LOGARITHMIC SHEAVES AND ARRANGEMENTS OF HYPERPLANES

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Abstract. This is the outline of the talk presented by the second author in the workshop *Configuration Spaces and Hyperplane Arrangements*, at the `Scuola Normale Superiore di Pisa`, on June 25, 2010.

One object naturally associated to a hyperplane arrangement $\mathcal{A}$ in $\mathbb{P}^n$ is the sheaf $\Omega^1(\mathcal{A})$ of logarithmic one-forms with poles along $\mathcal{A}$. We discuss here a certain sub-sheaf $\widetilde{\Omega}^1(\mathcal{A})$ of $\Omega^1(\mathcal{A})$, introduced by I. Dolgachev, and his notion of Torelli arrangement, that is, an arrangement $\mathcal{A}$ that can be uniquely reconstructed from the sheaf $\widetilde{\Omega}^1(\mathcal{A})$.

1. Logarithmic sheaves

The sheaves of logarithmic differential forms with poles along a divisor were defined by P. Deligne [1] for a divisor with normal crossings, and by K. Saito [6] for an arbitrary divisor. Let X be a smooth complex algebraic variety and D a divisor on X. The sheaf of logarithmic forms $\Omega^1_X(\log D)$ consists of the meromorphic differential forms ω on X such that both ω and $d\omega$ have at most a first-order pole along D. Deligne's $\Omega^1_X(\log D)$ is a locally free sheaf, whereas Saito's is not so in general, unless X is a surface.

In [2], Dolgachev introduced a sub-sheaf $\widetilde{\Omega}^1_X(\log D)$ of $\Omega^1_X(\log D)$, that although may not be locally free even for divisors on surfaces, enjoys other useful properties, particularly when D is a hyperplane arrangement divisor.

By definition, one always has a residue exact sequence

$$0 \to \Omega^1_X \to \widetilde{\Omega}^1_X(\log D) \to \nu_* \mathcal{O}_{\widetilde{D}} \to 0,$$

where $\nu : \widetilde{D} \to D$ is a resolution of singularities of D.

Since the cokernel of the inclusion $\widetilde{\Omega}^1_X(\log D) \to \Omega^1_X(\log D)$ is supported at a closed subset of codimension ≥ 2, we have that the double duals of the two sheaves coincide

$$\widetilde{\Omega}^1_X(\log D)^{**} \cong \Omega^1_X(\log D)^{**} = \Omega^1_X(\log D).$$

Moreover, if D is a normal crossing divisor in codimension ≤ 2, then

$$\widetilde{\Omega}^1_X(\log D) \cong \Omega^1_X(\log D).$$

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2. Torelli Arrangements

An arrangement of hyperplanes in $\mathbb{P}^n$ is a finite collection $\mathcal{A} = \{H_1, \dots, H_m\}$ of distinct hyperplanes. The union of the hyperplanes in $\mathcal{A}$ is a divisor D in $\mathbb{P}^n$. We will denote $\Omega^1(\mathcal{A}) := \Omega^1_{\mathbb{P}^n}(\log D)$, and $\widetilde{\Omega}^1(\mathcal{A}) := \widetilde{\Omega}^1_{\mathbb{P}^n}(\log D)$.

The subject of the Torelli type properties of arrangements was started by I. Dolgachev and M. Kapranov in [4]. They considered there divisors D in $\mathbb{P}^n$ which are unions of $m \geq 2n+3$ distinct hyperplanes in general position. The main result of [4] determines precisely which such hyperplane arrangements $\mathcal{A}$ can be uniquely reconstructed from the locally free sheaf $\Omega^1(\mathcal{A})$, that is, they are of Torelli type. This result has been later extended by J. Vallès in [7] to arrangements of $m \geq n+2$ hyperplanes.

Theorem 2.1 (Dolgachev-Kapranov, Vallès)**.** *An arrangement $\mathcal{A}$ of $m \geq n+2$ hyperplanes in $\mathbb{P}^n$ in general position can be uniquely reconstructed from $\Omega^1(\mathcal{A})$, unless the hyperplanes in $\mathcal{A}$ osculate the same rational normal curve C.*

In [2], Dolgachev revisited and generalized the previous approach to the Torelli type arrangements using the logarithmic sheaves $\widetilde{\Omega}^1(\mathcal{A})$.

