Characteristic varieties

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The character group

- Throughout, X will be a connected CW-complex, with finite k-skeleton, for some $k \geq 1$. We may assume X has a single 0-cell, call it x_0.
- Let $G = \pi_1(X, x_0)$ be the fundamental group of X.
- The *character group*,

  $$\widehat{G} = \mathrm{Hom}(G, \mathbb{C}^{\times}),$$

  is an algebraic group, with multiplication $\rho \cdot \rho'(g) = \rho(g)\rho'(g)$, and identity $G \to \mathbb{C}^{\times}$, $g \mapsto 1$.
- Let $G_{\mathrm{ab}} = G/G' \cong H_1(X, \mathbb{Z})$ be the abelianization of G. The projection $\mathrm{ab}\colon G \to G_{\mathrm{ab}}$ induces an isomorphism $\widehat{G}_{\mathrm{ab}} \xrightarrow{\simeq} \widehat{G}$.

- The identity component, $\widehat{G}^0$, is isomorphic to a complex algebraic torus of dimension $n = \operatorname{rank} G_{\text{ab}}$.

- The other connected components are all isomorphic to $\widehat{G}^0 = (\mathbb{C}^\times)^n$, and are indexed by the finite abelian group $\operatorname{Tors}(G_{\text{ab}})$.

- $\widehat{G}$ parametrizes rank 1 local systems on X:

$$\rho\colon G \to \mathbb{C}^\times \quad \rightsquigarrow \quad \mathcal{L}_\rho$$

  the complex vector space $\mathbb{C}$, viewed as a right module over the group ring $\mathbb{Z}G$ via $a \cdot g = \rho(g)a$, for $g \in G$ and $a \in \mathbb{C}$.

The equivariant chain complex

- Let $p\colon \widetilde{X} \to X$ be the universal cover. The cell structure on X lifts to a cell structure on $\widetilde{X}$.
- Fixing a lift $\tilde{x}_0 \in p^{-1}(x_0)$ identifies $G = \pi_1(X, x_0)$ with the group of deck transformations of $\widetilde{X}$.
- Thus, we may view the cellular chain complex of $\widetilde{X}$,

$$\cdots \longrightarrow C_{i+1}(\widetilde{X}, \mathbb{Z}) \xrightarrow{\tilde{\partial}_{i+1}} C_i(\widetilde{X}, \mathbb{Z}) \xrightarrow{\tilde{\partial}_i} C_{i-1}(\widetilde{X}, \mathbb{Z}) \longrightarrow \cdots,$$

as a chain complex of left $\mathbb{Z}G$-modules.
- The homology groups of X with coefficients in $\mathcal{L}_\rho$ are defined as

$$H_*(X, \mathcal{L}_\rho) = H_*(\mathcal{L}_\rho \otimes_{\mathbb{Z}G} C_\bullet(\widetilde{X}, \mathbb{Z})).$$

- In concrete terms, $H_*(X, \mathcal{L}_\rho)$ may be computed from the chain complex of $\mathbb{C}$-vector spaces,

$$\cdots \longrightarrow C_{i+1}(X,\mathbb{C}) \xrightarrow{\tilde{\partial}_{i+1}(\rho)} C_i(X,\mathbb{C}) \xrightarrow{\tilde{\partial}_i(\rho)} C_{i-1}(X,\mathbb{C}) \longrightarrow \cdots,$$

