Box splines

Let C_N be the N dimensional hypercube:

$$C_N := \{t_1, t_2, \ldots, t_N; 0 \leq t_i \leq 1\}$$

We slice it with the hyperplane:

$$H_t = \{\sum_{i=1}^{N} t_i = t\}$$

$$Box_N(t) = volume(C_N \cap H_t).$$

Then $Box_N(t)$ is supported on $0 \leq t \leq N$. On each interval $[k, k+1]$ given by a polynomial of degree $N-1$. But different polynomials on each interval. Their $N-2$ first derivatives agree at the extreme of intervals.

Remark the symmetry:
$Box_N(t) = Box_N(N - t)$
We also see that

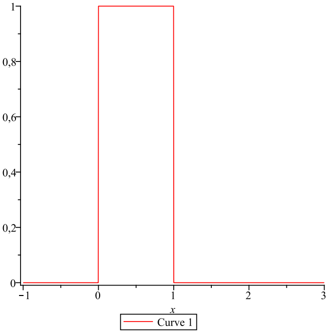
$$\frac{d}{dt} Box_N(t) = Box_{N-1}(t) - Box_{N-1}(t-1)$$

That is

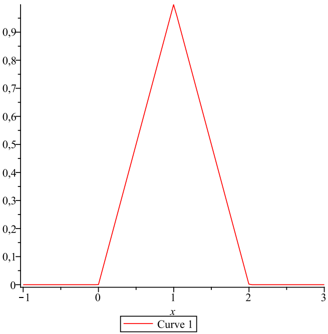
$$\frac{d}{dt} Box_N(t) = (\nabla Box_{N-1})(t)$$

where ∇ is the difference operator. (We will not use this equation in this elementary talk: it holds in the distribution sense , for $N > 1$, because of C^{N-2}-differentiability properties)

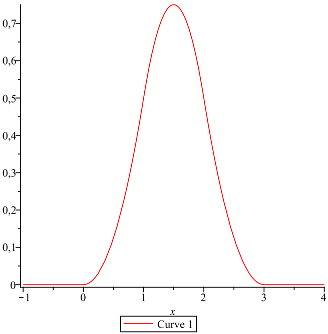
Example: $Box_1(t)$



$Box_2(t)$



$Bot_3(t)$

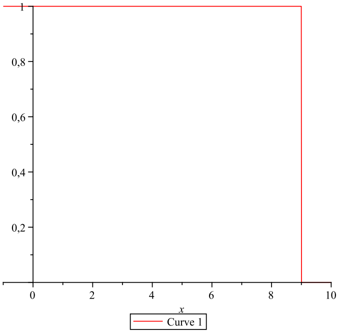


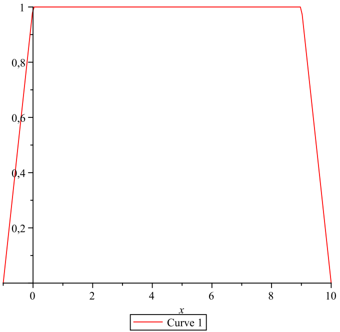
Wonderful properties of Box splines

For example

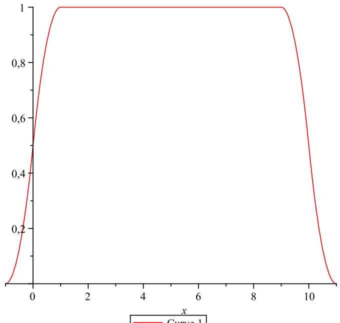
$$\sum_{n\in\mathbb{Z}} Box_N(t-n) = 1$$

The following pictures (see Procesi document) shows the sum of $Box_1(t-n), Box_2(t-n), Box_3(t-n)$ over the integers $n = -1, 0, \ldots, 8$.





