Jump loci of hyperplane arrangements

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Kähler manifolds

A compact, connected, complex manifold M^m is *Kähler* if there is a Hermitian metric h such that $\omega = \mathfrak{Im}(h)$ is a closed 2-form.

- Examples: smooth, complex projective varieties.

The Kähler condition puts strong restrictions on M, and on $G = \pi_1(M)$:

1. Hodge decomposition on $H^i(M, \mathbb{C})$
2. Lefschetz isomorphism, and Lefschetz decomposition on $H^i(M, \mathbb{R})$
3. Betti numbers b_{2i+1} must be even, and increasing for $2i + 1 \leq m$. Betti numbers b_{2i} must be increasing for $2i \leq m$.
4. M is formal, i.e., $(\Omega(M), d) \simeq (H^*(M, \mathbb{R}), 0)$ (DGMS 1975)

1. $b_1(G)$ is even
2. G is 1-formal, i.e., its Malcev Lie algebra $\mathfrak{m}(G)$ is quadratic
3. G cannot split non-trivially as a free product (Gromov 1989)

Quasi-Kähler manifolds

A manifold X is called *quasi-Kähler* if $X = \overline{X} \setminus D$, where $\overline{X}$ is Kähler and D is a divisor with normal crossings.

- Examples: smooth, quasi-projective complex varieties, such as complements of hypersurfaces in $\mathbb{CP}^n$.
- Every quasi-projective variety X admits a mixed Hodge structure $(W_\bullet, F^\bullet)$ on cohomology.
- X q.-p., $W_1(H^1(X, \mathbb{Q})) = 0 \Rightarrow \pi_1(X)$ is 1-formal. (Morgan 1978)
- There are smooth, quasi-projective varieties X for which $\pi_1(X)$ is *not* 1-formal: let $\mathbb{C}^* \to X \to \mathbb{C}^* \times \mathbb{C}^*$ be the bundle with $c_1 = 1$; then $\pi_1(X)$ is the Heisenberg group, thus not 1-formal.
- $X = \mathbb{CP}^n \setminus \{\text{hypersurface}\} \Rightarrow \pi_1(X)$ is 1-formal. (Kohno 1983)
- $X = \mathbb{CP}^n \setminus \{\text{hyperplane arr.}\} \Rightarrow X$ is formal. (Brieskorn 1973)

Characteristic varieties

Theorem (Arapura 1997)

Let $X = \overline{X} \setminus D$ be a quasi-Kähler manifold. Then:

1. *Each component of $\mathcal{V}^1_1(X)$ is either an isolated unitary character, or of the form $\rho \cdot f^*(H^1(C, \mathbb{C}^\times))$, for some torsion character ρ and some admissible map $f\colon X \to C$.*
2. *If either $X = \overline{X}$ or $b_1(\overline{X}) = 0$, then, for all $i \geq 0$ and $d \geq 1$, each component of $\mathcal{V}^i_d(X)$ is of the form $\rho \cdot f^*(H^1(T, \mathbb{C}^\times))$, for some unitary character ρ and some holomorphic map $f\colon X \to T$ to a complex torus.*

In particular, all the components of $\mathcal{V}^i_d(X)$ passing through 1 are subtori in $\mathrm{Hom}(\pi_1(X), \mathbb{C}^\times)$, provided X is Kähler, or $W_1(H^1(X, \mathbb{C})) = 0$.

Alexander polynomials

Theorem (DPS 2008)

Let G be a quasi-Kähler group. Set $n = b_1(G)$, and let Δ_G be the Alexander polynomial of G.

- *If $n \neq 2$, then the Newton polytope of Δ_G is a line segment.*
- *If G is actually a Kähler group, then $\Delta_G \doteq \text{const}$.*

If $n \geq 3$, we may write

$$\Delta_G(t_1, \dots, t_n) \doteq cP(t_1^{e_1} \cdots t_n^{e_n}),$$

for some $c \in \mathbb{Z}$, some polynomial $P \in \mathbb{Z}[t]$ equal to a product of cyclotomic polynomials, and some exponents $e_i \geq 1$ with $\gcd(e_1, \dots, e_n) = 1$.

