Dwyer–Fried invariants

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Minicourse on *Cohomology jumping loci and homological finiteness properties*

Centro Ennio De Giorgi
Pisa, Italy
May 11, 2010

Free abelian covers

- Let X be a connected CW-complex with finite 1-skeleton, $G = \pi_1(X, x_0)$.
- Consider the connected, regular covering spaces of X, with group of deck transformations a free abelian group of fixed rank r.
- Model situation: the r-dimensional torus $T^r = K(\mathbb{Z}^r, 1)$ and its universal cover, $\mathbb{R}^r \to T^r$, with group of deck transformations $\mathbb{Z}^r$.
- Any epimorphism $\nu\colon G \twoheadrightarrow \mathbb{Z}^r$ gives rise to a $\mathbb{Z}^r$-cover, by pull back:

$$\begin{array}{ccc} X^{\nu} & \longrightarrow & \mathbb{R}^r \\ \big\downarrow & & \big\downarrow \\ X & \xrightarrow{\ f\ } & T^r, \end{array}$$

  where f realizes ν at the level of fundamental groups.
- Note: (homotopy fiber of f) $\simeq X^{\nu}$.
- All connected, regular $\mathbb{Z}^r$-covers of X arise in this manner.

- The map ν factors as

$$G \xrightarrow{\text{ab}} G_{\text{ab}} \xrightarrow{\nu_*} \mathbb{Z}^r,$$

where ν_* may be identified with the induced homomorphism

$$f_*\colon H_1(X,\mathbb{Z}) \to H_1(T^r,\mathbb{Z}).$$

- Passing to the homomorphism in $\mathbb{Q}$-homology, we see that the cover $X^\nu \to X$ is determined by the kernel of

$$\nu_*\colon H_1(X,\mathbb{Q}) \to \mathbb{Q}^r.$$

- Conversely, every codimension-r linear subspace of $H_1(X,\mathbb{Q})$ can be realized as

$$\ker(\nu_*\colon H_1(X,\mathbb{Q}) \to \mathbb{Q}^r).$$

for some $\nu\colon G \twoheadrightarrow \mathbb{Z}^r$, and thus gives rise to a cover $X^\nu \to X$.

- Let $\mathrm{Gr}_r(H^1(X,\mathbb{Q}))$ be the Grassmanian of r-planes in the finite-dimensional, rational vector space $H^1(X,\mathbb{Q})$.
- Using the dual map $\nu^*\colon \mathbb{Q}^r \to H^1(X,\mathbb{Q})$ instead, we obtain:

Proposition (Dwyer–Fried 1987)

The connected, regular covers of X whose group of deck transformations is free abelian of rank r are parametrized by the rational Grassmannian $\mathrm{Gr}_r(H^1(X,\mathbb{Q}))$, via the correspondence

$$\{\mathbb{Z}^r\text{-covers } X^\nu \to X\} \longleftrightarrow \{r\text{-planes } P_\nu := \mathrm{im}(\nu^*) \text{ in } H^1(X,\mathbb{Q})\}.$$

The Dwyer–Fried sets

Moving about the rational Grassmannian, and recording how the Betti numbers of the corresponding covers vary leads to:

Definition

The *Dwyer–Fried invariants* of X are the subsets

$$\Omega^i_r(X) = \{ P_\nu \in \mathrm{Gr}_r(H^1(X,\mathbb{Q})) \mid b_j(X^\nu) < \infty \text{ for } j \leq i \},$$

defined for all $i \geq 0$ and all $r > 0$, with the convention that $\Omega^i_r(X) = \emptyset$ if $r > b_1(X)$.

In particular, if $b_1(X) = 0$, then all the Ω-invariants of X are empty. For a fixed $r > 0$, the Dwyer–Fried invariants form a descending filtration of the Grassmanian of r-planes,

$$\mathrm{Gr}_r(H^1(X,\mathbb{Q})) = \Omega^0_r(X) \supseteq \Omega^1_r(X) \supseteq \Omega^2_r(X) \supseteq \cdots .$$

The Ω-sets are homotopy-type invariants of X:

Lemma

Suppose $X \simeq Y$. For each $r > 0$, there is then an isomorphism $\mathrm{Gr}_r(H^1(Y,\mathbb{Q})) \cong \mathrm{Gr}_r(H^1(X,\mathbb{Q}))$ sending each subset $\Omega^i_r(Y)$ bijectively onto $\Omega^i_r(X)$.

Proof.