1
0,8
0,6
0,4
0,2
0
2
4
6
8
10
x
Curve 1



1
0,8
0,6
0,4
0,2
0
2
4
6
8
10
x

This property follows right away from the geometric definition:
Example: Box_2: By drawing... Compute $\sum_n Box_2(t-n)$: We have to sum all the volumes of $x_1 + x_2 = t - n$ for any n.
Now x_1 ranges between 0 and 1. Put $x_2 = t - x_1 - n$ where n is the integer $floor(t - x_1)$ so that x_2 is between 0 and 1. So it is just parametrized by $0 \leq x_1 \leq 1$.
More generally, we will see that:

Theorem

For any polynomial P of degree strictly less than N,

$$t -> \sum_n P(n) Box_N(t-n)$$

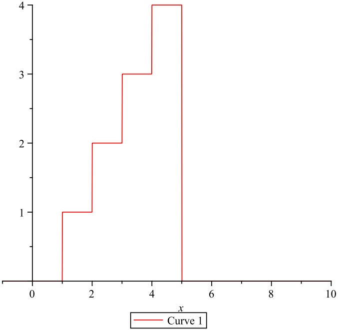
is a polynomial function of t.

$N = 1$

- Constant function: already seen.
- $P(t) = t$ of degree too big:

Drawing: I think I sum over $n = 0, 1, 2, 3, 4$

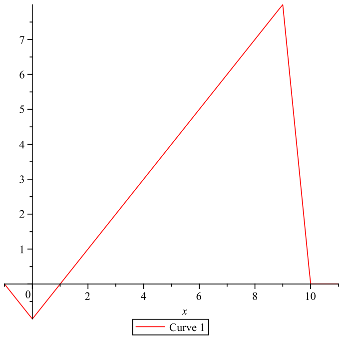
Compute $\sum_n nBox_1(t - n)$ not a polynomial !!



$N = 2$

- Constant function: already seen.
-

Function $P(t) = t$ of degree 1: true
Drawing: I think I summed over $n = -1, 0, 1, 2, 3, 4, .., 8$



Series of differential operators

If $\mathbf{T}(\frac{d}{dt}) = \sum_{n=0}^{\infty} t_n(\frac{d}{dt})^n$ is a series of differential operators, we can act on polynomials.
$(\mathbf{T}(\frac{d}{dt})P)(t)$ can be computed as for n large $(\frac{d}{dt})^n P = 0$.
We will use the series

$$(\frac{(1 - exp(-\frac{d}{dt}))}{\frac{d}{dt}}) = 1 - \frac{1}{2}\frac{d}{dt} + \frac{1}{3!}(\frac{d}{dt})^2$$

and the inverse

Definition

We define the Todd operator

$$\mathbf{Todd}(\frac{d}{dt}) = \frac{\frac{d}{dt}}{(1 - exp(-\frac{d}{dt}))}$$

$$= 1 + \frac{1}{2}\frac{d}{dt} + \frac{1}{12}(\frac{d}{dt})^2 - \frac{1}{720}(\frac{d}{dt})^4 + \cdots$$

Integration against the Box Spline

Let *test* be a smooth function of t. Then we can integrate $test(t_1 + t_2 + \cdots + t_N)$ over the hypercube. By Fubini theorem, we obtain

$$\int_0^1 \int_0^1 test(t_1 + t_2 + \cdots + t_N) d\mathbf{t} = \int_{\mathbb{R}} test(t) Box_N(t) dt$$

Definition

If *test* is a smooth function of t, define

- The usual convolution

$$(Box_N *_c test)(t) = \int_{u \in \mathbb{R}} test(u) Box_N(t-u) du$$

- The semi-discrete convolution

$$(Box_N *_d test)(t) = \sum_{u \in \mathbb{Z}} test(u) Box_N(t-u)$$

If *test* is a **polynomial**, we have (use Taylor expansion)

$$\int_0^1 test(t-u)du = (\frac{1-e^{-d/dt}}{d/dt})test(t).$$

Thus

$$(Box_N *_c test)(t) = \int_{u\in\mathbb{R}} test(u)Box_N(t-u)du$$

$$= \int_{u_1=0}^{1} \cdots \int_{u_N=0}^{1} test(t-(u_1+u_2+\cdots+u_N))du_1\cdots du_N$$

