

Mustafin Varieties

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Abstract

A Mustafin variety is a degeneration of projective space induced by a point configuration in a Bruhat-Tits building. The special fiber is reduced and Cohen-Macaulay, and its irreducible components form interesting combinatorial patterns. For configurations that lie in one apartment, these patterns are regular mixed subdivisions of scaled simplices, and the Mustafin variety is a twisted Veronese variety built from such a subdivision. This connects our study to tropical and toric geometry. For general configurations, the irreducible components of the special fiber are rational varieties, and any blow-up of projective space along a linear subspace arrangement can arise. A detailed study of Mustafin varieties is undertaken for configurations in the Bruhat-Tits tree of $PGL(2)$ and in the two-dimensional building of $PGL(3)$. The latter yields the classification of Mustafin triangles into 38 combinatorial types.

1 Introduction

Let K be a field with a discrete valuation $v: K^* \rightarrow \mathbb{Z}$, and let R be its ring of integers and k its residue field. For example, K could be the field of rational functions $k(t)$ or the field of formal Laurent series $k((t))$ over any ground field k , or it could be the field \mathbb{Q}_p of p -adic numbers for some prime number p . We fix a prime element π in the ring of integers R , i.e. π is an element of the field K having minimal positive valuation.

Let V be a vector space of dimension $d \geq 2$ over K and denote by $\mathbb{P}(V) = \text{Proj Sym } V^*$ the corresponding projective space, where V^* is the dual space of V . The projective space $\mathbb{P}(V)$ parametrizes lines through the origin in V . We regard V as an R -module, and a *lattice* in V is any R -submodule $L \subset V$ that is free of rank d . If L is a lattice in V , we denote by $\mathbb{P}(L) = \text{Proj Sym } L^*$ the corresponding projective space over the ring of integers R . Here, $L^* = \text{Hom}_R(L, R)$ denotes the dual R -module.

Definition 1.1. Let $\Gamma = \{L_1, \dots, L_n\}$ be a set of lattices in V . Then $\mathbb{P}(L_1), \dots, \mathbb{P}(L_n)$ are projective spaces over R whose generic fibers are canonically isomorphic to the projective space $\mathbb{P}(V) \simeq \mathbb{P}_K^{d-1}$. The open immersions $\mathbb{P}(V) \hookrightarrow \mathbb{P}(L_i)$ give rise to a map

$$\mathbb{P}(V) \longrightarrow \mathbb{P}(L_1) \times_R \dots \times_R \mathbb{P}(L_n).$$

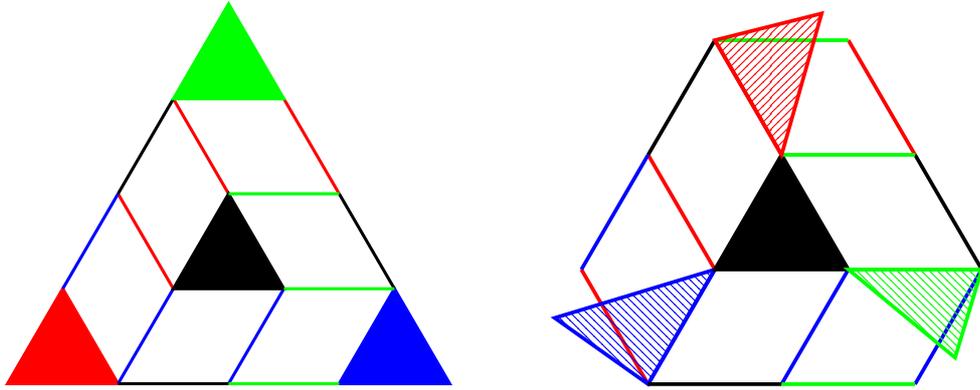


Figure 1: Special fibers of Mustafin varieties for $d = 3$ are degenerations of the projective plane. The two schemes depicted above arise from configurations of $n = 4$ points in \mathfrak{B}_3 .

Let $\mathcal{M}(\Gamma)$ be the closure of the image endowed with the reduced scheme structure. We call $\mathcal{M}(\Gamma)$ the *Mustafin variety* associated to the set of lattices Γ . Note that $\mathcal{M}(\Gamma)$ is a scheme over R whose generic fiber is $\mathbb{P}(V)$. Its special fiber $\mathcal{M}(\Gamma)_k$ is a scheme over k .

The construction of the Mustafin variety $\mathcal{M}(\Gamma)$ depends only on the homothety classes of the lattices L_i , so throughout this paper we regard Γ as a configuration in the Bruhat-Tits building \mathfrak{B}_d associated with the group $PGL(V)$. Varieties of the form $\mathcal{M}(\Gamma)$ were investigated by Mustafin [Mus] in order to generalize Mumford’s seminal work [Mu] on uniformization of curves to higher dimension. Mustafin primarily considered the case of convex subsets Γ , as defined in the text prior to Theorem 2.10.

In the present paper we are interested in arbitrary configurations Γ . The resulting Mustafin varieties have worse singularities but their combinatorial structure is richer. We note that every Mustafin variety is dominated by one from a convex configuration. Indeed, any inclusion $\Gamma \subset \Gamma'$ specifies a surjective morphism $\mathcal{M}(\Gamma') \rightarrow \mathcal{M}(\Gamma)$ and the set of lattice points in the convex hull of Γ is a finite set, as seen in [Fa, JSY].

The term “Mustafin variety” is used here for the first time. In the previous discussions of these objects in [CS, Fa, KT], Mustafin varieties had been called “Deligne schemes”, since for a convex set of vertices the corresponding Mustafin variety represents the so-called Deligne functor. We decided to name them after G. A. Mustafin, to recognize the contributions of [Mus], and we opted for “variety” over “scheme” because $\mathcal{M}(\Gamma)$ and its special fiber $\mathcal{M}(\Gamma)_k$ are reduced for all lattice configurations $\Gamma \subset \mathfrak{B}_d$.

Figure 1 shows two pictures representing Mustafin varieties for $d = 3$ and $n = 4$. In both cases, the special fiber $\mathcal{M}(\Gamma)_k$ is a surface with ten irreducible components, namely four copies of \mathbb{P}_k^2 and six copies of $\mathbb{P}_k^1 \times \mathbb{P}_k^1$. The left picture is *planar* (it is a *regular mixed subdivision*) because the configuration Γ lies in a single apartment of \mathfrak{B}_3 , while the right picture represents a *non-planar* case when Γ is not in one apartment.

This article is organized as follows. In Section 2 we develop the general theory of

Mustafin varieties, including their representation in terms of the polynomial ideals seen in [CS]. Theorem 2.3 summarizes the main geometric results, including the fact that the special fibers are reduced, Cohen-Macaulay, and have rational components.

Section 3 concerns the case $d = 2$, which was first studied by Mumford in [Mu, § 2]. Every configuration Γ in the Bruhat-Tits tree \mathfrak{B}_2 determines a finite phylogenetic tree T_Γ which is an invariant of the isomorphism type of $\mathcal{M}(\Gamma)$. In Theorem 3.5 we determine the reduction complex of $\mathcal{M}(\Gamma)_k$ in terms of T_Γ , and in Proposition 3.8 we characterize configurations whose Mustafin variety is defined by a monomial ideal.

The situation when Γ lies in a single apartment of \mathfrak{B}_d is investigated in Section 4. Theorem 4.4 realizes $\mathcal{M}(\Gamma)$ as a twisted Veronese variety. The special fiber $\mathcal{M}(\Gamma)_k$ is the toric degeneration of \mathbb{P}_k^{d-1} represented by a regular mixed subdivisions of scaled simplices, as seen on the left in Figure 1. The fact that any two points of \mathfrak{B}_d lie in one apartment leads to the classification of Mustafin varieties for $n = 2$ in Theorem 4.7.

In Section 5 we study the geometry of the irreducible components of the special fiber $\mathcal{M}(\Gamma)_k$. We distinguish between primary components, which are indexed by Γ itself, and secondary components, such as the bichromatic parallelograms in Figure 1. Both types of components are rational but they can be singular. Theorem 5.3 characterizes primary components as the blow-ups of projective spaces along linear subspace arrangements.

Section 6 offers a detailed study of the case $n=d=3$, centering around the algebro-geometric implications of the rich structure of triangles in the two-dimensional building \mathfrak{B}_3 . Our main result is the classification in Theorem 6.1 of Mustafin triangles into 38 combinatorial types, namely, the 18 planar types in Figure 6, and 20 non-planar types.

2 Structure of Mustafin Varieties

We denote by \mathfrak{B}_d the Bruhat-Tits building associated to the group $PGL(V)$. It can be obtained by gluing certain real vector spaces, the *apartments*. Let T be a maximal torus in $PGL(V)$. There is a basis e_1, \dots, e_d of V such that T is given by the group of diagonal matrices with respect to e_1, \dots, e_d . The apartment in \mathfrak{B}_d corresponding to T is defined as $A = X_*(T) \otimes_{\mathbb{Z}} \mathbb{R}$, where $X_*(T) = \text{Hom}(\mathbb{G}_m, T)$ is the cocharacter group of T . We write η_i for the cocharacter of T induced by mapping λ to the diagonal matrix with entry λ in the i th place and entries 1 in the other places. The map $A \rightarrow \mathbb{R}^d / \mathbb{R}(1, \dots, 1)$ that takes $\sum_i r_i \eta_i$ to the residue class of (r_1, \dots, r_d) is an isomorphism of vector spaces.

The apartment A is the geometric realization of a simplicial complex on the vertex set $X_*(T) \simeq \mathbb{Z}^d / \mathbb{Z}(1, \dots, 1)$. This uses the isomorphism above. Its simplices are the cells in the infinite hyperplane arrangement that consists of the affine hyperplanes

$$H_m^{(ij)} = \left\{ \sum_{\ell=1}^d r_\ell \eta_\ell \in A : r_i - r_j = m \right\} \quad \text{for } 1 \leq i < j \leq d \text{ and } m \in \mathbb{Z}. \quad (1)$$

The building \mathfrak{B}_d and its simplicial structure can be described in the following way. We write $[L] = \{\alpha L : \alpha \in K^*\}$ for the *homothety class* of a lattice L . Two lattice classes

$[L']$ and $[M']$ are called *adjacent* if there exist representatives L and M satisfying

$$\pi L \subset M \subset L. \quad (2)$$

Let $[L]$ be a lattice class such that L is in diagonal form with respect to the basis e_1, \dots, e_d , i.e. $L = \pi^{m_1} R e_1 + \dots + \pi^{m_d} R e_d$ for some $m_1, \dots, m_d \in \mathbb{Z}$. We associate to $[L]$ the point $\sum_i (-m_i) \eta_i$ in the apartment A . This is the standard bijection between the set of lattice classes in diagonal form with respect to e_1, \dots, e_d and the vertices of the simplicial complex above. This bijection preserves adjacency. Hence the simplicial complex on A is the flag complex of the adjacency graph on diagonal lattice classes.

We write \mathfrak{B}_d^0 for the set of all lattice classes $[L]$ in V . Putting all apartments together, we see that the building \mathfrak{B}_d is a geometric realization of a simplicial complex on \mathfrak{B}_d^0 , namely, the flag complex of the graph on all lattice classes defined by the adjacency relation from (2). The group $PGL(V)$ acts in the natural way on \mathfrak{B}_d and its vertex set \mathfrak{B}_d^0 . If the residue field k is a finite field containing q elements, then \mathfrak{B}_2 is an infinite regular tree of valency $q + 1$. More generally, the link of any vertex in \mathfrak{B}_d is isomorphic to the order complex of the poset of subspaces in k^d . This follows from

Lemma 2.1. *Every neighbor of a vertex $[M]$ in \mathfrak{B}_d^0 has the form $[L]$ for a lattice L with $\pi M \subset L \subset M$, where both inclusions are strict. Hence the quotient $L/\pi M$ is a non-trivial subspace of the k -vector space $M/\pi M$. In this way, we get a bijection*

$$\{\text{Neighbors of } [M]\} \longrightarrow \{\text{Non-trivial linear subspaces of } M/\pi M\},$$

mapping adjacent neighbors of $[M]$ to nested subspaces.

Now let us choose coordinates and describe the polynomial ideal that cuts out a Mustafin variety. This ideal will be multihomogeneous in the sense of the paper [CS] whose notation and setup we shall adopt. The image of the diagonal map $\Delta: \mathbb{P}(V) \rightarrow \mathbb{P}(V)^n = \mathbb{P}(V) \times_K \dots \times_K \mathbb{P}(V)$ is the subvariety of the product $\mathbb{P}(V)^n$ given by the ideal $I_2(X)$ generated by the 2×2 -minors of a matrix $X = (x_{ij})_{\substack{i=1, \dots, d \\ j=1, \dots, n}}$ of unknowns, where the j th column of this matrix represents coordinates on the j th factor.

Every $g \in GL(V)$ is represented by a matrix in $K^{d \times d}$, and it determines a dual (transpose) map ${}^t g: V^* \rightarrow V^*$ and a morphism $g: \mathbb{P}(V) \rightarrow \mathbb{P}(V)$. This induces the usual action of $PGL(V)$ on $\mathbb{P}(V)$. If g_1, \dots, g_n are elements of $GL(V)$, the image of

$$\mathbb{P}(V) \xrightarrow{\Delta} \mathbb{P}(V)^n \xrightarrow{g_1^{-1} \times \dots \times g_n^{-1}} \mathbb{P}(V)^n$$

is the subvariety of the product $\mathbb{P}(V)^n$ given by the multihomogeneous prime ideal

$$I_2((g_1, \dots, g_n)(X)) \subset K[X].$$

Here $(g_1, \dots, g_n)(X)$ is the $d \times n$ -matrix whose j th column equals

$$g_j \begin{pmatrix} x_{1j} \\ \vdots \\ x_{dj} \end{pmatrix}.$$

Consider the reference lattice $L = Re_1 + \dots + Re_d$. For any set of vertices $\Gamma = \{[L_1], \dots, [L_n]\}$ in the building \mathfrak{B}_d we choose matrices $g_1, \dots, g_n \in GL(V)$ such that $g_i L = L_i$ for all i . The following diagram commutes:

$$\begin{array}{ccc} \mathbb{P}(V) & \xrightarrow{(g_1^{-1}, \dots, g_n^{-1}) \circ \Delta} & \mathbb{P}(V)^n \\ \downarrow & & \downarrow \\ \prod_R \mathbb{P}(L_i) & \xrightarrow{(g_1^{-1}, \dots, g_n^{-1})} & \mathbb{P}(L)^n \end{array}$$

Hence the Mustafin variety $\mathcal{M}(\Gamma)$ is isomorphic to the subscheme of $\mathbb{P}(L)^n \simeq (\mathbb{P}_R^{d-1})^n$ cut out by the multihomogeneous ideal $I_2((g_1, \dots, g_n)(X)) \cap R[X]$ in $R[X]$.

Example 2.2 ($d = n = 3$). Let $K = \mathbb{Q}((t))$ and Γ the configuration determined by

$$g_1 = \text{diag}(t^2, t, 1), \quad g_2 = \text{diag}(t^4, t^2, 1), \quad g_3 = \text{diag}(t^6, t^3, 1).$$

Thus Γ lies in the apartment specified by our choice of basis. The generic fiber of the Mustafin variety $\mathcal{M}(\Gamma)$ is the subscheme of $(\mathbb{P}_K^2)^3$ defined by the 2×2 -minors of

$$(g_1, g_2, g_3)(X) = \begin{pmatrix} x_{11}t^2 & x_{12}t^4 & x_{13}t^6 \\ x_{21}t & x_{22}t^2 & x_{23}t^3 \\ x_{31} & x_{32} & x_{33} \end{pmatrix},$$

where x_{1j} , x_{2j} , and x_{3j} are the homogeneous coordinates of the j th factor of $(\mathbb{P}_K^2)^3$. The Mustafin variety $\mathcal{M}(\Gamma)$ itself is the intersection of this ideal with the ring $R[x_{ij}]$. The special fiber $\mathcal{M}(\Gamma)_{\mathbb{Q}}$ is the subscheme of $(\mathbb{P}_{\mathbb{Q}}^2)^3$ defined by the monomial ideal

$$\begin{aligned} & \langle x_{11}x_{22}, x_{11}x_{32}, x_{21}x_{32}, x_{11}x_{23}, x_{11}x_{33}, x_{21}x_{33}, x_{12}x_{23}, x_{12}x_{33}, x_{22}x_{33} \rangle \\ = & \langle x_{11}, x_{21}, x_{12}, x_{22} \rangle \cap \langle x_{11}, x_{21}, x_{23}, x_{33} \rangle \cap \langle x_{22}, x_{32}, x_{23}, x_{33} \rangle \\ & \cap \langle x_{11}, x_{21}, x_{12}, x_{33} \rangle \cap \langle x_{11}, x_{32}, x_{12}, x_{33} \rangle \cap \langle x_{11}, x_{32}, x_{33}, x_{23} \rangle. \end{aligned}$$

The first three components are isomorphic to $\mathbb{P}_{\mathbb{Q}}^2$, and the last three components are isomorphic to $\mathbb{P}_{\mathbb{Q}}^1 \times \mathbb{P}_{\mathbb{Q}}^1$. This special fiber is the *planar* monomial scheme in row 4 of [CS, Table 1] and it is an instance of the tropical cyclic polytopes in [BY, § 4].