Resonance varieties

Theorem (DPS 2009)

Let X be a quasi-Kähler manifold, and $G = \pi_1(X)$. Let $\{L_\alpha\}_\alpha$ be the non-zero irred components of $\mathcal{R}_1(G)$. If G is 1-formal, then

1. *Each L_α is a linear subspace of $H^1(G, \mathbb{C})$.*
2. *Each L_α is p-isotropic (i.e., restriction of $\cup_G$ to L_α has rank p), with $\dim L_\alpha \geq 2p + 2$, for some $p = p(\alpha) \in \{0, 1\}$.*
3. *If $\alpha \neq \beta$, then $L_\alpha \cap L_\beta = \{0\}$.*
4. $\mathcal{R}_d(G) = \{0\} \cup \bigcup_{\alpha:\dim L_\alpha > d+p(\alpha)} L_\alpha$.

Furthermore,

4. *If X is compact, then G is 1-formal, and each L_α is 1-isotropic.*
5. *If $W_1(H^1(X, \mathbb{C})) = 0$, then G is 1-formal, and each L_α is 0-isotropic.*

Corollary

Let X be a smooth, quasi-projective variety with $W_1(H^1(X,\mathbb{C})) = 0$. Let $G = \pi_1(X)$, and suppose $\mathcal{R}_1(G) \neq \{0\}$. Then G is not a Kähler group (though G is 1-formal).

Corollary

Let X be the complement of a hypersurface in $\mathbb{CP}^n$, and let $G = \pi_1(X)$. If $\mathcal{R}_1(G) \neq \{0\}$, then G is not a Kähler group.

The assumption $\mathcal{R}_1(G) \neq \{0\}$ is really necessary.
For example, take $X = \mathbb{C}^2 \setminus \{z_1 z_2 = 0\}$. Then $G = \mathbb{Z}^2$ is clearly a Kähler group, but $\mathcal{R}_1(G) = \{0\}$.

The complement of an arrangement

- $\mathcal{A} = \{H_1, \dots, H_n\}$ hyperplane arrangement in $\mathbb{C}^\ell$.
- Intersection lattice $L(\mathcal{A})$: poset of all non-empty intersections, ordered by reverse inclusion.
- Complement $X(\mathcal{A}) = \mathbb{C}^\ell \setminus \bigcup_{H \in \mathcal{A}} H$: a smooth, connected, quasi-projective variety.
- Cohomology ring $A(\mathcal{A}) = H^*(X(\mathcal{A}), \mathbb{C})$: the quotient $A = E/I$ of the exterior algebra E on classes dual to the meridians, modulo an ideal I determined by $L(\mathcal{A})$.
- Fundamental group $G(\mathcal{A}) = \pi_1(X(\mathcal{A}))$: computed from the braid monodromy read off a generic projection of a generic slice in $\mathbb{C}^2$. G has a (minimal) finite presentation with
  - Meridional generators $x_1, \dots, x_n$.
  - Commutator relators $x_i \alpha_j (x_i)^{-1}$, where $\alpha_j \in P_n$ are the (pure) braid monodromy generators, acting on F_n via the Artin representation.

  In particular, $G_{\text{ab}} = \mathbb{Z}^n$, with preferred basis $\{x_1, \dots, x_n\}$.

Example (The braid arrangement)

- $\mathcal{A}_\ell = \{H_{ij}\}_{1\le i<j\le \ell}$, with $H_{ij} = \{z \in \mathbb{C}^\ell \mid z_i - z_j = 0\}$.
- Intersection lattice $L(\mathcal{A}_\ell)$: the lattice of partitions of $\{1, \dots, \ell\}$, ordered by refinement (a supersolvable lattice).
- Complement $X(\mathcal{A}_\ell)$: the configuration space $F(\mathbb{C}, \ell)$ of ℓ ordered points in $\mathbb{C}$.
- $\pi_1(X(\mathcal{A}_\ell)) = P_\ell$, the pure braid group on ℓ strings.
- $X(\mathcal{A}_\ell)$ is aspherical, i.e., $X(\mathcal{A}) = K(P_\ell, 1)$.
- $H^*(X(\mathcal{A}))$ is a Koszul algebra, with Poincaré polynomial $P(t) = \prod_{j=1}^{\ell-1}(1 + jt)$.