$$= ((\frac{1-e^{-\frac{d}{dt}}}{\frac{d}{dt}})^N test)(t).$$

Theorem

- *If P is a polynomial function, then*

$$Box_N *_c P = ((\frac{1-e^{-\frac{d}{dt}}}{\frac{d}{dt}})^N P)(t).$$

- *If P is a polynomial function of degree $< N$, then*

$$Box_N *_c P = Box_N *_d P == ((\frac{1-e^{-\frac{d}{dt}}}{\frac{d}{dt}})^N P)(t).$$

Proof

Let Box_N: it is enough to prove this for $P(t) = t^{N-1}$

$$t -> \sum_n n^{N-1} Box_N(t-n)$$

is a polynomial; For smaller degree, we derivate, and we can use the recurrence formula.
I will compute

$$\sum_n P(n) Box_N(t-n)$$

not exactly for $P(t) = t^{N-1}$ but for the polynomial of degree $N-1$ given by

$$t \to \frac{(t+1)(t+2)\cdots(t+N-1)}{(N-1)!}$$

Consider the standard $(N-1)$-dimensional simplex (dilated)

$$S_{N-1}(t) = \{t_i \geq 0; t_1 + t_2 + \cdots + t_N = t.\}$$

$S_{N-1}(t)$ has volume $\frac{t^{N-1}}{(N-1)!}$.
If $t = n$ is an integer the number of integral points in $S_{N-1}(t)$ is

$$\mathcal{P}_N(n) = \frac{(n+1)\cdots(n+N-1)}{(N-1)!}.$$

Now let us integrate over the first quadrant $t_i > 0$, the function $e^{-(t_1+t_2+\cdots+t_N)y}$.

$$I = \int_{t_1>0} \cdots \int_{t_N>0} e^{-(t_1+t_2+\cdots+t_N)y} dt_1 dt_2 \cdots dt_N.$$

Using Fubini, we compute I by integrating first over the simplices $S_{N-1}(t)$ then over t, thus

$$I := \int_{t>0} \frac{t^{N-1}}{(N-1)!} e^{-ty} dt$$

But we can also decompose the quadrant in cubes $[n_1, n_2, n_3, \ldots, n_N] + Hypercube\ n_i = 0, 1, \ldots,\ O \leq t_i \leq 1$ and obtain that I is equal to

$$\sum_{\mathbf{n}} \int_{t_1=0}^{1} \cdots \int_{t_N=0}^{1} e^{-((n_1+t_1)-(n_2+t_2)-(n_3+t_3)-\cdots-(n_N+t_N))y} dt_1 dt_2 \cdots dt_N.$$

$$= \int_{t\in\mathbb{R}} \sum_{n_i} e^{-(\sum n_i)y} Box_N(t) e^{-ty} dt$$

$$= \int_{t\in\mathbb{R}} \sum_{n\geq 0} \mathcal{P}_N(n) e^{-ny} Box_N(t) e^{-ty} dt$$

$$= \int_{t\in\mathbb{R}} \sum_{n\geq 0} \mathcal{P}_N(n) Box_N(t-n) e^{-ty} dt.$$

We obtain thus that for every $y > 0$

$$I := \int_{t>0} \frac{t^{N-1}}{(N-1)!} e^{-ty} dt$$

and also

$$= \int_{t\in\mathbb{R}} \sum_{n\geq 0} \mathcal{P}_N(n) Box_N(t-n) e^{-ty} dt.$$

CONCLUSION: For $t > 0$, we have almost everywhere

$$\sum_{n\geq 0} \mathcal{P}_N(n) Box_N(t-n) = \frac{t^{N-1}}{(N-1)!}.$$

So we have it on each interval where Box_N is continuous.