  where the evaluation of $\tilde{\partial}_i$ at ρ is obtained by applying the ring homomorphism $\mathbb{Z}G \to \mathbb{C}$, $g \mapsto \rho(g)$ to each entry of $\tilde{\partial}_i$.
- Alternatively, consider the universal abelian cover, X^{ab}, and its equivariant chain complex, $C_\bullet(X^{\mathrm{ab}},\mathbb{Z}) = \mathbb{Z}G_{\mathrm{ab}} \otimes_{\mathbb{Z}G} C_\bullet(\widetilde{X},\mathbb{Z})$, with differentials $\partial_i^{\mathrm{ab}} = \mathrm{id} \otimes \widetilde{\partial}_i$. The homology of X with coefficients in the rank 1 local system given by $\rho \in \widehat{G}_{\mathrm{ab}} = \widehat{G}$ is computed from similar chain complex, with differentials $\partial_i^{\mathrm{ab}}(\rho) = \tilde{\partial}_i(\rho)$.
- The identity $1 \in \widehat{G}$ yields the trivial local system, $\mathcal{L}_1 = \mathbb{C}$, and $H_*(X,\mathbb{C})$ is the usual homology of X with $\mathbb{C}$-coefficients. Denote by $b_i(X) = \dim_{\mathbb{C}} H_i(X,\mathbb{C})$ the ith Betti number of X.

Homology jump loci

Definition

The *characteristic varieties* of X are the sets

$$\mathcal{V}^i_d(X) = \{\rho \in \widehat{G} \mid \dim_{\mathbb{C}} H_i(X, \mathcal{L}_\rho) \geq d\},$$

defined for all degrees $0 \leq i \leq k$ and all depths $d > 0$.

- For each i, get stratification $\widehat{G} \supseteq \mathcal{V}^i_1 \supseteq \mathcal{V}^i_2 \supseteq \cdots$
- $1 \in \mathcal{V}^i_d(X) \iff b_i(X) \geq d$.
- $\mathcal{V}^0_1(X) = \{1\}$ and $\mathcal{V}^0_d(X) = \emptyset$, for $d > 1$.
- $\mathcal{V}^1_d(X)$ depends only on G (in fact, only on G/G''), so we may write these sets as $\mathcal{V}_d(G)$.
- Define analogously $\mathcal{V}^i_d(X, \Bbbk) \subset \mathrm{Hom}(G, \Bbbk^\times)$, for arbitrary field $\Bbbk$. Then $\mathcal{V}^i_d(X, \Bbbk) = \mathcal{V}^i_d(X, \mathbb{K}) \cap \mathrm{Hom}(G, \Bbbk^\times)$, for any extension $\Bbbk \subseteq \mathbb{K}$.

Lemma

Each $\mathcal{V}^i_d(X)$ is a Zariski closed subset of the algebraic group $\widehat{G}$.

Proof.

Let $R = \mathbb{C}[G_{\mathrm{ab}}]$ be the coordinate ring of $\widehat{G} = \widehat{G}_{\mathrm{ab}}$. By definition, a character ρ belongs to $\mathcal{V}^i_d(X)$ if and only if

$$\operatorname{rank} \partial^{\mathrm{ab}}_{i+1}(\rho) + \operatorname{rank} \partial^{\mathrm{ab}}_{i}(\rho) \le c_i - d,$$

where $c_i = c_i(X)$ is the number of i-cells of X. Hence,

$$\mathcal{V}^i_d(X) = \bigcap_{r+s=c_i-d+1;\, r,s\ge 0} \{\rho \in \widehat{G} \mid \operatorname{rank} \partial^{\mathrm{ab}}_{i+1}(\rho) \le r-1 \text{ or } \operatorname{rank} \partial^{\mathrm{ab}}_{i}(\rho) \le s-1\}$$

$$= V\Bigg(\sum_{p+q=c_{i-1}+d-1;\, p,q\ge 0} E_p(\partial^{\mathrm{ab}}_i) \cdot E_q(\partial^{\mathrm{ab}}_{i+1}) \Bigg),$$

where $E_q(\varphi)$ = ideal of minors of size $a-q$ of $\varphi\colon R^b \to R^a$. □

The characteristic varieties are homotopy-type invariants of a space:

Lemma

Suppose $X \simeq X'$. For each $i \leq k$, there is an isomorphism $\widehat{G'} \cong \widehat{G}$, which restricts to isomorphisms $\mathcal{V}^i_d(X') \cong \mathcal{V}^i_d(X)$, for all $d > 0$.