Resonance varieties

Best understood are the varieties $\mathcal{R}_d(\mathcal{A}) = \mathcal{R}^1_d(X(\mathcal{A})) \subset \mathbb{C}^n$.
First assume $\mathcal{A} = \{\ell_1, \dots, \ell_n\}$ is an affine line arrangement in $\mathbb{C}^2$, for which no two lines are parallel. Then:

1. $\mathcal{R}_1(\mathcal{A})$ lies in the hyperplane $\Delta_n = \{x \in \mathbb{C}^n \mid \sum_{i=1}^n x_i = 0\}$.
2. Each component is a linear subspace of dimension ≥ 2.
3. Two distinct components of $\mathcal{R}_1(\mathcal{A})$ meet only at 0.
4. $\mathcal{R}_d(\mathcal{A})$ is the union of those subspaces of dimension $> d$.

Example

$\mathcal{A}$ a pencil of n lines, defined by $z_1^n - z_2^n = 0$. Then:

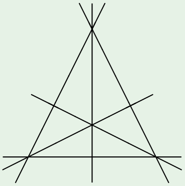
- $G = \langle x_1, \dots, x_n \mid x_1 \cdots x_n \text{ central}\rangle$.
- If $n = 1$ or 2, then $\mathcal{R}_1(\mathcal{A}) = \{0\}$.
- If $n \geq 3$, then $\mathcal{R}_1(\mathcal{A}) = \cdots = \mathcal{R}_{n-2}(\mathcal{A}) = \Delta_n$, and $\mathcal{R}_{n-1}(\mathcal{A}) = \{0\}$.

In general $\mathcal{R}_1(\mathcal{A})$ has

- Local components: to an intersection point $v_J = \bigcap_{j\in J} \ell_j$ of multiplicity $|J| \geq 3$, there corresponds a subspace $L_J = \{x \mid \sum_{j\in J} x_j = 0; x_i = 0 \text{ if } i \notin J$, of dimension $|J| - 1$.
- Non-local components come from “neighborly partitions" (or, multinets) of sub-arrangements of $\mathcal{A}$. These components have dimension either 2 or 3.

If $|\mathcal{A}| \leq 5$, then all components of $\mathcal{R}_1(\mathcal{A})$ are local. For $|\mathcal{A}| \geq 6$, though, the resonance variety $\mathcal{R}_1(\mathcal{A})$ may have interesting components.

Example (Braid arrangement $\mathcal{A}_4$)



$\mathcal{R}_1(\mathcal{A}) \subset \mathbb{C}^6$ has 4 local components (from triple points), and one non-local component, from neighborly partition $\Pi = (16|25|34)$:

$$L_{124} = \{x_1 + x_2 + x_4 = x_3 = x_5 = x_6 = 0\},$$
$$L_{135} = \{x_1 + x_3 + x_5 = x_2 = x_4 = x_6 = 0\},$$
$$L_{236} = \{x_2 + x_3 + x_6 = x_1 = x_4 = x_5 = 0\},$$
$$L_{456} = \{x_4 + x_5 + x_6 = x_1 = x_2 = x_3 = 0\},$$
$$L_{\Pi} = \{x_1 + x_2 + x_3 = x_1 - x_6 = x_2 - x_5 = x_3 - x_4 = 0\}.$$

Since all these components are 2-dimensional, $\mathcal{R}_2(\mathcal{A}) = \{0\}$.

- General case: $\mathcal{A} = \{\ell_1, \dots, \ell_n\}$ arrangement of affine lines in $\mathbb{C}^2$ (may contain parallel lines).
- Homogenizing, we get an arrangement $\bar{\mathcal{A}} = \{L_0, L_1, \dots, L_n\}$ in $\mathbb{CP}^2$, with $L_0 = \mathbb{CP}^2 \setminus \mathbb{C}^2$ the line at infinity.
- View $\bar{\mathcal{A}}$ as the projectivization of a central arrangement $\hat{\mathcal{A}}$ in $\mathbb{C}^3$. Taking a generic 2-section $\mathcal{B}$ of $\hat{\mathcal{A}}$, we're back to previous situation.
- Let $\mathcal{A}$ be an arrangement of n parallel lines in $\mathbb{C}^2$. Then:
  - $X(\mathcal{A}) = (\mathbb{C} \setminus \{n \text{ points}\}) \times \mathbb{C}, \quad G(\mathcal{A}) = F_n$.
  - $\mathcal{B}$ is a pencil of $n+1$ lines in $\mathbb{C}^2$, $\quad G(\mathcal{B}) \cong \mathbb{Z} \times F_n$.
  - This isomorphism identifies $\mathcal{R}_1(\mathcal{B}) = \Delta_n$ with $\mathcal{R}_1(\mathcal{A}) = \mathbb{C}^n$.
- In general:
  - (family of $k \geq 2$ parallel lines in $\mathcal{A}$) $\longleftrightarrow$ (pencil of $k+1$ lines in $\mathcal{B}$).
  - (k-dim local component of $\mathcal{R}_1(\mathcal{B})$) $\longleftrightarrow$ (k-dimensional component of $\mathcal{R}_1(\mathcal{A})$).
  - Similarly for non-local components.