- Let $f\colon X \to Y$ be a (cellular) homotopy equivalence.
- $f^*\colon H^1(Y,\mathbb{Q}) \to H^1(X,\mathbb{Q})$, defines isomorphisms $f^*_r\colon \mathrm{Gr}_r(H^1(Y,\mathbb{Q})) \to \mathrm{Gr}_r(H^1(X,\mathbb{Q}))$.
- It remains to show that $f^*_r(\Omega^i_r(Y)) \subseteq \Omega^i_r(X)$.
- For that, let $P \in \Omega^i_r(Y)$, and write $P = P_\nu$, for some $\nu\colon \pi_1(Y) \twoheadrightarrow \mathbb{Z}^r$. The map f lifts to a map $\bar{f}\colon X^{\nu\circ f_\sharp} \to Y^\nu$.
- Clearly, $\bar{f}$ is a homotopy equivalence. Thus, $b_i(X^{\nu\circ f_\sharp}) = b_i(Y^\nu)$, and so $f^*_r(P_\nu) = P_{\nu\circ f_\sharp}$ belongs to $\Omega^i_r(X)$.

□

In view of this lemma, we may extend the definition of the Ω-sets from spaces to groups.

Let G be a finitely-generated group. Pick a classifying space $K(G,1)$ with finite k-skeleton, for some $k \geq 1$.

Definition

The *Dwyer–Fried invariants* of G are the subsets

$$\Omega^i_r(G) = \Omega^i_r(K(G,1))$$

of $\mathrm{Gr}_r(H^1(G,\mathbb{Q}))$, defined for all $i \geq 0$ and $r \geq 1$.

Since the homotopy type of $K(G,1)$ depends only G, the sets $\Omega^i_r(G)$ are well-defined group invariants.

- Especially manageable situation: $r = n$, where $n = b_1(X) > 0$.
- In this case, $\mathrm{Gr}_n(H^1(X, \mathbb{Q})) = \{\mathrm{pt}\}$.
- This single point corresponds to the maximal free abelian cover, $X^\alpha \to X$, where $\alpha\colon G \twoheadrightarrow G_{\mathrm{ab}}/\operatorname{Tors}(G_{\mathrm{ab}}) = \mathbb{Z}^n$.
- The sets $\Omega^i_n(X)$ are then given by

$$\Omega^i_n(X) = \begin{cases} \{\mathrm{pt}\} & \text{if } b_j(X^\alpha) < \infty \text{ for } j \le i, \\ \emptyset & \text{otherwise.} \end{cases}$$

- Both situations may occur:

Example

Let $X = S^1 \vee S^k$, for some $k > 1$. Then $X^\alpha = X^{\mathrm{ab}}$ is homotopic to a countable wedge of k-spheres. Thus, $\Omega^i_1(X) = \{\mathrm{pt}\}$ for $i < k$, yet $\Omega^i_1(X) = \emptyset$, for $i \ge k$.

Remark

Finiteness of the Betti numbers of a free abelian cover X^ν does not imply finite-generation of the integral homology groups of X^ν. Thus, we cannot replace the condition "$b_i(X^\nu) < \infty$, for $i \le q$" by the (stronger) condition "$H_i(X^\nu, \mathbb{Z})$ is a finitely-generated group, for $i \le q$."

E.g., let K be a knot in S^3, with complement $X = S^3 \setminus K$, infinite cyclic cover X^{ab}, and Alexander polynomial $\Delta_K \in \mathbb{Z}[t^{\pm 1}]$. Then

$$H_1(X^{\mathrm{ab}}, \mathbb{Z}) = \mathbb{Z}[t^{\pm 1}]/(\Delta_K).$$

Hence, $H_1(X^{\mathrm{ab}}, \mathbb{Q}) = \mathbb{Q}^d$, where $d = \deg \Delta_K$. Thus,

$$\Omega^1_1(X) = \{\mathrm{pt}\}.$$

But, if Δ_K is not monic, $H_1(X^{\mathrm{ab}}, \mathbb{Z})$ need not be a f.g. $\mathbb{Z}$-module.

Example (Milnor 1968)

Let K be the 5_2 knot, with Alex polynomial $\Delta_K = 2t^2 - 3t + 2$. Then $H_1(X^{\mathrm{ab}}, \mathbb{Z}) = \mathbb{Z}[1/2] \oplus \mathbb{Z}[1/2]$ is not f.g., though $H_1(X^{\mathrm{ab}}, \mathbb{Q}) = \mathbb{Q} \oplus \mathbb{Q}$.