Proof.

Let $f \colon X \to X'$ be a (cellular) homotopy equivalence.

The induced homomorphism $f_\sharp \colon \pi_1(X, x_0) \to \pi_1(X', x_0')$, yields an isomorphism of algebraic groups, $\hat{f}_\sharp \colon \widehat{G'} \to \widehat{G}$.

Lifting f to a cellular homotopy equivalence, $\tilde{f} \colon \widetilde{X} \to \widetilde{X}'$, defines isomorphisms $H_i(X, \mathcal{L}_{\rho \circ f_\sharp}) \to H_i(X', \mathcal{L}_\rho)$, for each $\rho \in \widehat{G'}$.

Hence, $\hat{f}_\sharp$ restricts to isomorphisms $\mathcal{V}^i_d(X') \to \mathcal{V}^i_d(X)$. □

Example (The circle)

We have $\widetilde{S^1} = \mathbb{R}$.
Identify $\pi_1(S^1, *) = \mathbb{Z} = \langle t \rangle$ and $\mathbb{Z}\mathbb{Z} = \mathbb{Z}[t^{\pm 1}]$. Then:

$$C_\bullet(\widetilde{S^1}) : \; 0 \longrightarrow \mathbb{Z}[t^{\pm 1}] \xrightarrow{t-1} \mathbb{Z}[t^{\pm 1}] \longrightarrow 0$$

For $\rho \in \mathrm{Hom}(\mathbb{Z}, \mathbb{C}^\times) = \mathbb{C}^\times$, we get

$$C_\bullet(\widetilde{S^1}) \otimes_{\mathbb{Z}\mathbb{Z}} \mathcal{L}_\rho : \; 0 \longrightarrow \mathbb{C} \xrightarrow{\rho-1} \mathbb{C} \longrightarrow 0$$

which is exact, except for $\rho = 1$, when $H_0(S^1, \mathbb{C}) = H_1(S^1, \mathbb{C}) = \mathbb{C}$.
Hence:

$$\mathcal{V}_1^0(S^1) = \mathcal{V}_1^1(S^1) = \{1\}$$

$$\mathcal{V}_d^i(S^1) = \emptyset, \quad \text{otherwise.}$$

Example (The n-torus)

Identify $\pi_1(T^n) = \mathbb{Z}^n$, and $\operatorname{Hom}(\mathbb{Z}^n, \mathbb{C}^\times) = (\mathbb{C}^\times)^n$. Using the Koszul resolution $C_\bullet(\widetilde{T^n})$ as above, we get

$$\mathcal{V}^i_d(T^n) = \begin{cases} \{1\} & \text{if } d \leq \binom{n}{i}, \\ \emptyset & \text{otherwise.} \end{cases}$$

Example (Nilmanifolds)

More generally, let M be a nilmanifold. An inductive argument on the nilpotency class of $\pi_1(M)$, based on the Hochschild-Serre spectral sequence, yields (MP 2009)

$$\mathcal{V}^i_d(M) = \begin{cases} \{1\} & \text{if } d \leq b_i(M), \\ \emptyset & \text{otherwise} \end{cases}$$

Example (Wedge of circles)

Identify $\pi_1(\bigvee^n S^1) = F_n$, and $\mathrm{Hom}(F_n, \mathbb{C}^\times) = (\mathbb{C}^\times)^n$. Then:

$$\mathcal{V}^1_d(\bigvee^n S^1) = \begin{cases} (\mathbb{C}^\times)^n & \text{if } d < n, \\ \{1\} & \text{if } d = n, \\ \emptyset & \text{if } d > n. \end{cases}$$

Example (Orientable surface of genus $g > 1$)