By contrast, let us now consider the configuration Γ' in \mathfrak{B}_3 determined by

$$g_1 = M_1 \cdot \text{diag}(1, t, t^2), \quad g_2 = M_2 \cdot \text{diag}(1, t, t^2), \quad g_3 = M_3 \cdot \text{diag}(1, t, t^2),$$

where M_1, M_2 and M_3 are generic 3×3 -matrices over \mathbb{Q} . Then $\mathcal{M}(\Gamma')$ is the subscheme of $(\mathbb{P}_R^2)^3$ obtained by saturation from the ideal of 2×2 -minors of a matrix

$$\begin{pmatrix} \star x_{11} + \star x_{21}t + \star x_{31}t^2 & \star x_{12} + \star x_{22}t + \star x_{32}t^2 & \star x_{13} + \star x_{23}t + \star x_{33}t^2 \\ \star x_{11} + \star x_{21}t + \star x_{31}t^2 & \star x_{12} + \star x_{22}t + \star x_{32}t^2 & \star x_{13} + \star x_{23}t + \star x_{33}t^2 \\ \star x_{11} + \star x_{21}t + \star x_{31}t^2 & \star x_{12} + \star x_{22}t + \star x_{32}t^2 & \star x_{13} + \star x_{23}t + \star x_{33}t^2 \end{pmatrix},$$

where the stars indicate generic scalars in \mathbb{Q} . The special fiber $\mathcal{M}(\Gamma')_{\mathbb{Q}}$ is given by

$$\begin{aligned} & \langle x_{11}x_{12}, x_{11}x_{22}, x_{21}x_{12}, x_{11}x_{13}, x_{11}x_{23}, x_{13}x_{21}, x_{12}x_{13}, x_{12}x_{23}, x_{13}x_{22}, x_{21}x_{22}x_{23} \rangle \\ = & \langle x_{11}, x_{21}, x_{12}, x_{22} \rangle \cap \langle x_{12}, x_{22}, x_{13}, x_{23} \rangle \cap \langle x_{11}, x_{21}, x_{13}, x_{23} \rangle \\ & \cap \langle x_{11}, x_{21}, x_{12}, x_{13} \rangle \cap \langle x_{11}, x_{12}, x_{22}, x_{13} \rangle \cap \langle x_{11}, x_{12}, x_{13}, x_{23} \rangle. \end{aligned}$$

This monomial scheme, denoted Z in [CS, § 2], is the unique Borel-fixed point on the multigraded Hilbert scheme $H_{3,3}$ of the diagonal embedding $\mathbb{P}^2 \hookrightarrow \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$. \square

We have the following general structure theorem for Mustafin varieties.

Theorem 2.3. *For a finite subset Γ of \mathfrak{B}_d^0 , the Mustafin variety $\mathcal{M}(\Gamma)$ is an integral, normal, Cohen-Macaulay scheme which is flat and projective over R . Its generic fiber is isomorphic to the $(d-1)$ -dimensional projective space $\mathbb{P}(V)$, and its special fiber $\mathcal{M}(\Gamma)_k$ is reduced, Cohen-Macaulay and connected. All irreducible components of $\mathcal{M}(\Gamma)_k$ are rational varieties, and their number is at most $\binom{n+d-2}{d-1}$, where $n = |\Gamma|$.*

Proof. By construction, any Mustafin variety $\mathcal{M}(\Gamma)$ is irreducible, reduced and projective over R , and with generic fiber $\mathbb{P}(V)$. Since R is a discrete valuation ring, torsion-free implies flat, so $\mathcal{M}(\Gamma)$ is also flat over R . We show that the special fiber is connected by Zariski's Connectedness Principle [Liu, Theorem 5.3.15]. Since $\mathcal{M}(\Gamma)$ is proper over R , the group of global sections $\mathcal{O}_{\mathcal{M}(\Gamma)}(\mathcal{M}(\Gamma))$ is a finite R -module. As it is contained in $\mathcal{O}_{\mathbb{P}(V)}(\mathbb{P}(V)) = K$, and R is integrally closed, we find indeed that the push-forward of $\mathcal{O}_{\mathcal{M}(\Gamma)}$ is equal to \mathcal{O}_S . Thus, the special fiber is connected.

Each Mustafin variety $\mathcal{M}(\Gamma)$ corresponds to an R -valued point in the multigraded Hilbert scheme $H_{d,n}$ described in [CS], and its special fiber $\mathcal{M}(\Gamma)_k$ is a k -valued point of $H_{d,n}$. All k -valued points of $H_{d,n}$ are reduced and Cohen-Macaulay by Theorem 2.1 and Corollary 2.6 in [CS]. Since π is a non-zero divisor on $\mathcal{M}(\Gamma)$ such that the subscheme $\mathcal{M}(\Gamma)_k$ it defines is Cohen-Macaulay, $\mathcal{M}(\Gamma)$ is Cohen-Macaulay along $\mathcal{M}(\Gamma)_k$. Away from $\mathcal{M}(\Gamma)_k$, $\mathcal{M}(\Gamma)$ is regular, so $\mathcal{M}(\Gamma)$ is Cohen-Macaulay everywhere. Finally, $\mathcal{M}(\Gamma)$ is normal because it is flat over a discrete valuation ring with normal generic fiber and reduced special fiber [Liu, Lemma 4.1.18].

The number $\binom{n+d-2}{d-1}$ is the degree of the diagonal in the Segre embedding, and thus also the degree of $\mathcal{M}(\Gamma)_k$. Since $\mathcal{M}(\Gamma)_k$ is connected and Cohen-Macaulay, it is equidimensional, and thus it has at most $\binom{n+d-2}{d-1}$ components.

The only remaining point is that the components are rational varieties. That proof will be given in Section 5. The results in Sections 3 and 4 do not rely on it. \square

The following lemma enables us to take closer look at the components of $\mathcal{M}(\Gamma)_k$.

Lemma 2.4. *Let $\Gamma' \subset \Gamma$ be finite subsets of \mathfrak{B}_d^0 . For each irreducible component C of the special fiber $\mathcal{M}(\Gamma')_k$, there is a unique irreducible component of $\mathcal{M}(\Gamma)_k$ that maps birationally onto C via the natural projection $\mathcal{M}(\Gamma) \rightarrow \mathcal{M}(\Gamma')$.*

Proof. Let $\Gamma = \{[L_1], \dots, [L_n]\}$ and $\Gamma' = \{[L_1], \dots, [L_{n'}]\}$ be a subset with $n' \leq n$. The Chow ring of $(\mathbb{P}^{d-1})^n$ (over any field) is $\mathcal{A} = \mathbb{Z}[H_1, \dots, H_n]/\langle H_1^d, \dots, H_n^d \rangle$, where H_i represents the pullback of the hyperplane class from the i th factor. Up to change of coordinates, $\mathcal{M}(\Gamma)_K$ is embedded in $(\mathbb{P}_K^{d-1})^n$ as the diagonal. The codimension of this diagonal is $(d-1)(n-1)$ and its rational equivalence class is the sum over all monomials in \mathcal{A} of total degree $(d-1)(n-1)$ (see [Fu, Example 8.4.2 (c)] for the case $n = 2$, which generalizes easily). Since the special fiber $\mathcal{M}(\Gamma)_k$ is a specialization of $\mathcal{M}(\Gamma)_K$, as in [Fu, Section 20.3], $\mathcal{M}(\Gamma)_k$ has the same class in \mathcal{A} . This class is the sum of the classes of the components of $\mathcal{M}(\Gamma)_k$. Since each component is effective, its class must be a sum of non-negative multiples of monomials in \mathcal{A} , and since the class of $\mathcal{M}(\Gamma)_k$ is multiplicity-free, each component must be a sum of distinct monomials in \mathcal{A} .

Similarly, the class of the component C of $\mathcal{M}(\Gamma')_k$ is the sum of distinct monomials of degree $(d-1)(n'-1)$ in $\mathcal{A}' = \mathbb{Z}[H_1, \dots, H_{n'}]/\langle H_1^d, \dots, H_{n'}^d \rangle$. Let $H_1^{a_1} \cdots H_{n'}^{a_{n'}}$ with $a_1 + \cdots + a_{n'} = (d-1)(n'-1)$ be one of them. There is a unique component \tilde{C} in $\mathcal{M}(\Gamma)_k$ whose class contains the monomial $H_1^{a_1} \cdots H_{n'}^{a_{n'}} \cdot H_{n'+1}^{d-1} \cdots H_n^{d-1}$. Under the projection $\mathcal{M}(\Gamma)_k \rightarrow \mathcal{M}(\Gamma')_k$, this class pushes forward to $H_1^{a_1} \cdots H_{n'}^{a_{n'}}$. Since $\mathcal{M}(\Gamma)_K \rightarrow \mathcal{M}(\Gamma')_K$ is an isomorphism, under specialization, the rational equivalence class of $\mathcal{M}(\Gamma)_k$ pushes forward to the class of $\mathcal{M}(\Gamma')_k$. Thus, the projection of \tilde{C} contains the monomial $H_1^{a_1} \cdots H_{n'}^{a_{n'}}$. Since C is the only component of $\mathcal{M}(\Gamma')_k$ containing this monomial, it must be the image of \tilde{C} . Furthermore, since the coefficient of $H_1^{a_1} \cdots H_{n'}^{a_{n'}}$ is one, the map is birational and \tilde{C} is the unique component of $\mathcal{M}(\Gamma)_k$ with this property. \square

Corollary 2.5. *Let $\Gamma = \{[L_1], \dots, [L_n]\}$ be a finite subset of \mathfrak{B}_d^0 . For every index i , the special fiber $\mathcal{M}(\Gamma)_k$ has a unique irreducible component C_i with the property that C_i maps birationally to $\mathbb{P}(L_i)_k$ under the projection $\mathbb{P}(L_1)_k \times \cdots \times \mathbb{P}(L_n)_k \rightarrow \mathbb{P}(L_i)_k$.*

Proof. We take $\Gamma' = \{[L_i]\}$, so that $\mathcal{M}(\Gamma') = \mathbb{P}(L_i)$, and apply Lemma 2.4. \square

Definition 2.6. An irreducible component of $\mathcal{M}(\Gamma)_k$ mapping birationally to the special fiber of the factor $\mathbb{P}(L_i)$ for some $[L_i] \in \Gamma$ is called a *primary component*. All other components of the special fiber are called *secondary components*. In both ideal decompositions of Example 2.2, the first three components are primary and the last three components are secondary. For instance, the variety defined by $\langle x_{11}, x_{21}, x_{12}, x_{22} \rangle$ is $(0:0:1) \times (0:0:1) \times \mathbb{P}_k^2$, and this maps birationally (in fact, isomorphically) onto the third factor of $\mathbb{P}_k^2 \times \mathbb{P}_k^2 \times \mathbb{P}_k^2$.

Definition 2.7. By an *isomorphism of Mustafin varieties* $\mathcal{M}(\Gamma)$ and $\mathcal{M}(\Gamma')$ we mean an R -isomorphism between the schemes $\mathcal{M}(\Gamma)$ and $\mathcal{M}(\Gamma')$ which preserves the set of

primary components. Thus, an isomorphism of Mustafin varieties induces a bijection between the defining lattice configurations Γ and Γ' .

We note that two Mustafin varieties can be isomorphic as R -schemes without being isomorphic as Mustafin varieties. This is shown in Example 4.5, which exhibits a strict inclusion $\Gamma \subset \Gamma'$ such that the map $\mathcal{M}(\Gamma') \rightarrow \mathcal{M}(\Gamma)$ is an R -isomorphism. The following result characterizes the isomorphism classes of Mustafin varieties.

Theorem 2.8. *If $\mathcal{M}(\Gamma)$ and $\mathcal{M}(\Gamma')$ are isomorphic Mustafin varieties, then there exists an element g in $PGL(V)$ such that $\Gamma' = g \cdot \Gamma$ under the action on subsets of \mathfrak{B}_d .*

Proof. If $\Gamma = \{[L_1], \dots, [L_n]\}$ and $\Gamma' = g\Gamma$, then the isomorphism

$$(g, \dots, g): \mathbb{P}(L_1) \times_R \cdots \times_R \mathbb{P}(L_n) \longrightarrow \mathbb{P}(gL_1) \times_R \cdots \times_R \mathbb{P}(gL_n)$$

restricts to an isomorphism of Mustafin varieties $\mathcal{M}(\Gamma) \rightarrow \mathcal{M}(\Gamma')$, such that the induced map on the generic fiber $\mathbb{P}(V)$ is given by g . Suppose conversely that $\varphi: \mathcal{M}(\Gamma) \rightarrow \mathcal{M}(\Gamma')$ is an isomorphism of Mustafin varieties. The generic fiber of φ^{-1} is given by an element $g \in PGL(V)$. As we have just seen, g induces an isomorphism of Mustafin varieties $\mathcal{M}(\Gamma') \rightarrow \mathcal{M}(g\Gamma')$. Hence after replacing Γ' by $g\Gamma'$ we may assume that φ is the identity map on the generic fiber. We claim that in this case $\Gamma = \Gamma'$.

Let $[L]$ be a lattice class in Γ , and let C be the corresponding primary component of $\mathcal{M}(\Gamma)$. Since φ is an isomorphism of Mustafin varieties, it maps C to a primary component C' of $\mathcal{M}(\Gamma')$, which corresponds to some lattice class $[L'] \in \Gamma'$. We define the morphism $h: \mathcal{M}(\Gamma) \rightarrow \mathbb{P}(L) \times_R \mathbb{P}(L')$ as the product of the natural projection $\mathcal{M}(\Gamma) \rightarrow \mathbb{P}(L)$ and the composition $\mathcal{M}(\Gamma) \xrightarrow{\varphi} \mathcal{M}(\Gamma') \rightarrow \mathbb{P}(L')$, where $\mathcal{M}(\Gamma') \rightarrow \mathbb{P}(L')$ is the natural projection. Then the generic fiber of h is the diagonal embedding of $\mathbb{P}(V)$ into $\mathbb{P}(L) \times_R \mathbb{P}(L')$. Therefore, h induces a morphism from $\mathcal{M}(\Gamma)$ to the closure of $\mathbb{P}(V)$ in $\mathbb{P}(L) \times_R \mathbb{P}(L')$. Assuming that $[L]$ and $[L']$ are distinct lattice classes, the closure is the Mustafin variety $\mathcal{M}(\{[L], [L']\})$. Note that h maps the primary component C to the primary component D of $\mathcal{M}(\{[L], [L']\})$ corresponding to $[L]$. The following diagram is commutative and C maps birationally to $\mathbb{P}(L')_k$:

$$\begin{array}{ccc} C & \xrightarrow{\quad} & D \\ \downarrow & & \downarrow \\ \mathcal{M}(\Gamma)_k & \xrightarrow{h} & \mathcal{M}(\{[L], [L']\})_k \\ \downarrow \varphi & & \downarrow \\ \mathcal{M}(\Gamma')_k & \xrightarrow{\quad} & \mathbb{P}(L')_k \end{array}$$

We conclude that the component D is mapped birationally to $\mathbb{P}(L')_k$ under the projection on the right. However, by Corollary 2.5, D can't map birationally to both $\mathbb{P}([L'])_k$ and $\mathbb{P}([L])_k$, so $[L]$ and $[L']$ must be the same lattice point. Hence, $\Gamma = \Gamma'$. \square

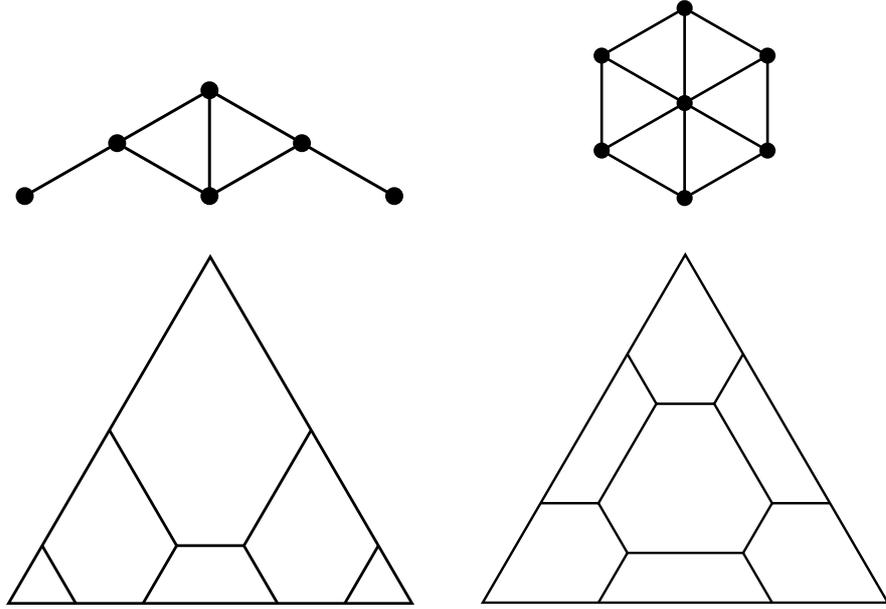


Figure 2: Convex configurations in \mathfrak{B}_3 and the special fibers of their Mustafin varieties

With every Mustafin variety $\mathcal{M}(\Gamma)$ we associate a simplicial complex representing the intersections between the irreducible components of its special fiber:

Definition 2.9. The *reduction complex* of $\mathcal{M}(\Gamma)$ is the simplicial complex with one vertex for each component of the special fiber $\mathcal{M}(\Gamma)_k$, where a set of vertices forms a simplex if and only if the intersection of the corresponding components is non-empty.

In Example 2.2, the reduction complex of $\mathcal{M}(\Gamma)$ is a tetrahedron with triangles attached at two adjacent vertices, while that of $\mathcal{M}(\Gamma')$ is the full 5-simplex. Reduction complexes for $d = 2$ are characterized in Theorem 3.5.