Ω-invariants and characteristic varieties

- Given an epimorphism $\nu\colon G \twoheadrightarrow \mathbb{Z}^r$, let $\hat{\nu}\colon \widehat{\mathbb{Z}^r} \to \widehat{G}$ be the induced morphism between character groups, given by $\hat{\nu}(\rho)(g) = \nu(\rho(g))$.
- Its image, $\mathbb{T}_\nu = \hat{\nu}(\widehat{\mathbb{Z}^r})$, is a complex algebraic subtorus of $\widehat{G}$, isomorphic to $(\mathbb{C}^\times)^r$.
- The following theorem was proved by Dwyer and Fried for a finite CW-complex X, using the support loci for the Alexander invariants of X. It was recast in a slightly more general context in (PS 2010), using the degree-1 characteristic varieties.

Theorem

Let X be a connected CW-complex with finite k-skeleton, $G = \pi_1(X)$. For an epimorphism $\nu\colon G \twoheadrightarrow \mathbb{Z}^r$, the following are equivalent:

1. *The vector space $\bigoplus_{i=0}^{k} H_i(X^\nu, \mathbb{C})$ is finite-dimensional.*
2. *The algebraic torus $\mathbb{T}_\nu$ intersects the variety $\mathcal{W}^k(X)$ in only finitely many points.*

Corollary

Suppose $\mathcal{W}^i(X)$ is finite. Then $\Omega^i_r(X) = \mathrm{Gr}_r(H^1(X,\mathbb{Q})), \ \forall r \leq b_1(X)$.

Example

Let M be a nilmanifold. Then $\Omega^i_r(M) = \mathrm{Gr}_r(\mathbb{Q}^n)$, for all $i \geq 0$ and $r \leq n = b_1(M)$.

Example

Suppose X is the complement of a knot in S^m, $m \geq 3$. Then $\Omega^i_1(X) = \{\mathrm{pt}\}$, for all $i \geq 0$.

Corollary

Let $n = b_1(X)$. Suppose $\mathcal{W}^i(X)$ is infinite, for some $i > 0$. Then $\Omega^q_n(X) = \emptyset$, for all $q \geq i$. In particular, $b_j(X^\alpha) = \infty$, for some $j \leq i$.

Example

Let S_g be a Riemann surface of genus $g > 1$. Then

$$\Omega^i_r(S_g) = \emptyset, \qquad \text{for all } i, r \geq 1$$
$$\Omega^n_r(S_{g_1} \times \cdots \times S_{g_n}) = \emptyset, \qquad \text{for all } r \geq 1$$

Example

Let $Y_m = \bigvee^m S^1$ be a wedge of m circles, $m > 1$. Then

$$\Omega^i_r(Y_m) = \emptyset, \qquad \text{for all } i, r \geq 1$$
$$\Omega^n_r(Y_{m_1} \times \cdots \times Y_{m_n}) = \emptyset, \qquad \text{for all } r \geq 1$$

The openess question

Question

For which spaces X, and for which indices i and r are the sets $\Omega^i_r(X)$ Zariski open subsets of $\mathrm{Gr}_r(H^1(X,\mathbb{Q}))$?

Write $n = b_1(X)$. Identify $H^1(X,\mathbb{Q}) = \mathbb{Q}^n$ and $\mathrm{Gr}_1(\mathbb{Q}^n) = \mathbb{QP}^{n-1}$.

Theorem (DF 1987)

Each $\Omega^i_1(X)$ is the complement of a finite union of projective subspaces in $\mathbb{QP}^{n-1}$. In particular, $\Omega^i_1(X)$ is a Zariski open set in $\mathbb{QP}^{n-1}$.

This subspace arrangement can be understood in terms of a more general construction, introduced in (DPS 2009).

Proposition (PS 2010)

Let $X^\nu \to X$ be a regular $\mathbb{Z}$-cover, classified by $\nu\colon \pi_1(X) \twoheadrightarrow \mathbb{Z}$. Let $\nu^*\colon H^1(\mathbb{Z},\mathbb{Z}) = \mathbb{Z} \to H^1(X,\mathbb{Z})$, and $\bar\nu = \nu^*(1)$. Then,

$$\sum_{i=1}^{k} b_i(X^\nu) < \infty \iff \bar\nu \notin \tau_1(\mathcal{W}^k(X)).$$

Here, if $W \subset (\mathbb{C}^\times)^n$ is a Zariski closed set, then

$$\tau_1(W) := \{z \in \mathbb{C}^n \mid \exp(\lambda z) \in W, \text{ for all } \lambda \in \mathbb{C}\}.$$

For $r > 1$, though, $\Omega^k_r(X)$ is not necessarily an open subset of $\mathrm{Gr}_r(\mathbb{Q}^n)$:

- Dwyer and Fried gave an example of a finite, 3-dimensional CW-complex for which $\Omega^2_2(X)$ is *not* open.
- In (S 2010), I give examples of finitely presented (Kähler) groups G for which $\Omega^1_2(G)$ is not open.