Write $\pi_1(S_g) = \langle x_1, \dots, x_g, y_1, \dots, y_g \mid [x_1, y_1] \cdots [x_g, y_g] = 1 \rangle$, and identify $\mathrm{Hom}(\pi_1(S_g), \mathbb{C}^\times) = (\mathbb{C}^\times)^{2g}$. Then:

$$\mathcal{V}^i_d(S_g) = \begin{cases} (\mathbb{C}^\times)^{2g} & \text{if } i = 1,\ d < 2g-1, \\ \{1\} & \text{if } i = 1,\ d = 2g-1, 2g, \text{ or } i = 2,\ d = 1, \\ \emptyset & \text{otherwise.} \end{cases}$$

Depth one characteristic varieties

Most important for us will be the depth-1 characteristic varieties, $\mathcal{V}^i_1(X)$, and their unions up to a fixed degree,

$$\mathcal{V}^i(X) = \bigcup_{j=0}^{i} \mathcal{V}^j_1(X) = \{\rho \in \widehat{G} \mid H_j(X, \mathcal{L}_\rho) \neq 0, \text{ for some } j \leq i\}.$$

Get ascending filtration of the character group,

$$\{1\} = \mathcal{V}^0(X) \subseteq \mathcal{V}^1(X) \subseteq \cdots \subseteq \mathcal{V}^k(X) \subseteq \widehat{G}.$$

These loci are the support varieties for the Alexander invariants of X. More precisely, view $H_*(X^{\mathrm{ab}}, \mathbb{C})$ as a module over the group-ring $\mathbb{C}[G_{\mathrm{ab}}]$. Then (PS 2010),

$$\mathcal{V}^i(X) = V\Big(\operatorname{ann}\Big(\bigoplus_{j\leq i} H_j(X^{\mathrm{ab}}, \mathbb{C})\Big)\Big).$$

We will also consider the varieties $\mathcal{W}^i_1(X) = \mathcal{V}^i_1(X) \cap \widehat{G}^0$ and

$$\mathcal{W}^i(X) = \bigcup_{j=0}^{i} \mathcal{W}^j_1(X) = \mathcal{V}^i(X) \cap \widehat{G}^0.$$

Get ascending filtration of the character torus of G,

$$\{1\} = \mathcal{W}^0(X) \subseteq \mathcal{W}^1(X) \subseteq \cdots \subseteq \mathcal{W}^k(X) \subseteq \widehat{G}^0.$$

Let $X^\alpha \to X$ be the maximal torsion-free abelian cover of X, corresponding to the canonical projection $\alpha\colon G \twoheadrightarrow H$, where

$$H = G_{\mathrm{ab}}/\operatorname{Tors}(G_{\mathrm{ab}}) = \mathbb{Z}^n, \qquad n = b_1(G).$$

Identify $\widehat{G}^0 = (\mathbb{C}^\times)^n$ and $\mathbb{C}[\mathbb{Z}^n] = \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$. Then,

$$\mathcal{W}^i(X) = V\Big(\operatorname{ann}\Big(\bigoplus_{j\le i} H_j(X^\alpha, \mathbb{C})\Big)\Big).$$

Products and wedges

The depth-1 characteristic varieties behave well with respect to products and wedges. More precisely:

- Let X_1 and X_2 be connected CW-complexes with finite k-skeleta, and with fundamental groups G_1 and G_2.
- Let $X = X_1 \times X_2$; set $G = \pi_1(X)$.
- Identify $G = G_1 \times G_2$, $\widehat{G} = \widehat{G}_1 \times \widehat{G}_2$, $\widehat{G}^0 = \widehat{G}_1^0 \times \widehat{G}_2^0$.

Proposition (PS 2010)

For all $i \le k$,

$$\mathcal{V}_1^i(X_1 \times X_2) = \bigcup_{p+q=i} \mathcal{V}_1^p(X_1) \times \mathcal{V}_1^q(X_2).$$

Proof.