We say that Γ is *convex* if whenever $[L]$ and $[L']$ are in Γ then any vertex of the form $[\pi^a L \cap \pi^b L']$ is also in Γ . This is the notion of convexity used in [Fa] and in [JSY]. The convex hull of $\Gamma \subset \mathfrak{B}_d^0$ is the smallest convex subset of \mathfrak{B}_d^0 containing Γ . We call Γ *metrically convex* if Γ is closed under taking geodesics in the natural graph metric on \mathfrak{B}_d^0 , i.e. if $[L]$ and $[L']$ are in Γ and $\text{dist}([L], [L'']) + \text{dist}([L''], [L']) = \text{dist}([L], [L'])$ then $[L'']$ is in Γ . This equality holds for $L'' = \pi^a L \cap \pi^b L'$, so metrically convex implies convex, but not conversely. Mustafin studied the varieties $\mathcal{M}(\Gamma)$ only for metrically convex configurations Γ . Note that in the context of Euclidean buildings there is yet another notion of convexity, which is induced from the Euclidean distances in apartments, but this notion of convexity does not play a role in our paper.

The following theorem about convex configurations is illustrated by Figure 2.

Theorem 2.10. *If Γ is a convex subset consisting of n lattice points in the building \mathfrak{B}_d , then the Mustafin variety $\mathcal{M}(\Gamma)$ is regular, and its special fiber $\mathcal{M}(\Gamma)_k$ consists of n smooth irreducible components that intersect transversely. In this case, the reduction complex of $\mathcal{M}(\Gamma)$ is isomorphic to the simplicial subcomplex of \mathfrak{B}_d induced by Γ .*

Proof. Mustafin [Mus, Proposition 2.2] established this result for configurations that are metrically convex, and we need to argue that it also holds for all configurations that are convex in the sense above. Under the convex hypothesis, Faltings [Fa] showed that $\mathcal{M}(\Gamma)$ is regular and that there are n components in the special fiber that intersect transversally. Note that Faltings uses the opposite convention for projective space of a lattice, so he also takes the dual notion of convexity, in terms of $L+L'$ instead of $L\cap L'$. Their smoothness follows from our Proposition 5.6 below.

We now prove the assertion about the reduction complex. Consider the simplicial complex on Γ induced from the simplicial structure on \mathfrak{B}_d^0 . The induced complex is always a subcomplex of the reduction complex of $\mathcal{M}(\Gamma)$, even if Γ is not convex. Indeed, if $\bar{\Gamma} \supset \Gamma$ is the metric convex closure of Γ , then for any simplex in Γ , the corresponding components in $\mathcal{M}(\bar{\Gamma})_k$ intersect by [Mus], and hence, so do their images in $\mathcal{M}(\Gamma)_k$.

Suppose Γ that is convex and the reduction complex contains a simplex that is not in the induced simplicial complex. Since the latter is a flag complex, we can assume that the simplex is an edge $\{[L_1], [L_2]\}$. The two corresponding primary components intersect in $\mathcal{M}(\Gamma)_k$, and hence so do their components in $\mathcal{M}(\Gamma')_k$ where Γ' is the convex hull of $[L_1]$ and $[L_2]$ in \mathfrak{B}_d . As in Proposition 4.7 below, we can fix an apartment that contains both $[L_1]$ and $[L_2]$. By construction, the tropical line segment spanned by $[L_1]$ and $[L_2]$ in that apartment has at least one additional lattice point $[L_3]$. Consider the mixed subdivision $\Delta_{\{[L_1],[L_2],[L_3]\}}$ as in Section 4. A combinatorial argument shows that the maximal cells indexed by $[L_1]$ and $[L_2]$ do not intersect in that subdivision. This contradicts to the assumption that their primary components do intersect. \square

For convex configurations Γ , the special fiber has only primary components, and no secondary components. Without convexity assumptions, a typical Mustafin variety has many secondary components. Theorem 2.3 implies the following upper bound:

$$\# \text{ secondary components of } \mathcal{M}(\Gamma)_k \leq \binom{n+d-2}{d-1} - n. \quad (3)$$

Note that for $d=2$, the special case of trees, the number above is zero.

Remark 2.11. The upper bound in (3) is attained when Γ is of monomial type, as defined below. This follows from the degree argument in the second-to-last paragraph of the proof of Theorem 2.3 and the fact that the special fiber is always reduced.

Definition 2.12. A configuration Γ in the Bruhat-Tits building \mathfrak{B}_d is of *monomial type* if there exist bases for the R -modules L_1, \dots, L_n such that the multihomogeneous ideal in $k[X]$ that defines $\mathcal{M}(\Gamma)_k$ is generated by monomials in the dual bases.

We believe that monomial type is a generic condition. To make this precise, we need to consider configurations of n points with \mathbb{Q} -rational coordinates in \mathfrak{B}_d , and the statement would be that rational configurations of monomial type are dense in the configuration space for n points in \mathfrak{B}_d . This would lead us to define a Gröbner fan structure on configuration spaces of buildings, a topic we hope to return to in the future.

3 Trees

The building \mathfrak{B}_2 is an infinite tree. Any two points v and w in \mathfrak{B}_2 can be connected by a unique path. The lattice points in \mathfrak{B}_2^0 determine the simplicial structure on \mathfrak{B}_2 . We regard \mathfrak{B}_2^0 as a metric space, where adjacent lattice points have distance one.

Given any finite configuration $\Gamma \subset \mathfrak{B}_2^0$, the induced metric is a *tree metric* on Γ . The tree that realizes this metric is the convex hull of Γ in \mathfrak{B}_2^0 with the induced metric. We denote this tree by T_Γ and we refer to it as the *phylogenetic tree* of Γ . Thus T_Γ is a metric tree with n labeled nodes that include the leaves. Tree metrics are studied in computational biology [PS, § 2.4], where it is well known that the phylogenetic tree T_Γ is uniquely determined by the metric on Γ . The *Neighbor-Joining Method* [PS, Algorithm 2.41] rapidly reconstructs the phylogenetic tree T_Γ from the $\binom{n}{2}$ pairwise distances.

We are interested in the Mustafin variety $\mathcal{M}(\Gamma)$ specified by the configuration $\Gamma \subset \mathfrak{B}_2^0$. First we show that the metric tree T_Γ can be read off the geometry of $\mathcal{M}(\Gamma)$. The following result is also proven in [Mu, Proposition 2.3] by a different argument.

Proposition 3.1. *If $\Gamma \subset \mathfrak{B}_2^0$, then each irreducible component of the special fiber $\mathcal{M}(\Gamma)_k$ is isomorphic to \mathbb{P}_k^1 , and these irreducible components are in bijection with Γ .*

Proof. By Theorem 2.3, the special fiber $\mathcal{M}(\Gamma)_k$ has at most n components, where $n = |\Gamma|$. By Corollary 2.5, there are precisely n primary components. We conclude that every component is primary. Also by Corollary 2.5, each primary component C maps birationally onto \mathbb{P}_k^1 . If \tilde{C} denotes the normalization of C , then the induced map $\tilde{C} \rightarrow C \rightarrow \mathbb{P}_k^1$ must be an isomorphism. Hence C is isomorphic to \mathbb{P}_k^1 . \square

Remark 3.2. Because of Proposition 3.1, in the rest of this section we will speak interchangeably of the elements of Γ and the components of the special fiber $\mathcal{M}(\Gamma)_k$.

If Γ consists of $n = 2$ points then their distance t has the following interpretation.

Lemma 3.3. *If $\Gamma = \{[L], [M]\}$, then the special fiber $\mathcal{M}(\Gamma)_k$ consists of two projective lines \mathbb{P}_k^1 that meet in one point. In a neighborhood of that point, the Mustafin variety $\mathcal{M}(\Gamma)$ is defined by a local equation of the form $xy = \pi^t$, where $t = \text{dist}([L], [M])$.*

Proof. Since the two lattice classes lie in a common apartment of the tree \mathfrak{B}_2 , there is a basis $\{e_1, e_2\}$ of V such that $L = Re_1 + Re_2$ and $M = Re_1 + \pi^t Re_2$. The Mustafin variety $\mathcal{M}(\Gamma)$ is the subscheme of $\mathbb{P}_R^1 \times \mathbb{P}_R^1$ defined by the bihomogeneous ideal $\langle x_2 y_1 - \pi^t x_1 y_2 \rangle$,

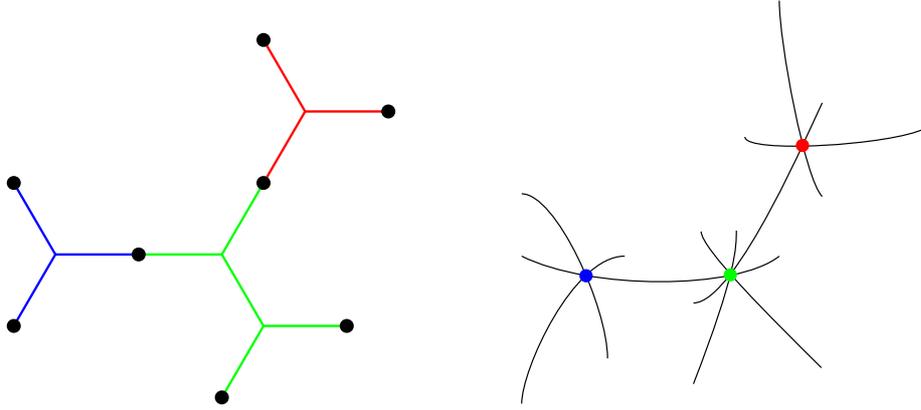


Figure 3: A configuration Γ of $n = 8$ points in \mathfrak{B}_2 whose associated phylogenetic tree T_Γ has six leaves. The corresponding special fiber is a tree of eight projective lines.

where $(x_1 : x_2)$ and $(y_1 : y_2)$ are coordinates on the two factors. Hence the special fiber consists of two copies of \mathbb{P}_k^1 meeting transversely in one point. In the affine coordinates $x = x_2/x_1$ and $y = y_1/y_2$, the equation of $\mathcal{M}(\Gamma)$ becomes $xy = \pi^t$. \square

The natural number t is known as the *thickness* of the singularity of $\mathcal{M}(\Gamma)$. The thickness t is invariant under changes of coordinates because in a minimal resolution of singularities, there are exactly $t - 1$ exceptional curves mapping to the singular point on $\mathcal{M}(\Gamma)$. This is shown in [Liu, Lemma 10.3.21].

We have the following formula for the thickness t in terms of two matrices $g, h \in GL_2(K)$ that represent $L = gL_0$ and $M = hL_0$ relative to a reference lattice $L_0 \subset V$:

$$t = \frac{v(\det(g_1 h_1) \det(g_2 h_2) - \det(g_1 h_2) \det(g_2 h_1))}{-2 \cdot \min\{v(\det(g_1 h_1)), v(\det(g_1 h_2)), v(\det(g_2 h_1)), v(\det(g_2 h_2))\}}. \quad (4)$$

Here g_i and h_j denote the columns of g and h . To prove (4), we note that $\mathcal{M}(\Gamma)_k$ is defined by $\langle \det(g_1 h_1) x_{11} x_{12} + \det(g_1 h_2) x_{11} x_{22} + \det(g_2 h_1) x_{21} x_{12} + \det(g_2 h_2) x_{21} x_{22} \rangle$, and we change coordinates on $\mathbb{P}_R^1 \times \mathbb{P}_R^1$ to eliminate the two middle terms. The formula is invariant under coordinate transformations that multiply g or h on the right by an element of $SL_2(R)$, but we do not know a simple direct argument for this invariance.

The following theorem describes the correspondence between Mustafin varieties $\mathcal{M}(\Gamma)$ and their phylogenetic trees T_Γ .

Theorem 3.4. *The isomorphism class of the Mustafin variety $\mathcal{M}(\Gamma)$ determines the tree T_Γ . Every phylogenetic tree whose maximal valency is at most one more than the cardinality of the residue field k arises in this manner from a configuration $\Gamma \subset \mathfrak{B}_2^0$.*

Proof. The first statement follows from Theorem 2.8. Alternatively, we can argue using Lemma 3.3. Given the Mustafin variety $\mathcal{M}(\Gamma)$ as an R -scheme, the components of

its special fiber $\mathcal{M}(\Gamma)_k$ are labeled by Γ . We then recover the tree metric d_Γ on the convex hull T_Γ as follows. For $v, w \in \Gamma$, the projection $\mathcal{M}(\Gamma) \rightarrow \mathcal{M}(\{v, w\})$ contracts all components of $\mathcal{M}(\Gamma)_k$ other than v and w . By [Liu, Proposition 8.3.28], this contraction morphism between normal fibered surfaces is unique up to unique isomorphism. In a neighborhood of the intersection point of these two components, $\mathcal{M}(\{v, w\})$ is defined by an equation of the form $xy - \pi^t$, as in Lemma 3.3. The exponent $t = d_\Gamma(v, w)$ is the distance between v and w and coincides with the thickness of the singularity in $\mathcal{M}(\{v, w\})$. Therefore we can construct the metric on Γ from the geometry of $\mathcal{M}(\Gamma)$.

Let T be any phylogenetic tree with n labeled leaves and positive integral edge lengths. Assuming that its maximal valency is smaller or equal to $|k| + 1$, we can embed T isometrically into the building \mathfrak{B}_2 in such a way that the leaves are mapped to lattice points in \mathfrak{B}_2^0 . However, different embeddings may lead to non-isomorphic Mustafin varieties. For example, if a vertex in Γ has degree four in T_Γ , then it intersects four other components, and the cross ratio between the coordinates of these intersection points is an invariant of $\mathcal{M}(\Gamma)_k$. Different cross ratios can occur for the same tree T_Γ . \square

We now discuss the special fiber $\mathcal{M}(\Gamma)_k$, starting with its reduction complex.

Theorem 3.5. *The maximal simplices of the reduction complex of $\mathcal{M}(\Gamma)$ correspond to the connected components of the punctured tree $T_\Gamma \setminus \Gamma$. The vertices in each maximal cell are the elements of Γ in the closure of the corresponding component. Thus, two irreducible components v and w of the special fiber $\mathcal{M}(\Gamma)_k$ intersect if and only if the unique geodesic between v and w in T_Γ does not contain any other vertex u in Γ .*

Proof. Let $\bar{\Gamma} = T_\Gamma \cap \mathfrak{B}_2^0$ be the set of all lattice points in the convex hull of Γ . Since $\Gamma \subset \bar{\Gamma}$, we have a projection from $\mathcal{M}(\bar{\Gamma})$ to $\mathcal{M}(\Gamma)$, and hence from $\mathcal{M}(\bar{\Gamma})_k$ to $\mathcal{M}(\Gamma)_k$. By Theorem 2.10, two components of the special fiber $\mathcal{M}(\bar{\Gamma})_k$ intersect if and only if their vertices are adjacent (i.e. have distance 1) in the simplicial structure on \mathfrak{B}_2 . Consider a connected component C of $T_\Gamma \setminus \Gamma$. If C is an edge in \mathfrak{B}_2 , then the adjacent vertices correspond to two intersecting components. Otherwise, the irreducible components of $\mathcal{M}(\Gamma)_k$ corresponding to the lattice points on C form a 1-dimensional connected subset of $\mathcal{M}(\Gamma)_k$. Each of these components is contracted in the projection, so the union of all components in C projects to a single point in $\mathcal{M}(\Gamma)_k$.

All irreducible components in $\mathcal{M}(\bar{\Gamma})_k$ corresponding to points of Γ lying in the closure of the connected component C in T_Γ intersect one of the components in C . Hence all these components intersect in a common point in $\mathcal{M}(\Gamma)_k$, and we conclude that the corresponding points of Γ form a simplex in the reduction complex.

It remains to be seen that there are no other simplices in the reduction complex. In other words, we must show that there are no other points of intersection between components of $\mathcal{M}(\Gamma)_k$. Suppose that $[L_1]$ and $[L_3]$ are not in the closure of a single connected component of $T_\Gamma \setminus \Gamma$. Then the path between them contains at least one other element $[L_2]$ of Γ . We explicitly compute the Mustafin variety of the triple $\Gamma' = \{[L_1], [L_2], [L_3]\}$ in order to show that $[L_1]$ and $[L_3]$ do not intersect in $\mathcal{M}(\Gamma')_k$. Since

all three points lie in a common apartment of \mathfrak{B}_2 , we can choose a basis $\{e_1, e_2\}$ for V such that $L_i = Re_1 \oplus \pi^{s_i}Re_2$ with $0 = s_1 < s_2 < s_3$. The ideal of $\mathcal{M}(\Gamma')$ is therefore

$$\langle \pi^{s_2}x_1y_2 - x_2y_1, \pi^{s_3}x_1y_3 - x_3y_1, \pi^{s_3-s_2}x_2y_3 - x_3y_2 \rangle,$$

where $(x_i : y_i)$ are the coordinates for $\mathbb{P}(L_i) \simeq \mathbb{P}_R^1$. The ideal of the special fiber is

$$\langle x_2y_1, x_3y_1, x_3y_2 \rangle = \langle x_2, x_3 \rangle \cap \langle y_1, x_3 \rangle \cap \langle y_1, y_2 \rangle.$$

The prime ideals on the right are the primary components for $[L_1]$, $[L_2]$ and $[L_3]$. It is easy to see that the lines defined by the first and last ideal do not intersect in $(\mathbb{P}_k^1)^3$. In the projection from $\mathcal{M}(\Gamma)$ to $\mathcal{M}(\Gamma')$, the component of $\mathcal{M}(\Gamma)_k$ indexed by $[L_i]$ maps to the corresponding component in $\mathcal{M}(\Gamma')_k$. The projective lines indexed by $[L_1]$ and $[L_3]$ are disjoint in $\mathcal{M}(\Gamma)_k$ because their images are disjoint in $\mathcal{M}(\Gamma')_k$. \square

Example 3.6. The reduction complex of the configuration in Figure 3 is a green tetrahedron which has two triangles, colored red and blue, attached at two of its vertices. \square

Our discussion leads to the following description of the singularities of $\mathcal{M}(\Gamma)_k$.