Let $\mathcal{A} = \{H_1, \dots, H_m\}$ be an arrangement of $m \geq n+2$ hyperplanes in $\mathbb{P}^n$. The sub-sheaf $\widetilde{\Omega}^1(\mathcal{A})$ of $\Omega^1(\mathcal{A})$ is characterized by the exact sequence

$$0 \to \Omega^1_{\mathbb{P}^n} \to \widetilde{\Omega}^1(\mathcal{A}) \to \oplus_{i=1}^m \mathcal{O}_{H_i} \to 0.$$

The following properties of $\widetilde{\Omega}^1(\mathcal{A})$ are derived by Dolgachev in [2]:

- $\widetilde{\Omega}^1(\mathcal{A})$ is a locally free sheaf if and only if $\mathcal{A}$ is a general position arrangement;
- $\widetilde{\Omega}^1(\mathcal{A}) = \Omega^1(\mathcal{A})$ if and only if $\mathcal{A}$ is a normal crossing divisor in codimension ≤ 2;
- $\widetilde{\Omega}^1(\mathcal{A})$ is a *Steiner sheaf*, admitting a projective resolution

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-1)^{m-n-1} \to \mathcal{O}_{\mathbb{P}^n}^{m-1} \to \widetilde{\Omega}^1(\mathcal{A}) \to 0,$$

  in particular, its Chern polynomial depends only on m and n.

In order to define the notion of Torelli for arbitrary hyperplane arrangements, Dolgachev uses Vallès's notion of unstable hyperplanes.

Definition 2.2. Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{P}^n$. The set of unstable hyperplanes $W(\mathcal{A})$ is defined as the following cohomology non-vanishing locus

$$W(\mathcal{A}) := \left\{ H \subset \mathbb{P}^n \mid H^{n-1}(H, \mathcal{F}(-n)_{|H}) \neq 0 \right\},$$

where $\mathcal{F}$ is the rank n Steiner sheaf $\widetilde{\Omega}^1(\mathcal{A})$.

Then Dolgachev [2] shows the inclusion $\mathcal{A} \subseteq W(\mathcal{A})$, and makes the following definition.

Definition 2.3. An arrangement $\mathcal{A}$ of $m \geq n+2$ hyperplanes in $\mathbb{P}^n$ is a *Torelli arrangement* if $W(\mathcal{A})$ consists of precisely the hyperplanes in $\mathcal{A}$.

In [2], the following conjecture is made on the Torelli arrangements $\mathcal{A}$ for which $\widetilde{\Omega}^1(\mathcal{A})$ is a semi-stable sheaf.

Conjecture 2.4 (Dolgachev). *A semi-stable arrangement of $m \geq n+2$ hyperplanes in $\mathbb{P}^n$ is Torelli unless the corresponding points in the dual space $\check{\mathbb{P}}^n$ lie on a stably normal rational curve C of degree n, that is a connected, reduced curve of arithmetic genus 0 whose components $C_1, \dots, C_s$ are smooth rational curves of degrees d_i such that $n = d_1 + \cdots + d_s$ and C_i spans a linear subspace of dimension d_i.*

The conjecture is verified in [2] for line arrangements in $\mathbb{P}^2$ of $m \leq 6$ lines.

3. Main theorem

Our main result is a resolution of Dolgachev's conjecture, with some necessary modifications of the statement.

Theorem 3.1. *Let $\mathcal{A}$ be an arrangement of $m \geq n+2$ hyperplanes in $\mathbb{P}^n$, and let Z be the corresponding set of m points in $\check{\mathbb{P}}^n$. Then the arrangement $\mathcal{A}$ is Torelli unless the set Z is contained in a subvariety of $\mathbb{P}^n$ of the form*

$$W = C \cup L_1 \cup \cdots \cup L_s,$$

where L_i are proper linear subspaces of dimension $n_i \geq 0$, and C is a smooth rational curve of degree d, with $0 \leq d \leq n$, spanning a linear subspace $L = \langle C \rangle$ of dimension d, such that

1. *space L meets each L_i at a single point of C,*
2. *spaces L_i are mutually disjoint and $n = d + n_1 + \cdots + d_s$,*

where $d = 0$ means $C = \{p\}$, and L_i meet only at the point p.

The key feature of the proof is reducing the unstable hyperplane condition to the existence of a global section, of a certain sheaf, which vanishes along the set Z. The vanishing of that section along Z is equivalent to Z being contained in the locus W cut by the 2×2 minors of a $2 \times n$ matrix M of linear forms

$$M = \begin{pmatrix} f_1 & f_2 & \cdots & f_n \\ g_1 & g_2 & \cdots & g_n \end{pmatrix}.$$

Such matrices M and their loci W, are classical objects, and in the generic case, are discussed for example in the monographs of D. Eisenbud [3] and J. Harris [5]. In general, the determinantal variety W is of the form described in our main theorem, as a consequence of the Kronecker-Weierstrass theory of canonical forms for pairs of matrices.

The conjecture proposed by Dolgachev needs to be modified due to examples of the following type.

Example 3.2. Let $Z = Z_1 \cup Z_2$ be a set of seven points in $\check{\mathbb{P}}^3$, five of which Z_1 are contained in a plane P, and the rest of two Z_2 span a line L transverse to P. Moreover, the points in Z_1 are situated on a conic $D \subset P$ disjoint from L, and no three of them are collinear. Then Z is contained in $W = L \cup P$, thus it is not Torelli, and the unstable set of Z consists of the entire line L.

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