- Let $\widetilde{X} = \widetilde{X}_1 \times \widetilde{X}_2$ be the universal cover. We have a G-equivariant isomorphism of chain complexes, $C_\bullet(\widetilde{X}) \cong C_\bullet(\widetilde{X}_1) \otimes_{\mathbb{Z}} C_\bullet(\widetilde{X}_2)$.
- Given a character $\rho = (\rho_1, \rho_2) \in \widehat{G}_1 \times \widehat{G}_2 = \widehat{G}$, we obtain an iso $C_\bullet(X, \mathcal{L}_\rho) \cong C_\bullet(X_1, \mathcal{L}_{\rho_1}) \otimes_{\mathbb{C}} C_\bullet(X_2, \mathcal{L}_{\rho_2})$.
- Hence, $H_i(X, \mathcal{L}_\rho) = \bigoplus_{s+t=i} H_s(X_1, \mathcal{L}_{\rho_1}) \otimes_{\mathbb{C}} H_t(X_2, \mathcal{L}_{\rho_2})$, and the conclusion follows.

□

Corollary

$$\mathcal{V}^i(X_1 \times X_2) = \bigcup_{p+q=i} \mathcal{V}^p(X_1) \times \mathcal{V}^q(X_2),$$

$$\mathcal{W}^i(X_1 \times X_2) = \bigcup_{p+q=i} \mathcal{W}^p(X_1) \times \mathcal{W}^q(X_2).$$

- Let $X = X_1 \vee X_2$ (taken at the unique 0-cells); set $G = \pi_1(X)$.
- Identify $G = G_1 * G_2$, $\widehat{G} = \widehat{G}_1 \times \widehat{G}_2$, $\widehat{G}^0 = \widehat{G}_1^0 \times \widehat{G}_2^0$.
- (PS 2010) Suppose X_1 and X_2 have positive first Betti numbers. Then, for all $1 \leq i \leq k$,

$$\mathcal{V}_1^i(X_1 \vee X_2) = \begin{cases} \widehat{G}_1 \times \widehat{G}_2 & \text{if } i = 1, \\ \mathcal{V}_1^i(X_1) \times \widehat{G}_2 \cup \widehat{G}_1 \times \mathcal{V}_1^i(X_2) & \text{if } i > 1. \end{cases}$$

- Hence, $\mathcal{V}^i(X_1 \vee X_2) = \widehat{G}$ and $\mathcal{W}^i(X_1 \vee X_2) = \widehat{G}^0$.
- The condition $b_1(X_s) > 0$ may be dropped if $i > 1$, but not if $i = 1$. E.g., take $X_1 = S^1$ and $X_2 = S^2$. Then $G_1 = \mathbb{Z}$, $G_2 = \{1\}$. Thus, $\widehat{G} = \mathbb{C}^\times$, yet $\mathcal{V}_1^1(S^1 \vee S^2) = \{1\}$.

The Alexander polynomial

- Recall the maximal torsion-free abelian cover, $q\colon X^{\alpha} \to X$, corresponding to $\alpha\colon G = \pi_1(X, x_0) \twoheadrightarrow H \cong \mathbb{Z}^n$.
- Define two modules over the Noetherian ring $\mathbb{Z}H \cong \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$:
  - The *Alexander module* $A_G = H_1(X^{\alpha}, q^{-1}(x_0); \mathbb{Z})$.
  - The *Alexander invariant* $B_G = H_1(X^{\alpha}, \mathbb{Z})$.
- These modules depend only on the group G:
  - $A_G = \mathbb{Z}H \otimes_{\mathbb{Z}G} I_G$, where $\epsilon\colon \mathbb{Z}G \to \mathbb{Z}$, $g \mapsto 1$ is the augmentation map, and $I_G = \ker \epsilon$.
  - $B_G = \ker(A_G \twoheadrightarrow I_H)$.
- Define the *Alexander polynomial* of G:

$$\Delta_G := \gcd(E_1(A_G)) \in \mathbb{Z}H.$$

- If $G = \langle x_1, \dots, x_q \mid r_1, \dots, r_m \rangle$ is finitely presented, Δ_G is the gcd of all minors of size $q - 1$ of the *Alexander matrix*,

$$\Phi_G = \left(\partial r_i / \partial x_j\right)^{\alpha} \colon \mathbb{Z}H^m \to \mathbb{Z}H^q.$$