Characteristic varieties

- Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be a central arrangement in $\mathbb{C}^\ell$, with $X = X(\mathcal{A})$, $G = G(\mathcal{A})$.
- Identify $\widehat{G} = H^1(X, \mathbb{C}^\times) = (\mathbb{C}^\times)^n$ and $H^1(X, \mathbb{C}) = \mathbb{C}^n$.
- Tangent cone formula:

$$\exp\colon (\mathcal{R}^i_d(X, \mathbb{C}), 0) \xrightarrow{\simeq} (\mathcal{V}^i_d(X), 1), \ \forall i, d > 0$$

  In particular, $\mathrm{TC}_1(\mathcal{V}^i_d(X)) = \mathcal{R}^i_d(X)$.
- Consequence: components of $\mathcal{V}_d(\mathcal{A}) = \mathcal{V}^1_d(X)$ passing through 1 are combinatorially determined:
  For each (linear) component $L \subset \mathbb{C}^n$ of $\mathcal{R}_d(\mathcal{A}) \rightsquigarrow$ a (torus) component $T = \exp(L) \subset (\mathbb{C}^\times)^n$ of $\mathcal{V}_d(\mathcal{A})$.
- Nevertheless, $\mathcal{V}_d(\mathcal{A})$ may contain translated subtori. It is still not clear whether these components are combinatorially determined.

Alexander polynomials of arrangements

- Let $\mathcal{A}$ be an arrangement of n lines in $\mathbb{C}^2$, with group $G = G(\mathcal{A})$.
- $H := \mathrm{ab}(G) \xrightarrow{\simeq} \mathbb{Z}^n$, with preferred basis corresponding to the oriented meridians around the lines.
- Get an identification of $\mathbb{Z}H$ with $\Lambda = \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$.

Definition

The *Alexander polynomial* of $\mathcal{A}$ is

$$\Delta_{\mathcal{A}} := \Delta_G \in \Lambda$$

Note that $\Delta_{\mathcal{A}}$ depends (up to normalization) only on the homeomorphism type of $X(\mathcal{A})$.

Example

- $\mathcal{A} = \mathcal{A}(n)$ a pencil of $n \geq 3$ lines, with $G = \langle x_1, \dots, x_n \mid x_1 \cdots x_n \text{ central} \rangle$. Then $\Delta_{\mathcal{A}} = (t_1 \cdots t_n - 1)^{n-2}$.
- $\mathcal{A}$ of type $\mathcal{A}(n-1, 1)$ ($n - 1 \geq 2$ parallel lines and a transverse line). Then $G = \langle x_1, \dots, x_n \mid x_n \text{ central} \rangle$, and $\Delta_{\mathcal{A}} = (t_n - 1)^{n-2}$.

For each $n \geq 3$, the corresponding pair of arrangements have isomorphic groups, but non-homeomorphic complements: the difference is picked up by the Alexander polynomial.

Theorem (DPS 2008, S 2009)

Let $\mathcal{A}$ be an arrangement of n lines in $\mathbb{C}^2$, with Alexander poly $\Delta_{\mathcal{A}}$.

1. *If $\mathcal{A}$ is a pencil and $n \geq 3$, then $\Delta_{\mathcal{A}} = (t_1 \cdots t_n - 1)^{n-2}$.*
2. *If $\mathcal{A}$ is of type $\mathcal{A}(n-1, 1)$ and $n \geq 3$, then $\Delta_{\mathcal{A}} = (t_n - 1)^{n-2}$.*
3. *For all other arrangements, $\Delta_{\mathcal{A}} \doteq \text{const}$.*

Proof.