Corollary 3.7. *Every point of the special fiber $\mathcal{M}(\Gamma)_k$ is locally isomorphic to a union of coordinate axes. Therefore $\mathcal{M}(\Gamma)_k$ has TAC singularities as defined in [To].*

Proof. The result is also shown in [Mu, Proposition 2.3]. Recall that $\mathcal{M}(\Gamma)_k$ is a subscheme of $\prod \mathbb{P}(L_i)_k$ and that each component projects isomorphically to exactly one of the factors. Therefore, the inverse of this isomorphism composed with one of the other projections must be constant. In other words, the i th component is embedded as the product of $\mathbb{P}(L_i)$ with a point from each of the other factors $\mathbb{P}(L_j)$ for $j \neq i$. Whenever two or more components meet, they can be written as the union of coordinate axes. \square

Our next goal is to explain the connection to the combinatorial data arising in the study of the multigraded Hilbert scheme $H_{2,n}$ in [CS, §4]. Among the k -points on that Hilbert scheme are the special fibers of any Mustafin variety for $d = 2$. Let us now characterize the configurations of monomial type. Recall that $\Gamma \subset \mathfrak{B}_2^0$ is of monomial type if the ideal of $\mathcal{M}(\Gamma)_k$ is generated by monomials in suitable bases.

Proposition 3.8. *A configuration $\Gamma \subset \mathfrak{B}_2^0$ has monomial type if and only if every element of Γ is either a leaf or is in the interior of an edge in the phylogenetic tree T_Γ .*

Proof. The configuration Γ is of monomial type if and only if there is a linear torus action on $\mathbb{P}(L_1)_k \times \cdots \times \mathbb{P}(L_n)_k$ such that the special fiber $\mathcal{M}(\Gamma)_k$ is invariant. On each factor $\mathbb{P}(L_i)_k$, picking a linear torus action is equivalent to picking two points 0 and ∞ . Such a torus action can be lifted to an action on $\mathbb{P}(L_1)_k \times \cdots \times \mathbb{P}(L_n)_k$ leaving the other factors invariant. This action fixes the component of $\mathcal{M}(\Gamma)_k$ that maps isomorphically to $\mathbb{P}(L_i)_k$. The other components however project to points on $\mathbb{P}(L_i)_k$. Hence a torus action on $\mathbb{P}(L_i)_k$ that leaves $\mathcal{M}(\Gamma)_k$ invariant exists if and only if there are at most two such points, and this happens if and only if $[L_i]$ has degree at most 2 in T_Γ . \square

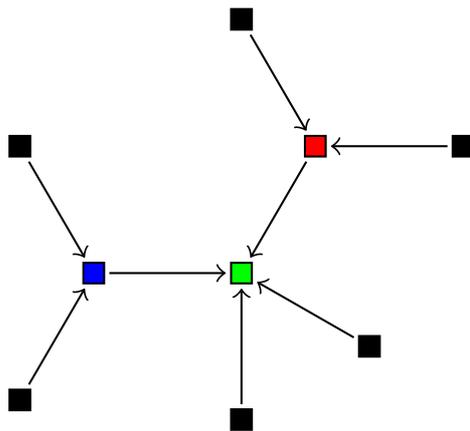


Figure 4: The Mustafin variety in Figure 3 is of monomial type. The monomial ideal of its special fiber is represented by a tree with 9 nodes and 8 directed edges, as in [DS].

According to [CS, Theorem 4.2], the Hilbert scheme $H_{2,n}$ has precisely $2^n(n+1)^{n-2}$ points that represent monomial ideals, and these are indexed by trees on $n+1$ unlabeled nodes with n labeled directed edges. For configurations Γ of monomial type, we can use Theorem 3.5 to derive the *monomial tree* representation of $\mathcal{M}(\Gamma)_k$ as in [CS, § 4] from the phylogenetic tree T_Γ . That combinatorial transformation of trees is as follows.

Assume that $n > 1$. We introduce one node for each leaf of T_Γ and one node for each connected component of $T_\Gamma \setminus \Gamma$. Thus the total number of nodes is $n+1$. For each $v \in \Gamma$ that is a leaf in T_Γ we introduce one edge between the node corresponding to v and the component of $T_\Gamma \setminus \Gamma$ that is incident to v . Each non-leaf $v \in \Gamma$ is incident to two components of $T_\Gamma \setminus \Gamma$, by Proposition 3.8, and we introduce an edge between these nodes as well. Thus the total number of edges is n . Now, the monomial tree is directed by orienting each of the n edges according to the choice of 0 and ∞ for that copy of \mathbb{P}^1 . In particular, if every point of Γ is a leaf in T_Γ , then the monomial tree is the star tree Z in [CS, Example 4.9]. By reversing this construction, we can show that each of the $2^n(n+1)^{n-2}$ monomial trees arises from a configuration in the Bruhat-Tits tree \mathfrak{B}_2 .

Corollary 3.9. *Every monomial ideal in the multigraded Hilbert scheme $H_{2,n}$ arises as the special fiber of a Mustafin variety $\mathcal{M}(\Gamma)$ for some n -element set $\Gamma \subset \mathfrak{B}_2^0$.*

4 Configurations in One Apartment

We will now investigate Mustafin varieties $\mathcal{M}(\Gamma)$ determined by configurations Γ that are contained in a single apartment A of the Bruhat-Tits building \mathfrak{B}_d . The apartment equals $A = X_*(T) \otimes_{\mathbb{Z}} \mathbb{R}$ for a maximal torus T , and we identify

$$A = \mathbb{R}^d / \mathbb{R}(1, 1, \dots, 1).$$

Scholars in tropical geometry have begun to use the term *tropical projective torus* for the apartment A together with its integral structure $A \cap \mathfrak{B}_d^0$. The *tropical projective space* is then the simplex obtained by compactifying A . We recall from Section 2 that

$$\{\pi^{m_1} Re_1 + \dots + \pi^{m_d} Re_d\} \mapsto (-m_1, \dots, -m_d) + \mathbb{R}(1, \dots, 1) \quad (5)$$

is a bijection between the set of lattice classes in diagonal form with respect to the basis e_1, \dots, e_d and the set of vertices in A . We define the distance between two points $u = (u_1, \dots, u_d)$ and $v = (v_1, \dots, v_d)$ in the apartment A as the variation

$$\text{dist}(u, v) = \max_i \{u_i - v_i\} - \min_i \{u_i - v_i\} = \max \{u_i - v_i - u_j + v_j : i \neq j\}.$$

If u and v are lattice points in A , then $\text{dist}(u, v)$ coincides with the combinatorial distance between lattice classes which we discussed in Section 2.

The apartment A is a *tropical semimodule* with tropical vector addition $\min\{u, v\} = (\min\{u_1, v_1\}, \dots, \min\{u_d, v_d\})$ and scalar multiplication $\lambda + u = (\lambda + u_1, \dots, \lambda + u_d)$. A subset S of A is *tropically convex* if, for all $u, v \in S$ and $\lambda, \mu \in \mathbb{R}$, the element $\min\{\lambda + u, \mu + v\}$ is also in S . For relevant basics on tropical convexity see [DS, JSY].

We fix a finite configuration $\Gamma = \{u^{(1)}, \dots, u^{(n)}\}$ of lattice points in $A \cap \mathfrak{B}_d^0$. The point $u^{(i)} = (u_{i1}, \dots, u_{id})$ represents the diagonal lattice $L^{(i)} = \pi^{-u_{i1}} Re_1 + \dots + \pi^{-u_{id}} Re_d$. Tropical convex combinations correspond to convex combinations of diagonal lattices in \mathfrak{B}_d . This is made precise by the following lemma. Recall (e.g. from [BY, DS, DJS, JSY]) that the *tropical polytope* or *tropical convex hull*, $\text{tconv}(\Gamma)$, is the smallest tropically convex subset of A containing Γ . The following lemma shows that this tropical convex hull corresponds to the convex hull of a set of lattice classes defined in Section 2.

Lemma 4.1. *The bijection (5) induces a bijection between the lattice points in $\text{tconv}(\Gamma)$ and the lattice classes in the convex hull of Γ in the Bruhat-Tits building \mathfrak{B}_d .*

Proof. Suppose L and L' are lattices that lie in the apartment A and let u and u' be the vectors in \mathbb{Z}^d that represent them. The lattice $\pi^s L \cap \pi^t L'$ is also in A , and is represented by $\min\{s + u, t + u'\}$. Therefore, the notions of convex combinations coincide. \square

We now consider the dual tropical structure on A given by the max-plus algebra. We encode Γ by the corresponding product of linear forms in the max-plus algebra:

$$P_\Gamma(X_1, \dots, X_d) = \sum_{i=1}^n \max(-u_{i1} + X_1, -u_{i2} + X_2, \dots, -u_{id} + X_d). \quad (6)$$

The tropical hypersurface $\mathcal{T}(P_\Gamma)$ defined by this expression is an arrangement of n tropical hyperplanes (see [AD]) in the $(d-1)$ -dimensional space A . Dual to this tropical hyperplane arrangement is a *mixed subdivision* [San, § 1.2] of the scaled standard simplex

$$n\Delta_{d-1} = \Delta_{d-1} + \Delta_{d-1} + \dots + \Delta_{d-1}. \quad (7)$$

We denote this mixed subdivision by Δ_Γ . Each cell of Δ_Γ has the form

$$\sigma = F_1 + F_2 + \cdots + F_n, \quad (8)$$

where F_i is a face of the i th summand Δ_{d-1} in the Minkowski sum (7).

The combinatorial relationship between the mixed subdivision, the tropical hyperplane arrangement and the tropical polytope $\text{tconv}(\Gamma)$ were introduced in [DS, § 5] and further developed in [AD, DJS]. The cells of the mixed subdivision Δ_Γ are in order-reversing bijection with the cells in the tropical hyperplane arrangement determined by $\mathcal{T}(P_\Gamma)$. The tropical polytope $\text{tconv}(\Gamma)$ is the union of the bounded cells in the arrangement $\mathcal{T}(P_\Gamma)$. These bounded cells correspond to the interior cells of Δ_Γ .

Example 4.2. For $d = 3$ there are many pictures of the above objects in the literature. For instance, for $n = 3$ consider the configuration Γ from Example 2.2 which is represented by the points $u^{(1)} = (-2, -1, 0)$, $u^{(2)} = (-4, -2, 0)$ and $u^{(3)} = (-6, -3, 0)$ in $A = \mathbb{R}^3/\mathbb{R}(1, 1, 1)$. That type of tropical triangle is obtained by moving the black point in [DJS, Figure 5] towards the southwestern direction. All 35 combinatorial types for $n = 4$ are depicted in [DS, Figure 6]. Note that the mixed subdivision Δ_Γ shown on the left in our Figure 1 corresponds to the configuration Γ that is labeled T [34] in the census of [DS]. Our Figure 2 shows configurations Γ that consist of *all* lattice points in a tropical polygon. Two cells in their mixed subdivisions Δ_Γ intersect if and only if the corresponding points in Γ are connected by an edge in the simplicial complex structure on $\text{tconv}(\Gamma)$. This is the statement about the reduction complex in Theorem 2.10. \square

Since the planar case ($d = 3$) has been amply visualized, we chose a configuration of three points in the three-dimensional apartment to serve as our example with picture.

Example 4.3. Fix $d = 4$, $n = 3$ and the vertices $u^{(1)} = (0, 0, 0, 0)$, $u^{(2)} = (0, -1, 0, -1)$, $u^{(3)} = (0, 0, -1, -1)$. This Γ defines an arrangement of three tropical planes in $A = \mathbb{R}^4/\mathbb{R}(1, 1, 1, 1)$ via

$$P_\Gamma = \max(X_1, X_2, X_3, X_4) + \max(X_1, X_2 + 1, X_3, X_4 + 1) + \max(X_1, X_2, X_3 + 1, X_4 + 1).$$

The expansion of this tropical product of linear forms is the tropical cubic polynomial

$$\begin{aligned} P_\Gamma = \max\{ & 3X_1 + c_{111}, 2X_1 + X_2 + c_{112}, \dots, X_1 + X_2 + X_3 + c_{123}, \dots, 3X_4 + c_{444}\}, \\ & \text{with } c_{111} = 0, \quad c_{112} = c_{113} = c_{114} = c_{122} = c_{133} = c_{222} = c_{333} = 1, \\ & c_{123} = c_{124} = c_{134} = c_{144} = c_{223} = c_{224} = c_{233} = c_{234} = c_{244} = c_{334} = c_{344} = c_{444} = 2. \end{aligned} \quad (9)$$

We associate these 20 coefficients with the lattice points in the tetrahedron $3\Delta_3$ of (7). The corresponding regular mixed subdivision Δ_Γ of the tetrahedron induced by lifting the points to heights c_{ijk} . It has four maximal cells (8) and is shown in Figure 5. \square

These combinatorial objects are important for the study of Mustafin varieties because of the following theorem, which is the main result in this section.

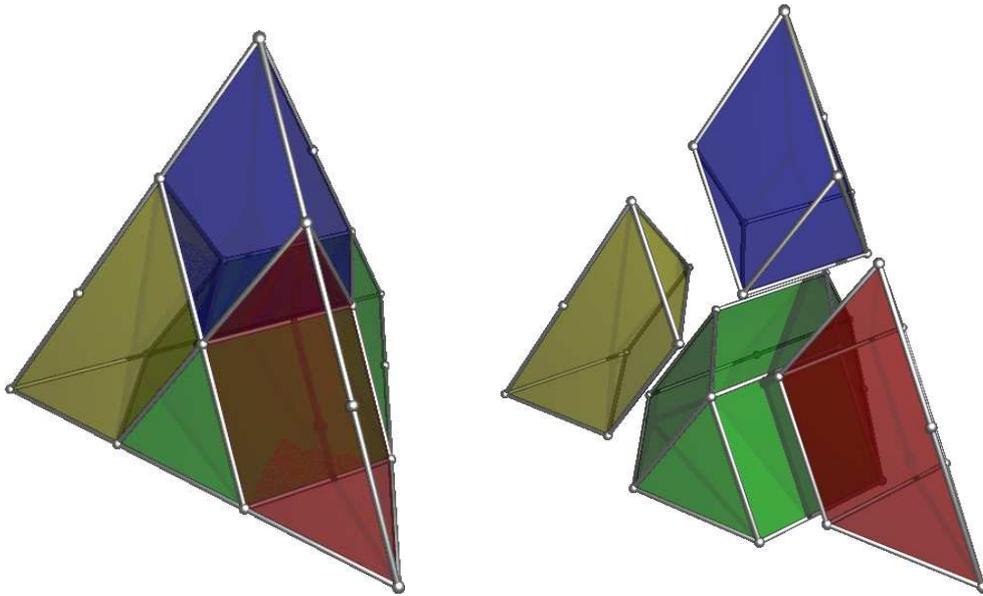


Figure 5: The special fiber of the Mustafin variety $\mathcal{M}(\Gamma)$ for the configuration Γ in Example 4.3 is represented by a mixed subdivision of the tetrahedron into four cells. The green cell represents a primary component of $\mathcal{M}(\Gamma)_k$ that is a singular toric 3-fold.

Theorem 4.4. *The Mustafin variety $\mathcal{M}(\Gamma)$ is isomorphic to the twisted n th Veronese embedding of the projective space \mathbb{P}_R^{d-1} determined by the tropical polynomial P_Γ . In particular, the special fiber $\mathcal{M}(\Gamma)_k$ equals the union of projective toric varieties corresponding to the cells in the regular mixed subdivision Δ_Γ of the simplex $n \cdot \Delta_{d-1}$.*

We now give a precise definition of the *twisted Veronese embedding* referred to above. Fix the lattice $L = R\{e_1, \dots, e_d\}$ and corresponding projective space $\mathbb{P}(L)$ with coordinates x_1, \dots, x_d . The n th symmetric power $\text{Sym}_n(L)$ of L is a free R -module of rank $\binom{n+d-1}{n}$. The corresponding projective space $\mathbb{P}(\text{Sym}_n(L))$ has coordinates $y_{i_1 i_2 \dots i_n}$ where $1 \leq i_1 \leq i_2 \leq \dots \leq i_n \leq d$. We can embed $\mathbb{P}(L)$ into $\mathbb{P}(\text{Sym}_n(L))$ by the n th *Veronese embedding* which is given in coordinates by $y_{i_1 i_2 \dots i_n} = x_{i_1} x_{i_2} \dots x_{i_n}$.

Consider any homogeneous tropical polynomial of degree n ,

$$C = \max\{-c_{i_1 i_2 \dots i_n} + X_{i_1} + X_{i_2} + \dots + X_{i_n} : 1 \leq i_1 \leq i_2 \leq \dots \leq i_n \leq d\}, \quad (10)$$

where the coefficients $c_{i_1 i_2 \dots i_n}$ are arbitrary integers. We define the *twisted Veronese embedding* $\mathbb{P}(V) \rightarrow_C \mathbb{P}(\text{Sym}_n(L))$ corresponding to the tropical polynomial C by

$$y_{i_1 i_2 \dots i_n} = x_{i_1} x_{i_2} \dots x_{i_n} \cdot \pi^{c_{i_1 i_2 \dots i_n}} \quad \text{for } 1 \leq i_1 \leq i_2 \leq \dots \leq i_n \leq d.$$

The closure of the image of this morphism in $\mathbb{P}(\text{Sym}_n(L))$ is an irreducible R -scheme. Its generic fiber is isomorphic to the n th Veronese embedding of \mathbb{P}_K^{d-1} .