- Recall $\mathcal{W}^1(G) = \mathcal{V}^1(G) \cap \widehat{G}^0$ is a subvariety of $\widehat{G}^0 = \widehat{H} = (\mathbb{C}^\times)^n$.
- Let $\check{\mathcal{W}}^1(G)$ be the union of all codim. 1 components of $\mathcal{W}^1(G)$.
- Let $V(\Delta_G)$ be the hypersurface in $\widehat{H} = (\mathbb{C}^\times)^n$ defined by Δ_G.

Theorem (DPS 2008)

1. $\Delta_G = 0 \iff \mathcal{W}^1(G) = \widehat{H}$. *In this case,* $\check{\mathcal{W}}^1(G) = \emptyset$.
2. *If* $b_1(G) \geq 1$ *and* $\Delta_G \neq 0$, *then*

$$\check{\mathcal{W}}^1(G) = \begin{cases} V(\Delta_G) & \text{if } b_1(G) > 1 \\ V(\Delta_G) \coprod \{1\} & \text{if } b_1(G) = 1. \end{cases}$$

3. *If* $b_1(G) \geq 2$, *then* $\check{\mathcal{W}}^1(G) = \emptyset \iff \Delta_G \doteq \text{const}$.

Knots, links, and 3-manifolds

- Let K be a non-trivial knot in S^3, with complement $X = S^3 \setminus K$, and $G = \pi_1(X, x_0)$.
- We have: $H = H_1(X, \mathbb{Z}) = \mathbb{Z}$, and $\Delta_G = \Delta_K \in \mathbb{Z}H = \mathbb{Z}[t^{\pm 1}]$ is the Alexander polynomial of the knot (J. Alexander 1928).
- Moreover, $\Delta_K(1) = \pm 1$. Thus, $\check{\mathcal{W}}^1 = \mathcal{W}^1 = \mathcal{V}^1 \subset \mathbb{C}^\times$.
- Hence:

$$\mathcal{V}^1(X) = \{z \in \mathbb{C}^\times \mid \Delta_K(z) = 0\} \cup \{1\}.$$

- More generally, let $L = (L_1, \dots, L_n)$ be a link in S^3, with complement $X = S^3 \setminus \bigcup_{i=1}^n L_i$. Then $H = \mathbb{Z}^n$ and

$$\mathcal{V}^1(X) = \{z \in (\mathbb{C}^\times)^n \mid \Delta_L(z) = 0\} \cup \{1\},$$

where $\Delta_L = \Delta_L(t_1, \dots, t_n)$ is the multi-variable Alex polynomial.

- Even more generally, let M be a compact, connected 3-manifold, with $G = \pi_1(M)$.

- Suppose either
  1. $\partial M \neq \emptyset$ and $\chi(\partial M) = 0$, or
  2. $\partial M = \emptyset$ and M is orientable.

- Theorem (DPS 2008), combined with results of (Eisenbud–Neumann 1985) and (McMullen 2002), yields:

$$\mathcal{V}^1(M) \setminus \{1\} = V(\Delta_G) \setminus \{1\}.$$

Toric complexes and right-angled Artin groups

- Given L simplicial complex on n vertices, define the *toric complex* $T_L = \mathcal{Z}_L(S^1, *)$ as the subcomplex of T^n obtained by deleting the cells corresponding to the missing simplices of L:

$$T_L = \bigcup_{\sigma \in L} T^\sigma, \quad \text{where } T^\sigma = \{x \in T^n \mid x_i = * \text{ if } i \notin \sigma\}.$$