Example (DF 1987)

- Let $Y = T^3 \vee S^2$. Then $\pi_1(Y) = H \cong \mathbb{Z}^3$, with generators x_1, x_2, x_3.
- Let $f\colon S^2 \to Y$ represent the element $x_1 - x_2 + 1$ of $\pi_2(Y) = \mathbb{Z}H$. Form the CW-complex $X = Y \cup_f D^3$, with $\pi_1(X) = H$ and $\pi_2(X) = \mathbb{Z}H/(x_1 - x_2 + 1)$.
- Identifying $\operatorname{Hom}(H, \mathbb{C}) = (\mathbb{C}^\times)^3$, we have $\mathcal{V}^1_1(X) = \{1\}$ and $\mathcal{V}^2_1(X) = \{z \in (\mathbb{C}^\times)^3 \mid z_1 - z_2 + 1 = 0\}$.
- Consider an algebraic 2-torus $T = \{z_1^{a_1} z_2^{a_2} z_3^{a_3} = 1\}$ in $(\mathbb{C}^\times)^3$.
- Then: $T \cap \mathcal{V}^2_1(X)$ is either empty (this happens precisely when $T = \{z_1 z_2^{-1} = 1\}$ or $T = \{z_2 = 1\}$), or is 1-dimensional.
- Thus, the locus in $\operatorname{Gr}_2(\mathbb{Q}^3) = \mathbb{QP}^2$ giving rise to algebraic 2-tori in $(\mathbb{C}^\times)^3$ having finite intersection with $\mathcal{V}^2_1(X)$ consists of 2 points.
- In particular, $\Omega^2_2(X)$ is not open in $\mathbb{QP}^2$, even in the usual topology.

Finiteness properties

Let G be a group, and k a positive integer.

- G has *property* F_k if it admits a classifying space $K(G,1)$ with finite k-skeleton.
  - F_1: G is finitely generated
  - F_2: G is finitely presentable.
- G has *property* FP_k if the trivial $\mathbb{Z}G$-module $\mathbb{Z}$ admits a projective $\mathbb{Z}G$-resolution which is finitely generated in all dimensions up to k.

The following implications (none of which can be reversed) hold:

$$\begin{aligned} G \text{ is of type } \mathrm{F}_k &\Rightarrow G \text{ is of type } \mathrm{FP}_k \\ &\Rightarrow H_i(G,\mathbb{Z}) \text{ is finitely generated, for all } i \le k \\ &\Rightarrow b_i(G) < \infty, \text{ for all } i \le k. \end{aligned}$$

Moreover, $\mathrm{FP}_k \;\&\; \mathrm{F}_2 \Rightarrow \mathrm{F}_k$.

Theorem

Let G be a finitely generated group, and $\nu\colon G \twoheadrightarrow \mathbb{Z}^r$ an epimorphism, with kernel Γ. Suppose $\Omega^k_r(G) = \emptyset$, and Γ is of type F_{k-1}. Then $b_k(\Gamma) = \infty$.

Hence, $H_k(\Gamma, \mathbb{Z})$ is not finitely generated, and Γ is not of type FP_k.

Proof.

Set $X = K(G, 1)$; then $X^\nu = K(\Gamma, 1)$.

Since Γ is of type F_{k-1}, we have $b_i(X^\nu) < \infty$ for $i \leq k-1$.

Since $\Omega^k_r(X) = \emptyset$, we must have $b_k(X^\nu) = \infty$. □

Corollary

Let G be a finitely generated group, and suppose $\Omega^3_1(G) = \emptyset$. Let $\nu\colon G \twoheadrightarrow \mathbb{Z}$ be an epimorphism. If the group $\Gamma = \ker(\nu)$ is finitely presented, then $H_3(\Gamma, \mathbb{Z})$ is not finitely generated.

Example

- Let $Y_2 = S^1 \vee S^1$ and $X = Y_2 \times Y_2 \times Y_2$. Clearly, X is a classifying space for $G = F_2 \times F_2 \times F_2$.
- Let $\nu \colon G \to \mathbb{Z}$ be the homomorphism taking each standard generator to 1. Set $\Gamma = \ker(\nu)$.
- Stallings (1963):

$$\Gamma = \langle a, b, c, x, y \mid [x, a], [y, a], [x, b], [y, b], [a^{-1}x, c], [a^{-1}y, c], [b^{-1}a, c] \rangle$$

Stallings showed, via a Mayer-Vietoris argument, that $H_3(\Gamma, \mathbb{Z})$ is not finitely generated.