Proof of Theorem 4.4. Our twisted Veronese embedding is a special case of the general construction of toric degenerations of projective toric varieties. That construction is well-known to experts, and it is available in the literature at various levels of generality, starting with the work on Gröbner bases of toric varieties, and the identification of initial monomial ideals with regular triangulations, presented in [Stu]. A complete treatment for the case of arbitrary polyhedral subdivisions, but still over \mathbb{C} , appears in [Hu, § 4]. The best reference for our setting of an arbitrary discretely valued field K seems to be Alexeev's construction of one-parameter families of stable-toric pairs in [Al, § 2.8]

The special fiber of the twisted Veronese embedding is a scheme over k . According to [Al, Lemma 2.8.4], its irreducible components are the projective toric varieties corresponding to the polytopes in the regular polyhedral subdivision of $n \cdot \Delta_{d-1}$ induced by the heights $c_{i_1 i_2 \dots i_n}$. These toric varieties are glued according to the dual cell structure given by the tropical hypersurface $\mathcal{T}(C)$. In the tropical literature this construction is known as *patchworking*. We note that the special fiber of the twisted Veronese can be non-reduced, even for $n = d = 2$, as with $\langle y_{12}^2 - \pi y_{11} y_{22} \rangle$.

In this proof we do not consider arbitrary tropical polynomials but only products of linear forms. Those encode configurations $\Gamma \subset A \cap \mathfrak{B}_d^0$. Their coefficients $c_{i_1 i_2 \dots i_d}$ are so special that they ensure the reducedness of the special fiber. Equating the tropical polynomial C in (10) with the tropical product of linear forms P_Γ in (6), we obtain

$$c_{i_1 i_2 \dots i_n} = \min \{ u_{i_1 \sigma_1} + u_{i_2 \sigma_2} + \dots + u_{i_n \sigma_n} : \sigma \in \mathfrak{S}_n \}.$$

Here \mathfrak{S}_n denotes the symmetric group on $\{1, 2, \dots, n\}$. The regular polyhedral subdivisions of $n \cdot \Delta_{d-1}$ defined by such choices of coefficients are the mixed subdivisions.

Consider the sequence of two embeddings

$$\mathbb{P}(V) \hookrightarrow \mathbb{P}(L_1) \times_R \dots \times_R \mathbb{P}(L_n) \hookrightarrow \mathbb{P}(L_1 \otimes_R \dots \otimes_R L_n).$$

The map on the left is the one in the definition of the Mustafin variety $\mathcal{M}(\Gamma)$. The map on the right is the classical *Segre embedding*, which is given in coordinates by

$$z_{i_1 i_2 \dots i_n} = x_{i_1, 1} x_{i_2, 2} \dots x_{i_n, n} \quad \text{for } 1 \leq i_1, i_2, \dots, i_n \leq d.$$

The Segre variety is the toric variety in $\mathbb{P}(L_1 \otimes_R \dots \otimes_R L_n)$ cut out by the equation

$$z_{i_1 \dots i_n} \cdot z_{j_1 \dots j_n} = z_{k_1 \dots k_n} \cdot z_{l_1 \dots l_n} \quad \text{whenever } \{i_\nu, j_\nu\} = \{k_\nu, l_\nu\} \text{ for } \nu = 1, 2, \dots, n.$$

The image of $\mathbb{P}(V)$ in the Segre variety is cut out by the linear equations

$$z_{j_1 j_2 \dots j_n} \pi^{u_{1, k_1} + \dots + u_{n, k_n}} = z_{k_1 k_2 \dots k_n} \pi^{u_{1, j_1} + \dots + u_{n, j_n}}$$

whenever the multisets $\{j_1, j_2, \dots, j_n\}$ and $\{k_1, k_2, \dots, k_n\}$ are equal. For any ordered sequence of indices $1 \leq i_1 \leq \dots \leq i_n \leq d$, we introduce the coordinates

$$y_{i_1 i_2 \dots i_n} := z_{j_1 j_2 \dots j_n} \pi^{c_{j_1 j_2 \dots j_n}}$$

where (j_1, \dots, j_n) is a permutation of (i_1, \dots, i_n) such that $c_{j_1 \dots j_n} = u_{1, i_1} + \dots + u_{n, i_n}$. Substituting the y -coordinates for the z -coordinates in the above equations, we find that the image of $\mathbb{P}(V)$ in $\mathbb{P}(L_1 \otimes_R \dots \otimes_R L_n)$ equals the twisted Veronese variety. \square

Example 4.5. Let $d = n = 3$ and $\Gamma = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$. The tropical convex hull of Γ is $\bar{\Gamma} = \Gamma \cup \{(0, 0, 0)\}$, as seen in [JSY, Figure 4]. The map $\mathcal{M}(\bar{\Gamma}) \rightarrow \mathcal{M}(\Gamma)$ is an isomorphism of schemes over R but it is not an isomorphism of Mustafin varieties. The special fiber $\mathcal{M}(\Gamma)_k$ has four irreducible components. It consists of the blow-up of \mathbb{P}_k^2 at three points with a copy of \mathbb{P}_k^2 glued along each exceptional divisor. The central component is primary in $\mathcal{M}(\bar{\Gamma})$ but it is demoted to being secondary in $\mathcal{M}(\Gamma)$. \square

The tropical polytope $\text{tconv}(\Gamma)$ comes with two natural subdivisions into classical convex polytopes. First, there is the *tropical complex* which is dual to the mixed subdivision Δ_Γ . Second, there is the simplicial complex induced from A on $\text{tconv}(\Gamma)$. This simplicial complex refines the tropical complex, and it is generally much finer. The vertices of the tropical complex on $\text{tconv}(\Gamma)$ correspond to the facets of Δ_Γ , and hence to the irreducible components of the special fiber $\mathcal{M}(\Gamma)_k$. As before, we distinguish between primary and secondary components, so the tropical complex on $\text{tconv}(\Gamma)$ has both primary and secondary vertices. The primary vertices are those contained in Γ , and the secondary vertices are all other vertices of the tropical complex.

The points in Γ are in *general position* if every maximal minor of the $d \times n$ -matrix (u_{ij}) is tropically non-singular. In this case the number of vertices in the tropical complex is $\binom{n+d-2}{d-1}$. That number is ten for $d = 3, n = 4$, as seen in [DJS, Figure 5], and the ten vertices correspond to the ten polygons in pictures as the left one in Figure 1. The points in Γ are in general position if and only if the subdivision Δ_Γ of $n\Delta_{d-1}$ is a *fine* mixed subdivision, arising from a *triangulation* of $\Delta_{d-1} \times \Delta_{n-1}$ [DS, Prop. 24].

Proposition 4.6. *For a set Γ of n elements in $A \cap \mathfrak{B}_d^0$ the following are equivalent:*

- (a) *The configuration Γ is in general position.*
- (b) *The special fiber $\mathcal{M}(\Gamma)_k$ is of monomial type.*
- (c) *$\mathcal{M}(\Gamma)_k$ is defined by a monomial ideal in $k[X]$ in our chosen coordinates.*
- (d) *The number of secondary components of $\mathcal{M}(\Gamma)_k$ equals $\binom{n+d-2}{d-1} - n$.*

Proof. Clearly, (c) implies (b), and Remark 2.11 states that (b) implies (d). By [San], the mixed subdivision of $n\Delta_{d-1}$ is equivalent to a subdivision of $\Delta_{d-1} \times \Delta_{n-1}$. That polytope is unimodular and has normalized volume $\binom{n+d-2}{d-1}$. Hence a polyhedral subdivision of $\Delta_{d-1} \times \Delta_{n-1}$ is a triangulation if and only if it has $\binom{n+d-2}{d-1}$ maximal cells, so (a) is equivalent to (d). The equivalence of (a) and (c) is proven in [BY, Prop. 4]. \square

Block and Yu [BY] showed that the tropical complex on $\text{tconv}(\Gamma)$ can be computed as a minimal free resolution in the sense of commutative algebra when Γ is in general

position. Namely, they show that the Alexander dual to the monomial ideal in (b) has a unique minimal cellular resolution, and the support of that resolution is the tropical complex on $\text{tconv}(\Gamma)$. This construction was extended to non-general tropical point configurations Γ in recent work of Dochtermann, Joswig and Sanyal [DJS].

Our results in this section apply to arbitrary Mustafin varieties when $n = 2$. Indeed, let $L^{(1)}$ and $L^{(2)}$ be any two lattices in V and consider $\Gamma = \{[L^{(1)}], [L^{(2)}]\}$. Then there exists an apartment A that contains Γ , and we can represent both $L^{(i)}$ by vectors $u^{(i)} \in \mathbb{Z}^d$ as above. The configuration $\{u^{(1)}, u^{(2)}\}$ is in general position if and only if the quantity $u_{1i} + u_{2j} - u_{2i} - u_{1j}$ are non-zero for all $1 \leq i < j \leq d$. This is equivalent to the statement that $u^{(1)}$ and $u^{(2)}$ are not contained in any affine hyperplane of the form (1) in A . Assuming that this is the case, the tropical complex on the line segment $\text{tconv}(\Gamma)$ consists of $d - 1$ edges and $d - 2$ secondary vertices between them. Each of the $d - 1$ edges is further subdivided into segments of unit length in the simplicial complex structure. If Γ is not in general position then some edges may have length zero.

Proposition 4.7. *For $n = 2$, isomorphism classes of Mustafin varieties are in bijection with lists of $d - 1$ non-negative integers, up to reversing the order. The elements of the list are the lengths of the segments of the one-dimensional tropical complex $\text{tconv}(\Gamma)$.*

Proof. We apply Theorem 2.8. Since any two points lie in a single apartment, their convex hull consists of a tropical line segment. The lengths along these line segments are an invariant of the configuration. \square

A coarser notion of isomorphism is given by the *combinatorial type* of the mixed subdivision Δ_Γ . Here, two configurations Γ and Γ' in $A \cap \mathfrak{B}_d^0$ have the same combinatorial type if and only if the special fibers $\mathcal{M}(\Gamma)_k$ and $\mathcal{M}(\Gamma')_k$ are isomorphic as k -schemes.

For fixed d and n , there are finitely many combinatorial types of configurations Γ . As mixed subdivisions of $n\Delta_{d-1}$ are in bijection with the triangulations of $\Delta_{n-1} \times \Delta_{d-1}$, the combinatorial types are classified by the faces of the *secondary polytope* $\Sigma(\Delta_{d-1} \times \Delta_{n-1})$. We illustrate this classification for the case $d = n = 3$ of triangles in the tropical plane.

Example 4.8 ($n = d = 3$). The *secondary polytope* of the direct product of two triangles, $\Sigma(\Delta_2 \times \Delta_2)$, is a four-dimensional polytope with f-vector $(108, 222, 144, 30)$. The 108 vertices correspond to the 108 triangulations of $\Delta_2 \times \Delta_2$. These come in five combinatorial types, first determined by A. Postnikov, and later displayed in [GKZ, Figure 39, p. 250]. The special fibers corresponding to these five types are listed in rows 1-5 of [DS, Table 1], and they are depicted in the first row of Figure 6 below. The second picture shows an orbit of size 12, and it represents a type which is incorrectly drawn in [GKZ, Figure 39]: the rightmost vertical edge should be moved to the left.

The 222 edges of $\Sigma(\Delta_2 \times \Delta_2)$ come in seven types, shown in the second and third row of Figure 6. The 144 2-faces come in five types, shown in the last two rows of Figure 6. In each case we report also the number of “bent lines,” by which we mean pairs of points in Γ whose tropical line segment disagrees with the classical line segment. Finally,

the 30 facets of $\Sigma(\Delta_2 \times \Delta_2)$ come in three types, corresponding to the coarsest mixed subdivisions of $3\Delta_2$. Two of them do not appear for us because they are degenerate in the sense that two of the three points in Γ are the identical. The only facet type that corresponds to a valid Mustafin triangle Γ is shown on the lower right in Figure 6. \square

5 Components of the Special Fiber

In this section, we describe the components of $\mathcal{M}(\Gamma)_k$. If $d = 2$ then each component is isomorphic to \mathbb{P}_k^1 , by Proposition 3.1. If Γ is contained in one apartment then each component is a toric variety, by Theorem 4.4. In general, we shall obtain the primary components of $\mathcal{M}(\Gamma)_k$ from projective space \mathbb{P}_k^{d-1} by blowing up linear subspaces.

Definition 5.1. Let W_1, \dots, W_m be linear subspaces in \mathbb{P}_k^{d-1} . Let $X_0 = \mathbb{P}_k^{d-1}$ and inductively define X_i to be the blowup of X_{i-1} at the preimage of W_i under $X_{i-1} \rightarrow \mathbb{P}_k^{d-1}$. We say that X is a *blow-up of \mathbb{P}_k^{d-1} at a collection of linear subspaces* if X is isomorphic to the variety X_m obtained by this sequence of blow-ups.

We now show that the order of the blow-ups does not matter in Definition 5.1:

Lemma 5.2. *Let $\pi: X \rightarrow \mathbb{P}_k^{d-1}$ be the blow-up at the linear spaces $\{W_1, \dots, W_m\}$. Then X is isomorphic to the blow-up of $\mathcal{I}_1 \mathcal{I}_2 \cdots \mathcal{I}_m$ where \mathcal{I}_i the ideal sheaf of W_i .*

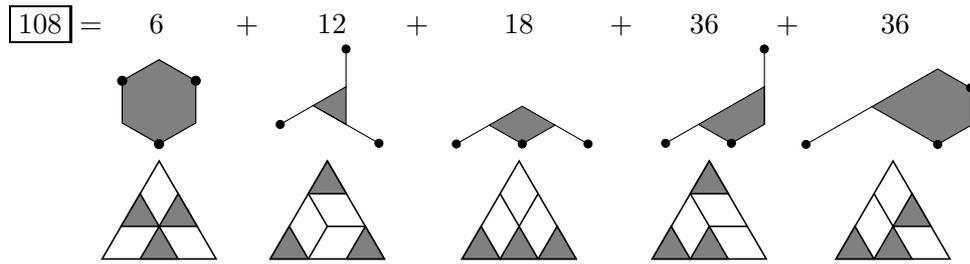
Proof. Let $\pi_i: X_i \rightarrow \mathbb{P}_k^{d-1}$ be the projection in the i th stage of Definition 5.1 and let $\rho: Y \rightarrow \mathbb{P}_k^{d-1}$ be the blow-up at the product ideal sheaf $\mathcal{I}_1 \cdots \mathcal{I}_m$. By the universal property of blowing up, $\pi_i^{-1}(\mathcal{I}_i) \cdot \mathcal{O}_{X_i}$ is locally free. Thus, locally, this sheaf is generated by one non-zero section of \mathcal{O}_{X_i} . Since $X \rightarrow X_i$ is surjective and X is integral, the pullback of this section is a non-zero section of \mathcal{O}_X , so $\pi^{-1}(\mathcal{I}_i) \cdot \mathcal{O}_X$ is also locally free. Therefore, the product $\pi^{-1}(\mathcal{I}_1) \cdots \pi^{-1}(\mathcal{I}_m) \cdot \mathcal{O}_X = \pi^{-1}(\mathcal{I}_1 \cdots \mathcal{I}_m) \cdot \mathcal{O}_X$ is locally free, and this defines a map from X to Y .

For the reverse map, we note that the product $\rho^{-1}(\mathcal{I}_1) \cdots \rho^{-1}(\mathcal{I}_m) \cdot \mathcal{O}_Y$ is locally free on Y , so each factor $\rho^{-1}(\mathcal{I}_i) \cdot \mathcal{O}_Y$ must be invertible. Thus, by the universal property of blowing up, $Y \rightarrow X_{i-1}$ inductively lifts to $Y \rightarrow X_i$. This defines a map $Y \rightarrow X$. Thus we have maps between Y and X which are isomorphisms over the complement of the linear spaces W_i , so they must be isomorphisms. \square

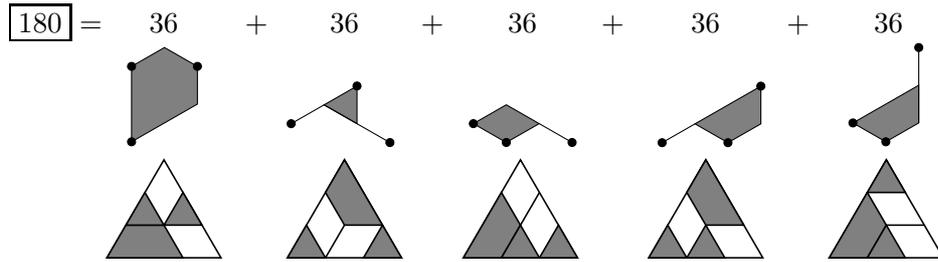
Theorem 5.3. *A projective variety X arises as a primary component of the special fiber $\mathcal{M}(\Gamma)_k$ for some configuration Γ of n lattice points in the Bruhat-Tits building \mathfrak{B}_d if and only if X is the blow-up of \mathbb{P}_k^{d-1} at a collection of $n - 1$ linear subspaces.*

Before proving this theorem, we shed some light on the only-if direction by describing the linear spaces in terms of the configuration Γ . Fix an index i and let C be the primary component of $\mathcal{M}(\Gamma)_k$ corresponding to the lattice class $[L_i]$. For any other point $[L_j]$ in Γ we choose the unique representative L_j such that $L_j \supset \pi L_i$ but $L_j \not\supset L_i$. Then the

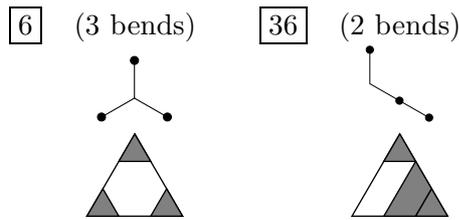
(i) 6 components, 3 bent lines, corresponding to vertices of the secondary polytope:



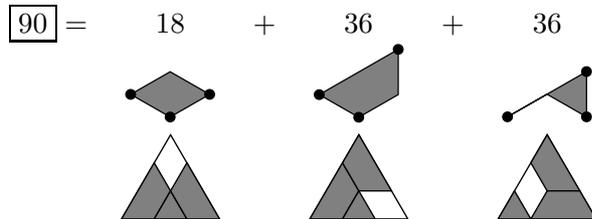
(ii) 5 components, 2 bent lines, corresponding to edges of the secondary polytope:



(iii) 4 components, 3 or 2 bent lines, corresponding to edges of the secondary polytope:



(iv) 4 components, 1 bent line, corresponding to 2-faces of the secondary polytope:



(v) 3 components, 1 or 0 bent lines, corresponding to 2-faces or facets:

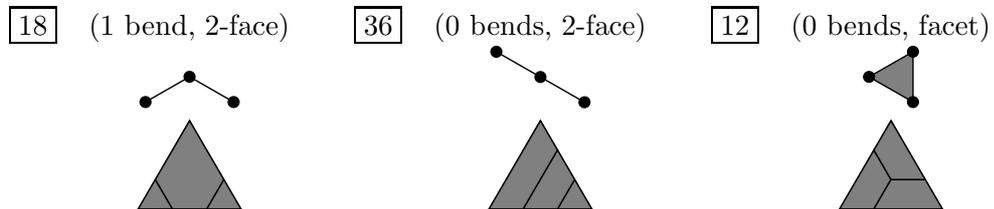


Figure 6: The 18 combinatorial types of planar Mustafin triangles

image of $L_j \cap L_i$ in the quotient $L_i/\pi L_i$ is a proper, non-trivial k -vector subspace, and we denote by W_j the corresponding linear subspace in $\mathbb{P}(L_i)_k$. The component C is the blow-up of $\mathbb{P}(L_i)_k$ at the linear subspaces W_j for all $j \neq i$.