- Let $\Gamma = (\mathsf{V}, \mathsf{E})$ be the graph with vertex set the 0-cells of L, and edge set the 1-cells of L. Then $\pi_1(T_L)$ is the *right-angled Artin group* associated to Γ:

$$G_\Gamma = \langle v \in V \mid vw = wv \text{ if } \{v, w\} \in E \rangle.$$

- Properties:
  - $\Gamma = \overline{K}_n \Rightarrow G_\Gamma = F_n$
  - $\Gamma = K_n \Rightarrow G_\Gamma = \mathbb{Z}^n$
  - $\Gamma = \Gamma' \coprod \Gamma'' \Rightarrow G_\Gamma = G_{\Gamma'} * G_{\Gamma''}$
  - $\Gamma = \Gamma' * \Gamma'' \Rightarrow G_\Gamma = G_{\Gamma'} \times G_{\Gamma''}$

- Identify character group $\widehat{G}_\Gamma = \mathrm{Hom}(G_\Gamma, \mathbb{C}^\times)$ with the algebraic torus $(\mathbb{C}^\times)^{\mathsf{V}} := (\mathbb{C}^\times)^n$.
- For each subset $\mathsf{W} \subseteq \mathsf{V}$, let $(\mathbb{C}^\times)^{\mathsf{W}} \subseteq (\mathbb{C}^\times)^{\mathsf{V}}$ be the corresponding subtorus; in particular, $(\mathbb{C}^\times)^{\emptyset} = \{1\}$.

Theorem (PS 2009)

$$\mathcal{V}^i_d(T_L) = \bigcup_{\substack{\mathsf{W}\subseteq \mathsf{V} \\ \sum_{\sigma\in L_{\mathsf{V}\setminus\mathsf{W}}} \dim_{\mathbb{C}} \widetilde{H}_{i-1-|\sigma|}(\mathrm{lk}_{L_{\mathsf{W}}}(\sigma),\mathbb{C})\geq d}} (\mathbb{C}^\times)^{\mathsf{W}},$$

where L_{W} is the subcomplex induced by L on W, and $\mathrm{lk}_K(\sigma)$ is the link of a simplex σ in a subcomplex $K \subseteq L$.

In particular:

$$\mathcal{V}^1_1(G_\Gamma) = \bigcup_{\substack{\mathsf{W}\subseteq \mathsf{V} \\ \Gamma_{\mathsf{W}} \text{ disconnected}}} (\mathbb{C}^\times)^{\mathsf{W}}.$$

Problem

Compute the Alexander polynomial of a right-angled Artin group.

For example, $\Delta_{F_n} = 0$, for $n \geq 1$, while $\Delta_{\mathbb{Z}^n} \doteq 1$, for $n > 1$.

Recall that the *connectivity* of a graph $\Gamma = (\mathsf{V}, \mathsf{E})$, denoted $\kappa(\Gamma)$, is the maximum integer r so that, for any subset $\mathsf{W} \subset \mathsf{V}$ with $|\mathsf{W}| < r$, the induced subgraph on $\mathsf{V} \setminus \mathsf{W}$ is connected.

Proposition (S 2009)

$$\Delta_{G_\Gamma} \not\doteq \text{const} \iff \kappa(\Gamma) = 1.$$

Proof.

- We know: $\mathcal{V}^1(G_\Gamma)$ consists of coordinate subspaces $(\mathbb{C}^\times)^{\mathsf{W}}$, indexed by maximal subsets $\mathsf{W} \subset \mathsf{V}$ such that Γ_{W} is disconnected.
- Thus, $\check{\mathcal{V}}^1(G_\Gamma)$ is non-empty if and only if Γ is connected and has a cut point, i.e., $\kappa(\Gamma) = 1$.
- If Γ has just 1 vertex, then $\kappa(\Gamma) = 0$; on the other hand, $G_\Gamma = \mathbb{Z}$, and so $\Delta_{G_\Gamma} = 0$.
- For all other graphs, $b_1(G_\Gamma) \geq 2$, and Theorem (DPS 2008) yields the desired conclusion.

□