if $P_1 \subseteq P_2 \subseteq \cdots$, then this sequence is eventually stationary
(since it is not hard to show that for increasing sequence $P_m \subset Q$.).

- The main ingredient in the proof of above results is the generic limit construction due and studied by deFernex, Ein, Mustata and Kollar:
  if $\mathfrak{a}_i$ is a sequence of ideals in $k[x_1, ..., x_n]$ such that $limlct(\mathfrak{a}_i) = c$ there exist an ideal $\mathfrak{a}$ in $K[[x_1, ..., x_n]]$ such that $lct(\mathfrak{a}) = c$. Here K is algebraically closed having countable transcendence degree over k.

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- Let $(\mathfrak{a}_1^{(m)}), \ldots, (\mathfrak{a}_r^{(m)})$ be sequences of r-tuples of ideals. Can assume that sequence I in the theorem is equal to $\{1, \ldots, s\}$ i.e. $Q = limP_m$ is not in $x_i = 0$, exactly for $i = 1, ...s$. Associated to the s sequences $(\mathfrak{a}_i^{(m)})$, with $1 \leq i \leq s$, we get s **generic limits** $\mathfrak{a}_1, \ldots, \mathfrak{a}_s$. These are ideals in $K[[x_1, \ldots, x_n]]$, where K is a suitable algebraically closed field extension of k. One show that all $\mathfrak{a}_i$ are nonzero and Hausdorff limit of polytopes of P_i is $LCT(\mathfrak{a}_1, ..., \mathfrak{a}_s)$.

- Using this one shows the key:
  Lemma If $\lambda \in LCT(\mathfrak{a}_1, ..., \mathfrak{a}_s) \cap \mathbf{Q}^s$ then there are infinitely many m such that $\lambda \in P_m$.

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  Lemma If $\lambda \in LCT(\mathfrak{a}_1, ..., \mathfrak{a}_s) \cap \mathbf{Q}^s$ then there are infinitely many m such that $\lambda \in P_m$.
- Key lemma plus straightforward arguments yield both: closeness of the set of LCT polytopes under Hausdorff limits and stablization of increasing sequences.