Since $[L_i]$ and $[L_j]$ are in a common apartment, there is a basis e_1, \dots, e_d of V such that $L_i = R\{e_1, \dots, e_d\}$ and $L_j = R\{\pi^{-s_1}e_1, \dots, \pi^{-s_d}e_d\}$, where $-1 = s_1 \leq \dots \leq s_d$, in order to satisfy the above condition on the representative. Then W_j is the linear space spanned by $\{e_i : s_i \geq 0\}$. In particular, if $\{W_i, W_j\}$ are in general position then W_j is the hyperplane spanned by e_2, \dots, e_d and the blow-up of W_j is trivial.

Example 5.4. Various classical varieties arise as primary components of some $\mathcal{M}(\Gamma)_k$. For instance, any del Pezzo surface (other than $\mathbb{P}^1 \times \mathbb{P}^1$) is the blow up of \mathbb{P}^2 at ≤ 8 general points, and thus arises for an appropriate configuration $\Gamma \in \mathfrak{B}_3$ with $|\Gamma| \leq 9$. \square

Proof of Theorem 5.3. First we suppose that Γ has only two elements, and we choose coordinates as in the discussion prior to Example 5.4. By Theorem 4.4, we can compute the special fiber from the arrangement of two tropical hyperplanes. We represent a point in $A = \mathbb{R}^d/\mathbb{R}(1, \dots, 1)$ by the last $d - 1$ entries of a vector in \mathbb{R}^d , after rescaling so that the first entry is 0. Thus, our tropical hyperplanes are centered at $(0, \dots, 0)$ and $(s_2 + 1, \dots, s_d + 1)$. The former point lies in the relative interior of the cone of the latter tropical hyperplane generated by $-e_t, \dots, -e_d$, where t is the smallest index such that $s_t \geq 0$. This containment creates a ray at $(0, \dots, 0)$ generated by the vector $e_t + \dots + e_d$, together with adjacent cones. The resulting complete fan corresponds to the toric blow-up of \mathbb{P}_k^{d-1} at the linear space spanned by e_t, \dots, e_d . This agrees with the description given after the statement of Theorem 5.3.

Now suppose Γ has $n > 2$ elements. We fix one element $[L_i]$ and let C denote the corresponding primary component of $\mathcal{M}(\Gamma)_k$. We claim that C is the blow-up of $\mathbb{P}(L_i)_k$ at the linear subspaces W_j described after the theorem. For each $[L_j] \in \Gamma \setminus \{[L_i]\}$, we have a projection $\mathcal{M}(\Gamma) \rightarrow \mathcal{M}(\{[L_i], [L_j]\})$ which sends C to the $[L_i]$ -primary component of $\mathcal{M}(\{[L_i], [L_j]\})_k$, and we denote this component by $C_j \subset \mathbb{P}(L_i)_k \times \mathbb{P}(L_j)_k$. We have shown C_j to be blow-up of $\mathbb{P}(L_i)_k$ at W_j . By taking the fiber product of these components with the base $\mathbb{P}(L_i)_k$ for all $j \neq i$, we get the closed immersion

$$C \rightarrow \prod_{j \neq i} C_j \rightarrow \prod_{j=1}^n \mathbb{P}(L_j)_k,$$

where the first product is the fiber product over $\mathbb{P}(L_i)_k$ and the second is over $\text{Spec } k$. Since each projection $C_j \rightarrow \mathbb{P}(L_i)_k$ is a birational morphism, the fiber product $\prod_{j \neq i} C_j$ contains an open subset which is mapped isomorphically to $\mathbb{P}(L_i)_k$. Since C is irreducible and birational with $\mathbb{P}(L_i)_k$, we conclude that C is isomorphic to the closure of this open set, which is necessarily the desired primary component.

On the other hand, let B be the blow-up of $\mathbb{P}(L_i)_k$ at the linear subspaces W_j , and we wish to show that C is isomorphic to B . Since B maps compatibly to each C_j , we

have a map to the fiber product $\prod_{j \neq i} C_j$ with base $\mathbb{P}(L_i)_k$. Since B is irreducible and birational with $\mathbb{P}(L_i)_k$, this map factors through a map $B \rightarrow C$. On the other hand, the pullbacks of the ideal sheaves of each W_j to C are all invertible, and this gives the inverse map from C to B . We conclude that C and B are isomorphic.

Finally, for any arrangement of $n - 1$ linear subspaces in \mathbb{P}_k^{d-1} we can choose a configuration $\Gamma \in \mathfrak{B}_d$ with $n = |\Gamma|$ which realizes the blow-up at these linear spaces as a primary component. To do this, we represent each linear space as a vector subspace W_j of k^d , and we let M_j denote the preimage in R^d under the residue map $R^d \rightarrow k^d$. Then we take our configuration to be the standard lattice R^d and the adjacent lattices $M_j + \pi R^d$. The component of $\mathcal{M}(\Gamma)_k$ corresponding to R^d is the desired blow-up. \square

It follows from Theorem 5.3 that the primary components are always smooth for $d \leq 3$ or $n = 2$. However, if $d \geq 4$ and $n \geq 3$, then we encounter primary components that are not smooth. These arise from the fact that the simultaneous blow-up of projective space at several linear subspaces may be singular. This was demonstrated in Figure 5.

Example 5.5. Let $V = K^4$ with basis $\{e_1, e_2, e_3, e_4\}$ and let $\Gamma = \{[L_1], [L_2], [L_3]\}$ be given by $L_1 = R\{e_1, e_2, e_3, e_4\}$, $L_2 = R\{e_1, \pi e_2, e_3, \pi e_4\}$ and $L_3 = R\{e_1, e_2, \pi e_3, \pi e_4\}$. The primary component C corresponding to $[L_1]$ is singular. It is obtained from \mathbb{P}_k^3 by blowing up the two lines spanned by $\{e_1, e_2\}$ and $\{e_1, e_3\}$ respectively. This configuration was studied in Example 4.3. Its special fiber is drawn in Figure 5, in which the polytope corresponding to C is the green cell. This polytope has a vertex that is adjacent to four edges. \square

Example 5.5 underscores the fact that the convex configurations considered by Mustafin [Mus] and Faltings [Fa] are very special. In the convex case, all primary components are smooth. This is a consequence of the following more general result.

Proposition 5.6. *Fix the lattice L_n and the linear spaces W_1, \dots, W_{n-1} in $L_n/\pi L_n$ as above. Suppose that this collection of linear spaces is closed under intersection, and that the order refines their order by inclusion, so that $W_j \subset W_i$ implies $j < i$. Then the primary component C corresponding to $[L_n]$ is formed by successively blowing up the strict transforms of W_1, \dots, W_{n-1} in that order. In particular, C is smooth.*

Proof. We know that C is formed by the successive blow-ups of the weak transforms, so we just need to show that these are equivalent to the blow-ups of the strict transforms. Let B_{i-1} be the blow-up of W_1, \dots, W_{i-1} . We describe the weak transform of W_i by tracing it through the previous blow-ups. For the blow-up at $W_j \subset W_i$, the weak transform of W_i consists of the strict transform together with the exceptional divisor of the blow-up. At every other step, the weak transform is identical to the strict transform. Therefore, the weak transform of W_i consists of the union of the strict transforms \tilde{W}_i together with the exceptional divisors from blow-ups of $W_j \subset W_i$. Since these exceptional divisors are defined by locally principal ideals, we can remove them

without changing the blow-up, so B_i is isomorphic to the blow-up of \tilde{W}_i . Since each strict transform \tilde{W}_i is smooth, its blow-up is also smooth, so C is smooth. \square

Example 5.7. The compactification $\overline{\mathcal{M}}_{0,6}$ of the moduli space of six points on \mathbb{P}_k^1 arises from \mathbb{P}_k^3 by blowing up five general points followed by blowing up the ten lines through these points [Ka, Theorem 4.3.3]. Using Proposition 5.6, there exists a configuration $\Gamma \in \mathfrak{B}_d$ with $|\Gamma| = 16$ such that $\overline{\mathcal{M}}_{0,6}$ is a component of $\mathcal{M}(\Gamma)_k$. For $\overline{\mathcal{M}}_{0,7}$, the Kapranov blow-up construction requires blowing up planes in \mathbb{P}_k^4 without first blowing up their point of intersection, which does not fit into the model of Proposition 5.6. \square

The isomorphism types of the secondary components of the special fibers are less restricted than that of the primary components, but they are still rational varieties.

Lemma 5.8. *Let C be a secondary component in $\mathcal{M}(\Gamma)_k$. Then there exists a vertex v in \mathfrak{B}_d^0 such that if $\Gamma' = \Gamma \cup \{v\}$, then $\mathcal{M}(\Gamma') \rightarrow \mathcal{M}(\Gamma)$ restricts to a birational morphism $\tilde{C} \rightarrow C$, where \tilde{C} is the primary component of $\mathcal{M}(\Gamma')_k$ corresponding to v .*

Proof. Let $\bar{\Gamma}$ be the set of all vertices in the convex closure of Γ . By Lemma 2.4, there is some component of $\mathcal{M}(\bar{\Gamma})_k$ mapping birationally onto C . By Theorem 2.10, the component of $\mathcal{M}(\bar{\Gamma})_k$ must be primary, and so corresponds to some vertex v . Let $\Gamma' = \Gamma \cup \{v\}$ and let \tilde{C} be the primary component corresponding to v . Since $\mathcal{M}(\bar{\Gamma}) \rightarrow \mathcal{M}(\Gamma)$ factors through $\mathcal{M}(\Gamma')$, \tilde{C} must map birationally onto C . \square

Corollary 5.9. *Every secondary component is a rational variety.*

Proof. This follows from Lemma 5.8 and Theorem 5.3. \square

Now we wish to describe the geometry of the secondary components in more detail. If C is a secondary component of $\mathcal{M}(\Gamma)_k$, we let C , \tilde{C} and Γ' be as in Lemma 5.8, and let π denote the projection $\mathcal{M}(\Gamma') \rightarrow \mathcal{M}(\Gamma)$. We identify \tilde{C} with the blow-up of \mathbb{P}_k^{d-1} at the linear spaces W_i as in Theorem 5.3.

Lemma 5.10. *If $\tilde{L} \subset \tilde{C}$ is the strict transform of a line L in \mathbb{P}_k^{d-1} , then $\pi|_{\tilde{L}}$ is either constant or a closed immersion. Moreover, $\pi|_{\tilde{L}}$ is constant if and only if L intersects all subspaces W_i . The restriction of π to an exceptional divisor is a closed immersion.*

Proof. Let L be a line in \mathbb{P}_k^{d-1} and \tilde{L} its strict transform. Consider any vertex $w_i \in \Gamma$ with W_i the corresponding linear space in \mathbb{P}_k^{d-1} . Then the projection of \tilde{L} onto the i th factor \mathbb{P}_k^{d-1} is constant if L intersects W_i and is a closed immersion if not. Since the projection of \tilde{L} to $\mathcal{M}(\Gamma)_k$ consists of the projection to the fiber product of these factors, we have the desired result. For any exceptional divisor, the projection to the factor of v is constant, so the projection to $\mathcal{M}(\Gamma)_k$ must be a closed immersion. \square

Proposition 5.11. *With the set-up as in Lemma 5.10, the exceptional locus of $\tilde{C} \rightarrow C$ is the union of the strict transforms of all lines which intersect all of the subspaces W_i .*

Proof. If a line $L \subset \mathbb{P}_k^{d-1}$ passes through all of the W_i , then its strict transform in \tilde{C} is contracted in C by Lemma 5.10, so any point on the line is in the exceptional locus. Conversely, suppose that x is in the exceptional locus, so there exists a point y in C such that $\pi(x) = \pi(y)$. By Lemma 5.10, x and y cannot be in an exceptional divisor. Thus, we take the projections of x and y to \mathbb{P}_k^{d-1} and let L be the line through them. Lemma 5.10 implies that \tilde{L} must be contracted to a point by π , and therefore L intersects all the linear spaces W_i . \square

Example 5.12. Fix a basis e_1, e_2, e_3 of V . Let $L_i = R\{\pi e_1, \pi e_2, \pi e_3, e_i\}$ and consider the secondary component associated to $v = [R\{e_1, e_2, e_3\}]$. Thus, $W_i = ke_i$. If $\Gamma = \{[L_1], [L_2]\}$, then C is the blow-up of \mathbb{P}_k^2 at the points W_1 and W_2 , followed by the blow-down of the line between them, yielding $\mathbb{P}_k^1 \times \mathbb{P}_k^1$. If $\Gamma = \{[L_1], [L_2], [L_3]\}$, then \tilde{C} and C are both isomorphic to the blow-up of \mathbb{P}_k^2 at the three points W_1, W_2 and W_3 , since there are no lines passing through all three points. This is the same configuration as in Example 4.5. \square

Example 5.13. Suppose that $L_1 = R\{e_1, \pi e_2, \pi e_3\}$, $L_2 = R\{\pi e_1, e_2, \pi e_3\}$ and $L_3 = R\{e_1 + e_2, \pi e_2, \pi e_3\}$, and let $\Gamma = \{[L_1], [L_2], [L_3]\}$. Then $\mathcal{M}(\Gamma)_k$ has a secondary component that is singular, indexed by $L_4 = R\{e_1, e_2, e_3\}$. The corresponding primary component of $\mathcal{M}(\Gamma \cup \{[L_4]\})_k$ is the blow-up of \mathbb{P}_k^2 at three collinear points $W_1 = ke_1$, $W_2 = ke_2$, $W_3 = k(e_1 + e_2)$. The secondary component in $\mathcal{M}(\Gamma)_k$ arises by blowing down the strict transform of the line through these three points. Algebraically, the ideal

$$\langle x_1, z_1, y_2, z_2 \rangle \cap \langle x_1, z_1, x_3, z_3 \rangle \cap \langle y_2, z_2, x_3, z_3 \rangle \cap \langle x_1, y_2, x_3, z_1 z_2 y_3 + z_1 x_2 z_3 - y_1 z_2 z_3 \rangle \quad (11)$$

represents $\mathcal{M}(\Gamma)_k$, where $(x_i : y_i : z_i)$ are the coordinates on the i th factor of $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$. The last prime ideal in (11) is the secondary component. It has a quadratic cone singularity at the point $((0 : 1 : 0), (1 : 0 : 0), (0 : 1 : 0))$. This special fiber looks like a *sailboat*: the secondary component is the boat; its sails are three projective planes attached at three of its lines. In the census of Theorem 6.1, this is the unique type which is not a union of toric surfaces, so it cannot be drawn as a 2-dimensional polyhedral complex. \square

6 Triangles

In this section we classify Mustafin varieties for $d = n = 3$. We refer to a triple Γ in \mathfrak{B}_3^0 as a *Mustafin triangle*. Two such triangles are said to have the same *combinatorial type* if the special fibers of the associated Mustafin varieties are isomorphic as k -schemes.

Theorem 6.1. *There are precisely 38 combinatorial types of Mustafin triangles. In addition to the 18 planar types (in Figure 6) there are 20 non-planar types (in Table 1).*

Number of components	Number of bent lines			
	0	1	2	3
3	$2 + \mathbf{0}$	$1 + \mathbf{0}$		
4		$3 + \mathbf{3}$	$1 + \mathbf{0}$	$1 + \mathbf{1}$
5			$5 + \mathbf{6}$	$0 + \mathbf{2}$
6				$5 + \mathbf{8}$

Table 1: Classification of the *18 planar* and **20 non-planar** types of Mustafin triangles.

The term *non-planar* is used as in [CS, § 5]. It refers to combinatorial types consisting entirely of configurations Γ that do not lie in a single apartment. The rest of this section is devoted to proving Theorem 6.1. Since the planar configurations were enumerated in Example 4.8, our task is to classify all non-planar Mustafin triangles, and to show that all types are realizable over any valuation ring (R, K, k) .

The convex hull of any two points v and w in \mathfrak{B}_3 is a tropical line segment which is contained in a single apartment. If this tropical line consists of a single Euclidean line segment, then we say that the line is *unbent*. Otherwise, the tropical line consists of two Euclidean line segments and we call the line *bent* and the junction of the two lines the *bend point*. Note that in this section the term *line* always means tropical line. By the bend points of a configuration Γ we mean the bend points of all pairs of points in Γ .