- Cases (1) and (2) have been dealt with. Thus, we may assume $\mathcal{A}$ does not belong to either class; in particular, $n \geq 3$.
- Intersection points of multiplicity $k+1 \geq 3$, and families of $k \geq 2$ parallel lines give rise to local components of $\mathcal{V}_1(\mathcal{A})$, of dimension k. In both situations, we must have $k \leq n-2$.
- If $n \leq 5$, then all components of $\mathcal{V}_1(\mathcal{A})$ are local, except if $\mathcal{A}$ is the deconed braid arrangement of 5 lines, in which case $\mathcal{V}_1(\mathcal{A})$ has a 2-dim global component.
- If $n \geq 6$, $\mathcal{V}_1(\mathcal{A})$ may have non-local components, but they all have dimension ≤ 4, by (PY 2008).
- Hence all components of $\mathcal{V}_1(\mathcal{A})$ must have codimension at least 2, that is, the codimension-1 stratum of this variety, $\check{\mathcal{V}}_1(\mathcal{A})$, is empty.
- Since $n \geq 2$, we conclude that $\Delta_{\mathcal{A}} \doteq \text{const}$.

□

Comparison with Kähler groups

Recall that arrangement groups are 1-formal, quasi-projective groups. But are they Kähler groups?
Of course, a necessary condition for $G = G(\mathcal{A})$ to be a Kähler group is that $b_1(G) = |\mathcal{A}|$ must be even.

Theorem (S. 2009)

Let $\mathcal{A}$ be an arrangement of lines in $\mathbb{C}^2$, with group $G = G(\mathcal{A})$. The following are equivalent:

1. *G is a Kähler group.*
2. *G is a free abelian group of even rank.*
3. *$\mathcal{A}$ consists of an even number of lines in general position.*

Proof.

(3) $\Rightarrow$ (2): follows from Zariski's theorem.
(2) $\Rightarrow$ (1): clear.
(1) $\Rightarrow$ (3): assume $G = G(\mathcal{A})$ is a Kähler group.

- If $\mathcal{A}$ is not in general position, then either $\mathcal{A}$ has an intersection point of multiplicity $k+1 \geq 3$, or $\mathcal{A}$ contains a family of $k \geq 2$ parallel lines.
- In either case, $\mathcal{R}_1(\mathcal{A})$ has a k-dimensional component; in particular, $\mathcal{R}_1(\mathcal{A}) \neq \{0\}$.
- Conclusion follows from a previous corollary.

□

Comparison with right-angled Artin groups

Arrangement groups also share many common features with right-angled Artin groups: if G is a group in either class, then

- G is a commutator-relators group;
- G is 1-formal;
- each resonance variety $\mathcal{R}_d(G)$ is a union of linear subspaces;
- the free groups F_n and the free abelian groups $\mathbb{Z}^n$ belong to both classes.

So what exactly is the intersection of these two classes of groups?

Theorem (DPS 2009)

Let Γ be a finite simple graph, and G_Γ the corresponding right-angled Artin group. Then:

1. *G_Γ is a quasi-Kähler group if and only if Γ is a complete multipartite graph $K_{n_1,\dots,n_r} = \overline{K}_{n_1} * \cdots * \overline{K}_{n_r}$, in which case $G_\Gamma = F_{n_1} \times \cdots \times F_{n_r}$.*
2. *G_Γ is a quasi-Kähler group if and only if Γ is a complete graph K_{2m}, in which case $G_\Gamma = \mathbb{Z}^{2m}$.*

Definition

Given a line arrangement $\mathcal{A}$, its *multiplicity graph*, $\Gamma(\mathcal{A})$, is the graph with vertices the intersection points with multiplicity at least 3, and edges the segments between multiple points on lines which pass through more than one multiple point.

Theorem (S. 2009)

The following are equivalent:

1. *G is a right-angled Artin group.*
2. *G is a finite direct product of finitely generated free groups.*
3. *The multiplicity graph $\Gamma(\mathcal{A})$ is a forest.*

Proof.

(1) $\Leftrightarrow$ (2) follows at once from previous theorem.
(3) $\Rightarrow$ (2) is proved in (Fan 1997).
(2) $\Rightarrow$ (3) is proved in (ELST 2008). $\square$

Using similar techniques, we can recover and sharpen a result of (Fan 2009).