- Alternate explanation: We have $\Omega^3_1(X) = \emptyset$. Thus, the desired conclusion follows from above Corollary.

Kollár's question

Two groups, G_1 and G_2, are said to be *commensurable up to finite kernels* if there is a zig-zag of groups and homomorphisms,

$$G_1 \leftarrow H_1 \rightarrow H_2 \leftarrow \cdots \cdots \leftarrow H_q \rightarrow G_2\ ,$$

with all arrows of finite kernel and cofinite image.

Question (J. Kollár 1995)

Given a smooth, projective variety M, is the fundamental group $\Gamma = \pi_1(M)$ commensurable, up to finite kernels, with another group, π, admitting a $K(\pi, 1)$ which is a quasi-projective variety?

Theorem (DPS 2009)

For each $k \geq 3$, there is a smooth, irreducible, complex projective variety M of complex dimension $k - 1$, such that the group $\Gamma = \pi_1(M)$ is of type F_{k-1}, but not of type FP_k.

Lemma (Bieri 1981)

Let π be a finite-index subgroup of G. Then G is of type FP_n if and only if π is.

Lemma (Bieri 1981)

Let $1 \to N \to G \to Q \to 1$ be an exact sequence of groups, and assume N is of type FP_∞. Then G is of type FP_n if and only if Q is.

Corollary

Suppose G_1 and G_2 are commensurable up to finite kernels. Then G_1 is of type FP_n if and only G_2 is of type FP_n.

Fact: every quasi-projective variety has the homotopy type of a finite CW-complex.

Hence, $\Gamma = \pi_1(M)$ is not commensurable (up to finite kernels) to any group π admitting a $K(\pi, 1)$ which is a quasi-projective variety.

Construction of M

- Let E be a complex elliptic curve, and fix an integer $g \geq 2$.
- Pick a subset $B \subset E$ of cardinality $|B| = 2g - 2$.
- Fix a basepoint $x_0 \in E \setminus B$, and for each point $b \in B$, choose a loop α_b in $E \setminus B$, circling in a positive direction around b.
- Finally, choose a homomorphism $\varphi\colon \pi_1(E \setminus B, x_0) \to \mathbb{Z}_2$ such that $\varphi(\alpha_b) = 1$, for all $b \in B$.
- With these choices, there is a smooth projective curve C of genus g, and a branched 2-fold cover, $f\colon C \to E$, which induces a bijection between the ramification locus $R \subset C$ and the branch locus $B \subset E$.
- The restriction $f\colon C \setminus R \to E \setminus B$ is the regular cover corresponding to φ.

- Now fix an integer $k \geq 3$, and set $X = C^{\times k}$.
- Let $s_2 \colon E^{\times 2} \to E$ be the group law of the elliptic curve, and extend it by associativity to a map $s_k \colon E^{\times k} \to E$.
- Composing this map with the product map $f^{\times k} \colon C^{\times k} \to E^{\times k}$, we obtain a surjective holomorphic map,

$$h = s_k \circ f \colon X \to E.$$

Lemma (DPS 2009)

Let M be the generic fiber of h. Then M is a smooth, complex projective variety of dimension $k-1$. Moreover,

1. *M is connected.*
2. $\pi_1(M) = \ker\left(h_\sharp \colon \pi_1(X) \twoheadrightarrow \pi_1(E)\right)$.
3. $\pi_2(M) = \cdots = \pi_{k-2}(M) = 0$.

Proof of Theorem.

- Set $G = \pi_1(X)$ and $\Gamma = \pi_1(M)$. Identify $\pi_1(E) = \mathbb{Z}^2$, and write $\nu = h_\sharp$.
- From lemma, parts (1) and (2), we have a short exact sequence,

$$1 \longrightarrow \Gamma \longrightarrow G \xrightarrow{\ \nu\ } \mathbb{Z}^2 \longrightarrow 1 .$$

- Since X is a k-fold product of surfaces of genus $g \geq 2$, we have that X is a $K(G, 1)$.
- We also know: $\Omega^k_2(G) = \emptyset$.
- By lemma, part (3), a classifying space $K(\Gamma, 1)$ can be obtained from M by attaching cells of dimension k and higher.
- Consequently, Γ is of type F_{k-1}.
- Finally, a previous theorem shows that Γ is not of type FP_k.

□