Proposition 6.2. *For a finite subset Γ of \mathfrak{B}_3^0 , the secondary components of $\mathcal{M}(\Gamma)_k$ are in bijection with the bend points of Γ which are not themselves elements of the set Γ .*

Proof. Let $\Gamma = \{v_1, \dots, v_n\}$ and let Γ' be the union of Γ and the set of bend points of Γ . By Lemma 2.4, each component C of $\mathcal{M}(\Gamma)_k$ is the image of a *unique* component C' of $\mathcal{M}(\Gamma')_k$ under the natural projection. Suppose that C is secondary in $\mathcal{M}(\Gamma)_k$. To establish the bijection, we must prove that C' is primary in $\mathcal{M}(\Gamma')_k$.

Recall from the proof of Lemma 2.4 that the cycle class of any component of $\mathcal{M}(\Gamma)_k$ is a sum of distinct monomials of $\mathbb{Z}[H_1, \dots, H_n]/\langle H_1^3, \dots, H_n^3 \rangle$ having degree $2n - 2$. Since C is not a primary component, every term in this cycle class involves every variable. Therefore, after permuting the factors, we can assume that the cycle class of C contains the term $H_1 H_2 H_3^2 \cdots H_n^2$. The image of C under the projection $\mathcal{M}(\Gamma) \rightarrow \mathcal{M}(\{v_1, v_2\})$ must be a component, and since C is a secondary component, so is its image. The vertices v_1 and v_2 lie in a common apartment, and Theorem 4.4 implies that the tropical convex hull of v_1 and v_2 must have a bend point, which we denote w , corresponding to the secondary component. By Lemma 2.4, the secondary component is the image of some component of $\mathcal{M}(\{v_1, v_2, w\})_k$. Using Theorem 4.4 we see that the only candidate is the w -primary component. The w -primary component of $\mathcal{M}(\Gamma')_k$ maps onto the w -primary component of $\mathcal{M}(\{v_1, v_2, w\})_k$ and then onto the secondary component of $\mathcal{M}(\{v_1, v_2\})_k$, so it equals the unique component C' that maps surjectively onto C . \square

Our classification of non-planar Mustafin triangles will proceed in two phases. For

special fibers with few components, the key result is Lemma 6.4 below. On the other hand, for special fibers with five or six components, Lemma 6.6 will imply that their ideals are monomial or “almost monomial”. Before getting to these technical phases, however, we first discuss all bold face entries in Table 1, starting with the last column.

Example 6.3 (Mustafin triangles with three bent lines). Let Γ be a triple in \mathfrak{B}_3^0 with three bent lines. The first row of Table 1 concerns types without any secondary component, which is not possible if there are three bent lines. In the second row we find types with one secondary component. There is one planar possibility, namely, the type , and one non-planar possibility, namely the “sailboat” in Example 5.13. Corollary 6.7 will take care of the last row in Table 1: these are the 13 monomial ideals in [CS, Table 1] that lie on the main component of the Hilbert scheme $H_{3,3}$. In the pictures offered in [CS, Figure 2] we recognize the 5 planar monomial types     , and the 8 non-planar types are obtained from these by *grafting triangles*.

An especially interesting entry in Table 1 is the rightmost entry $0 + \mathbf{2}$ in the third row. No planar types have two secondary components and three bent lines, but there are two non-planar types. Their pictures are shown in Figure 7. The three lattices

$$L_1 = R\{\pi e_1, \pi e_2, e_3\}, \quad L_2 = R\{e_1, \pi^2 e_2, \pi^2 e_3\}, \quad L_3 = R\{e_1 + \pi e_2, \pi^2 e_2, \pi^2 e_3\} \quad (12)$$

give a Mustafin variety whose special fiber is defined by the ideal

$$\langle y_1, z_1, x_2, z_2 \rangle \cap \langle y_1, z_1, x_3, z_3 \rangle \cap \langle x_2, \underline{y_2}, x_3, \underline{y_3} \rangle \cap \langle y_1, z_1, x_2, x_3 \rangle \cap \langle z_1, x_2, x_3, z_2 y_3 - y_2 z_3 \rangle. \quad (13)$$

The primary components of this special fiber are all isomorphic to \mathbb{P}_k^2 and embedded in $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$ as coordinate linear spaces. The two secondary components are isomorphic to $\mathbb{P}_k^1 \times \mathbb{P}_k^1$. One of these copies of $\mathbb{P}_k^1 \times \mathbb{P}_k^1$ is embedded as a coordinate linear space. The other is embedded as a coordinate linear space times the diagonal of the previous secondary component. Thus, the intersection of the two secondary components is the diagonal in the first and a line of one of the rulings in the second. Two of the primary components are attached along coordinate lines of the former secondary component. The final primary component is glued along the unique coordinate line of the diagonal secondary component which does not intersect the other two primary components.

Both initial ideals of (13) belong to isomorphism class 11 from [CS, § 5] which is the fifth picture in the second row of [CS, Figure 2]. The special fiber (13) is obtained from that picture by removing the two uppermost parallelograms and replacing them with a long rectangle which is attached to the diagonal of the lower parallelogram.

The other non-planar special fiber for the entry $0 + \mathbf{2}$ of Table 1 is the variant of the above configuration with $L'_1 = R\{\pi e_1, \pi e_3, e_2\}$ instead of L_1 in (12). Its ideal is

$$\langle y_1, z_1, x_2, z_2 \rangle \cap \langle y_1, z_1, x_3, z_3 \rangle \cap \langle x_2, \underline{z_2}, x_3, \underline{z_3} \rangle \cap \langle y_1, z_1, x_2, x_3 \rangle \cap \langle z_1, x_2, x_3, z_2 y_3 - y_2 z_3 \rangle. \quad (14)$$

The components of (14) are identical to those of (13) except that the final primary component is glued along the other coordinate line of the diagonal secondary component,

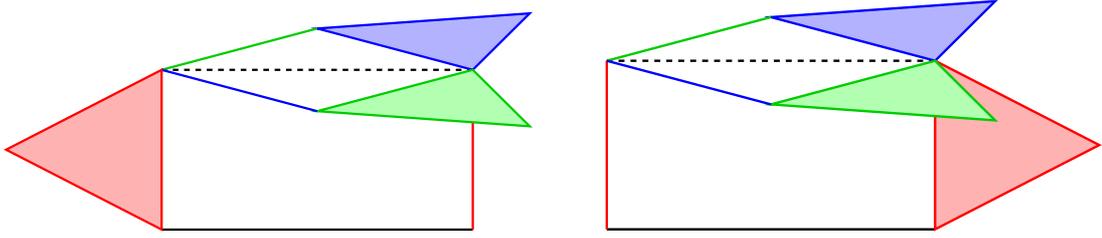


Figure 7: The two combinatorial types of Mustafin triangles with three bent lines and two secondary components. The ideal (13) is on the left while (14) is on the right.

so that all three primary components intersect. Its initial ideals are of isomorphism class 13 from [CS, § 5], which is the left picture in the third row of [CS, Figure 2]. The special fiber (14) is again obtained by replacing the two uppermost parallelograms with a long rectangle which is attached to the diagonal of the lower parallelogram. \square

We now come to the first technical phase in our classification of Mustafin triangles.

Lemma 6.4. *Let $[L_1]$, $[L_2]$ and $[L_3]$ be distinct points in \mathfrak{B}_3^0 such that the line between $[L_1]$ and $[L_i]$ is unbent for $i = 2, 3$. Either the three points lie in a single apartment, or there exists a basis e_1, e_2, e_3 of L_1 and integers $0 < t < s, u$ such that the other lattices can be written in one of the following forms (possibly after exchanging $[L_2]$ and $[L_3]$):*

- (i) $L_2 = R\{e_1, \pi^s e_2, \pi^s e_3\}$ and $L_3 = R\{e_1 + \pi^t e_2, \pi^u e_2, \pi^u e_3\}$,
- (ii) $L_2 = R\{e_1, \pi^s e_2, \pi^s e_3\}$ and $L_3 = R\{e_1 + \pi^t e_3, e_2, \pi^u e_3\}$,
- (iii) $L_2 = R\{e_1, e_2, \pi^s e_3\}$ and $L_3 = R\{e_1 + \pi^t e_3, e_2, \pi^u e_3\}$.

Proof. We can choose representatives $L_2 = M_2 + \pi^s L_1$ and $L_3 = M_3 + \pi^u L_1$ such that $s, u > 0$ and the R -modules M_i are direct summands of L_1 . The R -module $M_2 \cap M_3$ is also a direct summand of L_1 , and its rank is smaller or equal to the k -dimension of $\overline{M_2} \cap \overline{M_3}$. Here $\overline{M_2}$ and $\overline{M_3}$ are the subspaces of $L_1/\pi L_1 \simeq k^3$ induced by M_2 and M_3 .

Suppose the rank of $M_2 \cap M_3$ equals the k -dimension of $\overline{M_2} \cap \overline{M_3}$. We claim that the lattices are in a single apartment. We pick a k -basis $\bar{e}_1, \bar{e}_2, \bar{e}_3$ for $L_1/\pi L_1$ such that $\overline{M_2}$, $\overline{M_3}$ and $\overline{M_2} \cap \overline{M_3}$ are all spanned by subsets of this basis. By our assumption on the dimension, we have $\overline{M_2} \cap \overline{M_3} = \overline{M_2} \cap \overline{M_3}$. Hence we can lift the basis elements in $\overline{M_2} \cap \overline{M_3}$ to $M_2 \cap M_3$. We also lift the other basis elements, choosing representatives in M_2 and M_3 for the elements in $\overline{M_2}$ and $\overline{M_3}$, respectively. By Nakayama's Lemma, L_1 is generated by the lifts of all three basis elements, and M_2 and M_3 are generated by subsets thereof. Thus, $\{[L_1], [L_2], [L_3]\}$ lies in the apartment defined by this basis.

We now assume that the rank of $M_2 \cap M_3$ is strictly smaller than the k -dimension of $\overline{M_2} \cap \overline{M_3}$. Note that the ranks of M_2 and M_3 are either one or two. If M_2 and M_3 both have rank one, our assumption implies that $M_2 \cap M_3 = 0$ and $\dim_k(\overline{M_2} \cap \overline{M_3}) = 1$,

so that $\overline{M_2} = \overline{M_3}$. Let e_1 be a generator of M_2 . We lift $\overline{e_1}$ to a generator $e_1 + \pi^t e_2$ of M_3 , where $t \geq 1$ and $e_2 \in L_1 \setminus \pi L_1$. Since $M_2 \cap M_3 = 0$, we can assume that $\overline{e_1}, \overline{e_2}$ are linearly independent. Hence they can be completed to a basis of $L_1/\pi L_1$, which lifts to a basis e_1, e_2, e_3 of L_1 . If $t \geq u$, then $L_3 = M_3 + \pi^u L_1 = R\{e_1, \pi^u e_2, \pi^u e_3\}$, and all three lattice classes lie in the apartment given by e_1, e_2, e_3 . If $t \geq s$, then $L_2 = M_2 + \pi^s L_1 = R\{e_1 + \pi^t e_3, \pi^s e_2, \pi^s e_3\}$ and all three lattice classes lie in the apartment given by $e_1 + \pi^t e_3, e_2, e_3$. If $t < s$ and $t < u$, then we are in case (i).

If M_2 has rank one and M_3 has rank two, the dimension of $\overline{M_2} \cap \overline{M_3}$ is at most one. Our assumption implies that it is one and that $M_2 \cap M_3 = 0$. Let e_1 be a generator for M_2 . We fix $e_2 \in M_3$ such that $\overline{e_1}, \overline{e_2}$ is a basis of $\overline{M_3}$. We choose e'_3 to complete e_1 and e_2 to a basis for L_1 . Then M_3 is generated by e_2 and an element of the form $e_1 + \pi^t u e'_3$, where u is a unit in R . By replacing e'_3 with $e_3 = u e'_3$ in our basis, M_3 is generated by e_2 and $e_1 + \pi^t e_3$. If $t \geq s$ or $t \geq u$, the three lattice classes lie in one apartment by the same argument as in the previous case, and if $t < s$ and $t < u$, we are in case (ii).

If M_2 and M_3 both have rank two, then $M_2 \cap M_3$ has rank one, since $M_2 \neq M_3$. Our assumption implies that $\overline{M_2} \cap \overline{M_3}$ is two-dimensional, so that $\overline{M_2} = \overline{M_3}$. Choose a generator e_2 of $M_2 \cap M_3$. Since $M_2 \cap M_3$ is a split submodule of M_2 , we can complete it to a basis e_1, e_2 of M_2 . As in the previous case, we choose some e'_3 such that e_1, e_2, e'_3 is a basis for L_1 . Then M_3 can be generated by e_2 and an element of the form $e_1 + \pi^t u e'_3$, where u is a unit in R . By replacing e'_3 with $e_3 = u e'_3$ in our basis, M_3 is generated by e_2 and $e_1 + \pi^t e_3$. Hence we are in case (iii) if $t < u$ and $t < s$. Otherwise, the same argument as above shows that the three lattice classes lie in one apartment. \square

Corollary 6.5. *If Γ has no bent lines, then Γ lies in a single apartment.*

Proof. The exceptional cases in Lemma 6.4 each have a bent line between L_2 and L_3 . \square

The Chow ring of the product $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$ is $\mathbb{Z}[H_1, H_2, H_3]/\langle H_1^3, H_2^3, H_3^3 \rangle$. As seen in the proof of Lemma 2.4, the special fiber $\mathcal{M}(\Gamma)_k$ of a Mustafin triple Γ has the class

$$H_2^2 H_3^2 + H_1^2 H_3^2 + H_1^2 H_2^2 + H_1 H_2 H_3^2 + H_1 H_2^2 H_3 + H_1^2 H_2 H_3. \quad (15)$$

The cycle class of each component of $\mathcal{M}(\Gamma)_k$ is a sum of a subset of these monomials.

Lemma 6.6. *There exist coordinates on $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$ such that the projection of each component of $\mathcal{M}(\Gamma)_k$ is a coordinate linear space. Each component whose cycle class is one of the monomials in (15) is a product of coordinate linear spaces in this basis.*

Proof. Let $\Gamma = \{v_1, v_2, v_3\}$ and C_1, C_2, C_3 the corresponding primary components of $\mathcal{M}(\Gamma)_k$. Let C_{12} denote the unique component whose cycle class includes the monomial $H_1 H_2 H_3^2$ and similarly for the other pairs of indices. In this manner, every component of $\mathcal{M}(\Gamma)_k$ has a label, which may not be unique, e.g. $C_1 = C_{12}$ or $C_{12} = C_{13}$ are allowed.

For each component C other than C_1 , we project to $\mathcal{M}(\{v_1\})_k \simeq \mathbb{P}_k^2$. If the image of C is positive-dimensional then it meets every line in \mathbb{P}_k^2 , so $H_1 \cdot [C]$ is a non-zero

cycle. In this case, C must be C_{12} or C_{13} , because the other monomials in (15) are annihilated by H_1 . Thus, each component other than C_1 , C_{12} and C_{13} maps to a point in $\mathcal{M}(\{v_1\})_k \simeq \mathbb{P}_k^2$. We shall describe the images of these components more explicitly.

Since v_1 and v_2 lie in a single apartment, Theorem 4.4 implies that $\mathcal{M}(\{v_1, v_2\})_k$ is either the union two copies of \mathbb{P}_k^2 and $\mathbb{P}_k^1 \times \mathbb{P}_k^1$, or the union of \mathbb{P}_k^2 and the blow-up of \mathbb{P}_k^2 at a point. Looking at the cycle classes, we see that C_{12} maps onto $\mathbb{P}_k^1 \times \mathbb{P}_k^1$ in the first case and onto the blow-up in the second case. If C_{12} is distinct from C_1 , then a copy of \mathbb{P}_k^2 in $\mathcal{M}(\{v_1, v_2\})_k$ maps isomorphically onto $\mathcal{M}(\{v_1\})_k$. Thus, the image of C_{12} in $\mathcal{M}(\{v_1\})_k$ must be their line of intersection in $\mathcal{M}(\{v_1, v_2\})_k$. Also by Lemma 2.4, the image of C_2 must be one of the components in $\mathcal{M}(\{v_1, v_2\})_k$, so it meets the image of C_{12} . If C_2 is different from C_{12} , then we already say that its image in $\mathcal{M}(\{v_1\})_k$ is a point, and since its image intersects C_{12} , it must be a point in the image of C_{12} . Suppose that C_{23} has cycle class $H_1^2 H_2 H_3$, and so has no other label. By Theorem 2.3, $\mathcal{M}(\Gamma)_k$ is Cohen-Macaulay and hence connected in codimension 1. The curves on C_{23} all have cycle classes which are linear combinations of $H_1^2 H_2^2 H_3$ and $H_1^2 H_2 H_3^2$. Thus, C_{23} intersects either C_2 or C_3 in codimension 1. If the image of either C_2 or C_3 is a point, then the image of C_{23} must be the same point.

In conclusion, the set of images in $\mathcal{M}(\{v_1\})_k \simeq \mathbb{P}_k^2$ of the components other than C_1 consists of at most two lines and at most one point in each of the lines. We can always choose coordinates such that each of these is a coordinate linear space. Repeating this for each of the projections gives the desired system of coordinates on $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$.

For the second assertion of Lemma 6.6, we choose coordinates as above and let C be a component whose cycle class is a monomial in the Chow ring. Then C must be the product of linear subspaces, so C is equal to the product of its images on the projections, which must be coordinate linear spaces by the above argument. \square

Corollary 6.7. *A Mustafin triangle Γ is of monomial type if and only if the special fiber $\mathcal{M}(\Gamma)_k$ has six irreducible components.*

Proof. Monomial type implies six irreducible components by Remark 2.11. Conversely, if $\mathcal{M}(\Gamma)_k$ has six irreducible components, then each component has a unique monomial from (15) as its cycle class, so Lemma 6.6 implies that $\mathcal{M}(\Gamma)_k$ is of monomial type. \square

At this point in our derivation, the following facts about Table 1 have been proved. All entries below the main diagonal are zero: by Proposition 6.2, the number of secondary components cannot exceed the number of bent lines. Corollary 6.5 confirms the first column. Lemma 6.4 confirms the second column. Corollary 6.7 shows that $5 + 8$ is an upper bound for the lower right entry. We now prove the matching lower bound.