Theorem (S 2009)

Let $\mathcal{A} = \{\ell_1, \dots, \ell_n\}$ be an arrangement of lines in $\mathbb{C}^2$, with group $G = G(\mathcal{A})$. The following are equivalent:

1. *The group G is a free group.*
2. *The characteristic variety $\mathcal{V}_1(\mathcal{A})$ coincides with $(\mathbb{C}^\times)^n$.*
3. *The resonance variety $\mathcal{R}_1(\mathcal{A})$ coincides with $\mathbb{C}^n$.*
4. *The lines $\ell_1, \dots, \ell_n$ are all parallel.*

Boundary manifolds

Let $\mathcal{A} = \{\ell_0, \dots, \ell_n\}$ be an arrangement of lines in $\mathbb{CP}^2$.
The *boundary manifold* $M = M(\mathcal{A})$ is obtained by taking the boundary of a regular neighborhood of $\bigcup_{i=0}^n \ell_i$ in $\mathbb{CP}^2$.

- $\mathcal{A}$ a pencil of $n+1$ lines $\implies M = \sharp^n S^1 \times S^2$.
- $\mathcal{A}$ a near-pencil of $n+1$ lines $\implies M = S^1 \times \Sigma_{n-1}$.

The boundary manifold $M(\mathcal{A})$ is a graph manifold. The underlying graph Γ has

- a vertex v_i for each line ℓ_i;
- a vertex v_J for each intersection point $\bigcap_{j\in J} \ell_j$ of multiplicity $|J| \geq 3$;
- an edge $e_{i,j}$ from v_i to v_j, $i < j$, if the ℓ_i and ℓ_j are transverse;
- an edge $e_{J,i}$ from v_J to v_i if $\ell_i \supset F_J$.

Since M is a graph manifold, the group $G = \pi_1(M)$ may be realized as the fundamental group of a graph of groups. The resulting presentation for G may be simplified to a commutator-relators presentation.

The cohomology jump loci of boundary manifolds of arrangements were (partially) computed in (CS 2006, 2008).

- The Alexander polynomial of $G = \pi_1(M(\mathcal{A}))$ is

$$\Delta_G = \prod_{v \in V(\Gamma)} (t_v - 1)^{m_v - 2},$$

  where m_v denotes the degree of the vertex v, and $t_v = \prod_{i \in v} t_i$.
- Hence, the first characteristic variety is

$$\mathcal{V}_1(G) = \bigcup_{v \in V(\Gamma)\,:\, m_v \geq 3} \{t_v - 1 = 0\}.$$

- The first resonance variety:

$$\mathcal{R}_1(G) = \begin{cases} \mathbb{C}^n & \text{if } \mathcal{A} \text{ is a pencil,} \\ \mathbb{C}^{2(n-1)} & \text{if } \mathcal{A} \text{ is a near-pencil,} \\ H^1(G, \mathbb{C}) & \text{otherwise.} \end{cases}$$

- The higher-depth resonance varieties are more complicated.

Example

- Let $\mathcal{A}$ be an arrangement of 4 lines in $\mathbb{CP}^2$ in general position, and set $G = \pi_1(M(\mathcal{A}))$.
- Then $H^1(G, \mathbb{C}) = \mathbb{C}^{10}$, and $\mathcal{R}_7(G) = Q \times \mathbb{C}^4$, where

$$Q = \{z \in \mathbb{C}^6 \mid z_1 z_6 - z_2 z_5 + z_3 z_4 = 0\},$$

  which is an irreducible quadric, with an isolated singularity at 0.
- On the other hand, $\mathcal{V}_d(G) \subseteq \{1\}$, for all $d \geq 1$.
- Consequently, $\mathrm{TC}_1(\mathcal{V}_7(G)) \neq \mathcal{R}_7(G)$.
- Hence, G is not 1-formal, and thus $M(\mathcal{A})$ is not formal.

Theorem (CS 2008, DPS 2008)

Let $\mathcal{A} = \{\ell_0, \dots, \ell_n\}$ be an arrangement of lines in $\mathbb{CP}^2$, and let M be the corresponding boundary manifold. The following are equivalent:

1. *The manifold M is formal.*
2. *The group $G = \pi_1(M)$ is 1-formal.*
3. *The group G is quasi-projective.*
4. *The group G is quasi-Kähler.*
5. *$\mathcal{A}$ is either a pencil or a near-pencil.*
6. *M is either $\sharp^n S^1 \times S^2$ or $S^1 \times \Sigma_{n-1}$.*