Proposition 6.8. *Each of the 13 isomorphism classes of monomial ideals on the main component of the Hilbert scheme $H_{3,3}$ arises as the special fiber of a Mustafin variety.*

Proof. For any integer vector (a, b, c, d, e, f, g, h) consider the three matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad G = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \pi^a & 0 \\ 0 & 0 & \pi^b \end{pmatrix} \quad \text{and} \quad H = \begin{pmatrix} \pi^c & \pi^d & (1 + \pi)\pi^e \\ 0 & \pi^f & \pi^g \\ 0 & 0 & \pi^h \end{pmatrix}$$

in $GL_3(K)$. Consider the configuration $\Gamma = \{[L_1], [GL_1], [HL_1]\}$ in the building \mathfrak{B}_3 where $L_1 \simeq R^3$ is a reference lattice. The generic fiber of the Mustafin variety $\mathcal{M}(\Gamma)$ is defined by the 2×2 -minors of the matrix

$$\begin{pmatrix} x_1 & x_2 & x_3\pi^c + y_3\pi^d + z_3(1 + \pi)\pi^e \\ y_1 & y_2\pi^a & y_3\pi^f + z_3\pi^g \\ z_1 & z_2\pi^b & z_3\pi^h \end{pmatrix} \quad (16)$$

The following list proves that each of the 13 isomorphism classes of monomial ideals in $H_{3,3}$ can be realized as a special fiber for some Γ . We use the labeling in [CS, Table 1]:

type	(a, b, c, d, e, f, g, h)	monomial ideal defining the special fiber
1	$(-1, 1, 0, 1, 0, 1, 0, -1)$	$\langle y_2z_3, x_2z_3, y_1z_3, x_1z_3, x_3y_2, x_3y_1, y_2z_1, x_1y_2, x_2z_1 \rangle$
2	$(-1, 3, -1, 0, 1, 0, 1, 1)$	$\langle y_2z_3, x_2z_3, y_3z_1, x_3y_2, x_3z_1, x_3y_1, y_2z_1, x_1y_2, x_2z_1 \rangle$
3	$(-1, 2, -1, 0, 0, 1, 1, 0)$	$\langle y_2z_3, x_2z_3, y_1z_3, x_3y_2, x_3z_1, x_3y_1, y_2z_1, x_1y_2, x_2z_1 \rangle$
4	$(-1, 1, 1, 1, 2, -1, 0, 0)$	$\langle y_2z_3, x_2z_3, x_1z_3, x_2y_3, y_3z_1, x_1y_3, y_2z_1, x_1y_2, x_2z_1 \rangle$
5	$(1, -2, 1, 0, 2, 2, 4, 1)$	$\langle y_1z_3, y_3z_2, y_2y_3, y_3z_1, y_1y_3, x_3z_2, z_2y_1, z_2x_1, x_2y_1, x_1y_2z_3 \rangle$
8	$(1, -2, 2, 0, 2, -1, 0, 0)$	$\langle z_2z_3, x_1z_3, y_3z_2, x_2y_3, y_3z_1, x_1y_3, z_2y_1, z_2x_1, x_2y_1 \rangle$
9	$(1, 3, 1, 0, 0, 2, -1, 0)$	$\langle y_2z_3, x_2z_3, z_3z_1, x_1z_3, y_2y_3, y_3z_1, y_2z_1, x_2z_1, x_2y_1 \rangle$
10	$(2, 1, 0, -1, -2, 1, -1, 0)$	$\langle z_2z_3, x_2z_3, z_3z_1, y_1z_3, y_3z_2, y_3z_1, z_2y_1, x_2z_1, x_2y_1 \rangle$
11	$(-4, 1, 3, 4, 1, 1, 0, 3)$	$\langle z_2z_3, y_2z_3, z_3z_1, x_1z_3, y_3z_1, x_3y_2, y_2z_1, x_1y_2, x_2z_1, y_3z_2x_1 \rangle$
13	$(-3, -6, 6, 7, 3, 4, 3, 0)$	$\langle z_2z_3, y_2z_3, y_1z_3, x_1z_3, y_3z_2, x_1y_3, z_2y_1, z_2x_1, x_1y_2, x_3y_2y_1 \rangle$
14	$(3, 1, 0, 1, -1, 3, 1, 0)$	$\langle z_2z_3, x_2z_3, z_3z_1, y_1z_3, x_3z_2, x_3y_1, z_2y_1, x_2z_1, x_2y_1 \rangle$
15	$(-1, 4, 1, -1, -2, 1, -1, 1)$	$\langle y_2z_3, x_2z_3, z_3z_1, y_1z_3, y_2y_3, y_3z_1, y_2z_1, x_1y_2, x_2z_1, y_1x_2y_3 \rangle$
16	$(2, -2, 1, 0, 0, 1, 0, -2)$	$\langle z_2z_3, x_2z_3, y_1z_3, x_1z_3, y_3z_2, y_1y_3, z_2y_1, z_2x_1, x_2y_1, y_3x_2x_1 \rangle$

We argue below that these realizations are independent of the choice of (R, K, k) . \square

Remark 6.9. Our proof of Proposition 6.8 relies on the *computation* of the special fiber of a Mustafin variety *over an arbitrary discrete valuation ring*. For this, we work over the 2-dimensional base ring $T = S^{-1}\mathbb{Z}[t]$, where S is the multiplicative set $\{1 + r \mid r \in t \cdot \mathbb{Z}[t]\}$. We take the ideal of 2×2 -minors of (16), with $t = \pi$ and saturate with respect to t to obtain an ideal I in $T' = T[x_i, y_i, z_i : 1 \leq i \leq 3]$. The essential check is that the resulting quotient ring T'/I is flat over S . For any discrete valuation ring R with uniformizer π , there is a unique homomorphism $f: S \rightarrow R$ which sends t to π . Since the subscheme of $(\mathbb{P}_R^2)^3$ defined by $f(I)$ is flat over R , it coincides with

the desired Mustafin variety $\mathcal{M}(\Gamma)$. The special fiber is defined by the image of I in $k \otimes_{\mathbb{Z}} (T'/\langle t \rangle) = k[x_i, y_i, z_i : 1 \leq i \leq 3]$.

There are no well-developed tools for computing in T directly, so we use `Singular` with a local term order in the ring $\mathbb{Q} \otimes T = \mathbb{Q}[t]_{(t)}$. We compute $I_{\mathbb{Q}}$ as the saturation of the ideal of the 2×2 minors of (16) with respect to t . As long as our standard basis for $I_{\mathbb{Q}}$ has integer coefficients and each leading term has coefficient 1, we can define I in T' to be the ideal generated by the same polynomials. Each reduction of these generators to $\mathbb{Z}/p \otimes T'$ is also standard basis. Thus, for each prime p , $\mathbb{Z}/p \otimes (T'/I)$ is flat over $\mathbb{Z}/p \otimes T = (\mathbb{Z}/p)[t]_{(t)}$ with the same Hilbert function, so T'/I is flat over T .

Our next lemma bounds the entry **5+6** in Table 1.

Lemma 6.10. *If a Mustafin triangle Γ has two distinct bend points, but is not contained in one apartment, then the $\mathcal{M}(\Gamma)_k$ is one of the following six ideals:*

$$\begin{aligned}
(-2, 2, 0, 1, -1, -2, -1, 0) & \langle y_3 z_1, x_2 z_3, y_2 z_3, x_1 y_3, x_1 y_2, y_2 z_1, z_3 z_1, x_2 z_1, x_3 y_2 - x_2 y_3 \rangle \\
(-3, -1, 4, 5, 4, 1, 0, 1) & \langle x_2 z_3, z_1 z_3, y_2 z_3, x_1 y_3, z_1 y_2, x_1 y_2, x_1 z_3, y_2 x_3 - x_2 y_3, x_1 z_2 \rangle \\
(-1, -3, 1, 4, 2, 2, 0, -1) & \langle y_1 z_3, x_1 z_3, x_1 y_2, x_3 y_1, y_1 z_2, z_2 x_1, z_2 x_3, z_2 z_3, x_2 z_3 - x_3 y_2 \rangle \\
(-3, -4, 3, 2, 3, 0, -1, 0) & \langle x_1 z_3, x_1 y_2, x_1 y_3, y_3 z_2, y_1 z_2, z_2 x_1, z_3 z_2, y_2 y_3 - x_2 z_3, z_3 z_1 \rangle \\
(-3, -1, 2, 1, 2, -1, 0, 0) & \langle z_1 y_2, z_2 x_1, y_2 z_3, x_1 z_3, x_1 y_2, y_2 y_3, y_3 z_1, x_1 y_3, y_3 z_2 - x_2 z_3 \rangle \\
(-2, -4, 3, 2, 3, 1, 0, -1) & \langle y_1 z_3, x_1 z_3, x_1 y_2, x_1 y_3, y_1 z_2, y_3 z_2, z_2 x_1, z_3 z_2, y_2 y_3 - x_2 z_3 \rangle
\end{aligned}$$

Proof. By Proposition 6.2, $\mathcal{M}(\Gamma)_k$ has five components. Lemma 6.6 implies that four of these components are defined by monomial ideals, and the fifth, C , is a surface in $\mathbb{P}_k^1 \times \mathbb{P}_k^2$. We fix coordinates $(x_1:y_1)$ on \mathbb{P}_k^1 and coordinates $(x_2:y_2:z_2)$ on \mathbb{P}_k^2 . The equation for C has the form $[x_1 \ y_1] M [x_2 \ y_2 \ z_2]^t$, where M is a 2×3 matrix. If all the non-zero entries of M lie in a single row or a single column, then C would be reducible. Without loss of generality, we assume that the coefficients of both $x_1 x_2$ and $y_1 y_2$ are non-zero. In particular, the point defined by $x_1 = x_2 = z_2 = 0$ is not in C .

Fix any term order with $x_1, x_2 > y_1, y_2, z_2$. The initial ideal of C equals $\langle x_1 x_2 \rangle$. Let B denote the union of the other components. The monomial ideal defining B is its own initial ideal. By [CS, Thm. 2.1], the initial ideal of the special fiber is radical. Its Hilbert function is sum of the Hilbert functions of $\text{in}(C)$ and of $\text{in}(B) = B$ minus that of $\text{in}(C) \cap \text{in}(B)$. However, the Hilbert function is constant under taking an initial ideal and $C \cap B$ is already a monomial ideal, so $\text{in}(C) \cap \text{in}(B)$ equals $C \cap B$. In particular, B does not contain the point $x_1 = x_2 = z_2 = 0$. Thus, $\mathcal{M}(\Gamma)_k$ has an initial monomial ideal in which C degenerates to the union of \mathbb{P}_k^2 and $\mathbb{P}_k^1 \times \mathbb{P}_k^1$, whose common line $(x_1 = x_2 = 0)$ contains a coordinate point $(x_1 = x_2 = z_2 = 0)$ not in any other component.

When examining the pictures of the 13 monomial schemes on the main component of $H_{3,3}$ [CS, Figure 2], we find that there are, up to symmetry, 22 coordinate points on the line between a \mathbb{P}^2 and a $\mathbb{P}^1 \times \mathbb{P}^1$, but not on any other component. Each of these points yields a scheme with 5 components by replacing the \mathbb{P}^2 and $\mathbb{P}^1 \times \mathbb{P}^1$ with the blow-up of

\mathbb{P}^2 at a point. However, there are two possible ways of obtaining each scheme, so there are 11 schemes, 5 of which are planar. The remaining **6** special fibers, not achievable by a configuration in one apartment, are those listed in the statement.

The computational methods outlined in Remark 6.9 show that the ideals can be realized as the special fiber $\mathcal{M}(\Gamma)_k$ by a configuration as in Proposition 6.8, where the vector of integers (a, b, c, d, e, f, g, h) is listed to the left of each ideal. \square

Proof of Theorem 6.1. Bearing in mind the planar classification in Figure 6, we enumerate all possibilities based on the number of bent lines. If that number is zero or one then Lemma 6.4 proves the claim.

Next suppose that Γ has two bent lines and one unbent line, say, between $[L_2]$ and $[L_3]$. If the two bent lines have the same bend point v , then none of the bend points can lie in Γ , so $\mathcal{M}(\Gamma)_k$ has four components. Then $\{v, [L_2], [L_3]\}$ is a configuration with no bent lines, and thus, by Corollary 6.5, it must lie in a single apartment. The first lattice point after v on the line to either $[L_2]$ or $[L_3]$ corresponds to a line in $L_v/\pi L_v$ under the bijection of Lemma 2.1, so the angle between the two edges must be either 0 or 120 degrees. Since the line between $[L_2]$ and $[L_3]$ is unbent, the angle must be 0, i.e. the three vertices lie on a straight line. Without loss of generality, we assume that $[L_3]$ lies between v and $[L_2]$. Therefore, if we choose an apartment containing $[L_1]$ and $[L_2]$, then it will contain $[L_3]$ as well. If Γ has two bent lines whose bend points do not coincide, then $\mathcal{M}(\Gamma)_k$ has five components by Proposition 6.2. By Lemma 6.10, there are six non-planar types in this case.

What remains to be proved is the last column in Table 1. The top entry $1 + \mathbf{0}$ is correct because here each of the three lines from Γ is bent at the third point, and the only possibility for this is $\Gamma = \blacktriangleleft$ with $\mathcal{M}(\Gamma)_k = \blacktriangle$. The fourth entry $5 + \mathbf{8}$ in the last column of Table 1 is correct by Corollary 6.7 and Proposition 6.8.

Suppose that Γ has three bent lines but there is only one bend point $v = [L_0]$. By Proposition 6.2, there is only one secondary component, indexed by v . Consider the first step on the line from v to any of the vertices in Γ . The first steps give us one-dimensional subspaces of $L_0/\pi L_0 \simeq k^3$, hence points in \mathbb{P}_k^2 . These points must be distinct or else the convex hull of two lattice points $[L_i]$ and $[L_j]$ would not pass through v . Each L_i can be chosen to be of the form $M_i + \pi^{t_i} L_0$, where each M_i is a rank 1 direct summand of L_0 . Up to automorphisms of \mathbb{P}_k^2 , there are two possibilities for three distinct lines in \mathbb{P}_k^2 : either they are collinear or not. In the latter case, we fix a generator for each of M_1 , M_2 and M_3 . Since the images of these three elements in $L_0/\pi L_0$ are linearly independent, they form a basis for L_0 and the configuration lies in the corresponding apartment. Hence $\Gamma = \blacktriangleright$ and $\mathcal{M}(\Gamma)_k = \blacktriangle$. In the former case, we choose generators e'_1 and e'_2 of M_1 and M_2 , respectively, and let e_3 be an element completing these to a basis. Then M_1 is generated by $ue'_1 + ve'_2 + re_3$, where r is in the maximal ideal and u and v are units by the assumption that $\overline{M_3}$ is distinct from the other two reductions. By substituting e_1 and e_2 for ue'_1 and ve'_2 respectively, M_1 and M_2

remain generated by e_1 and e_2 respectively, and M_1 is generated by $e_1 + e_2 + re_3$. It can be checked that any configuration of this form leads to the “sailboat” of Example 5.13.

We now assume that Γ has three bent lines and exactly two of the bend points coincide. By Proposition 6.2, Γ has five components. We must show that (13) and (14) are the only possibilities. Suppose v is the common bend point of the lines determined by $\{[L_1], [L_2]\}$ and $\{[L_1], [L_3]\}$. Consider the triple $\{v, [L_2], [L_3]\}$. If it lies in a single apartment, then either it lies on an unbent line or v is the bend point between $[L_2]$ and $[L_3]$, both of which contradict our assumptions. Thus, $\{v, [L_2], [L_3]\}$ is one of the non-planar configurations in Lemma 6.4. Since the first step from v to either $[L_2]$ or $[L_3]$ defines a point (and not a line) in \mathbb{P}_k^2 , the only possibility is case (i), and we have $v = [R\{e_1, e_2, e_3\}]$, $L_2 = R\{e_1, \pi^s e_2, \pi^s e_3\}$, $L_3 = R\{e_1 + \pi^t e_2, \pi^u e_2, \pi^u e_3\}$ with $0 < t < s, u$.

The lattice L_1 must have the form $M_1 + \pi^r R\{e_1, e_2, e_3\}$ where M_1 is a rank 1 direct summand of $R\{e_1, e_2, e_3\}$. Since v is the bend point of $\{L_1, L_2\}$, the first steps from v to L_1 and L_2 cannot coincide. Hence $\overline{M}_1 \neq k\{e_1\}$, so M_1 is generated by $ae_1 + be_2 + ce_3$, where b and c do not both have positive valuation. If c is a unit, then we can take the change of basis $e'_3 = ae_1 + be_2 + ce_3$ and then our lattices become

$$L_1 = R\{\pi^r e_1, \pi^r e_2, e'_3\}, \quad L_2 = R\{e_1, \pi^s e_2, \pi^s e'_3\}, \quad L_3 = R\{e_1 + \pi^t e_2, \pi^u e_2, \pi^u e'_3\}.$$

The corresponding special fiber is (13) from Example 6.3. On the other hand, if c is not a unit, then b must be a unit, and we take $e'_1 = (1 - \pi^t a/b)e_1$ and $e'_2 = (a/b)e_1 + e_2$ to get the lattices

$$L_1 = R\{\pi^r e'_1, \pi^r e_3, e'_2 + (c/b)e_3\}, \quad L_2 = R\{e'_1, \pi^s e'_2, \pi^s e_3\}, \quad L_3 = R\{e'_1 + \pi^t e'_2, \pi^u e'_2, \pi^u e_3\},$$

where c/b has positive valuation. Now, the special fiber is (14) from Example 6.3. \square

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