Recall that we want to compute

$$\sum_{n\in\mathbb{Z}} P(n) Box_N(t-n)$$

for the polynomial

$$t \to \frac{(t+1)(t+2)\cdots(t+N-1)}{(N-1)!}$$

The sum is over the integers n such that $t-n \leq N$, as Box_N is supported on $[0, N]$. thus over the integers $-(N-1), -(N-2), -(N-3), \ldots, -1, 0$. But my polynomial vanishes there, and for $n \geq 0$ coincide with $\mathcal{P}_N(n)$. Thus I obtain For $t \geq 0$

$$\sum_{n\in\mathbb{Z}} P(n) Box_N(t-n) = \frac{t^{N-1}}{(N-1)!}$$

Same calculation for $t < 0$, using $Box_N(t) = Box_N(N - t)$

$$\sum_{n \in \mathbb{Z}} P(n) Box_N(t - n) = \sum_{n \in \mathbb{Z}} P(n) Box_N(N + n - t)$$

$$= \sum_{n \in \mathbb{Z}} P(-m - N) Box_N(-t - m)$$

Remark that $P(-m - N) = (-1)^{(N-1)} P(m)$
and we obtain the same formula
For $t < 0$

$$\sum_{n \in \mathbb{Z}} P(n) Box_N(t - n) = \frac{t^{N-1}}{(N-1)!}.$$

It remains to see that

$$(\frac{(1-e^{-\partial_t})}{\partial_t})^N binomial(t+N-1, N-1) = \frac{t^{N-1}}{(N-1)!}.$$

For example, by induction.
Using distributions, we could have seen directly that

$$(\frac{d}{dt})^N (Box_N *_d P) = 0$$

so that the result is a polynomial of degree $< N$.
as

$$\frac{d}{dt} * (Box_N *_d P) = (\nabla Box_{N-1}) *_d P = Box_{N-1} * \nabla P.$$

Consequence of this theorem

Let f be any function on $\mathbb{Z}$. Consider the function on $\mathbb{R}$

$$F(t) := (Box_N *_d f)(t) := \sum_n f(n) Box_N(t-n).$$

Then F is a locally polynomial function of t (on each interval it is given by a polynomial function of t). We can derivate F over any open interval by any series of differential operator $P(\frac{d}{dt})$.

The Todd operator

Theorem

Let f be any function on $\mathbb{Z}$*:*
Let

$$F_N(t) = (Box_N *_d f)(t) := \sum_n f(n) Box_N(t-n).$$

Then $F_N(t)$ *is a function on* $\mathbb{R}$*, polynomial on each interval.*
Then $f(n)$ *(n an integer) is obtained by the limit from the right of*

$$f(n) = \lim_{\epsilon > 0, \epsilon -> 0} ((Todd(d/dt))^N (Box_N *_d f))(n+\epsilon)$$

Why we want to do it

For some very interesting cases, the function $(Box_N *_d f)$ is known and related to the "classical" geometry.
If we know regularity properties of $(Box_N *_d f)$, then we deduce regularities properties for f.
Going from F_N to f is going from the classical mechanics to quantum mechanics.
We will see examples later.

Proof

We want to prove this equation, for $n = 0$ (enough). Then $F_N(t)$ for $t > 0$ near 0 (HERE I USE THE LIMIT on the RIGHT) involves only the values $f(0), f(-1), f(-2), \ldots f(-(N-1))$ of f, indeed $Box_N(t+N) = 0$ for $t > 0$, as Box_N is supported on $[0, N]$.
We can choose a unique polynomial P of degree $N-1$ which coincide with f at $0, -1, \ldots, -(N-1)$. Near $t > 0$, small,

$$(Box_N *_d f)(t) = (Box_N *_d P)(t)$$

Differentiate with the reverse operator, we obtain our identity $f(0) = (Todd(\frac{d}{dt}))^N (Box_N *_d f)(0)$.

A wonderful property of the Box spline

Apply this to $f(n) = 0$ except for $n = 0$ where $f(0) = 1$.
THAT IS

$$f = \delta_0.$$

Then

Theorem

Consider the locally polynomial function

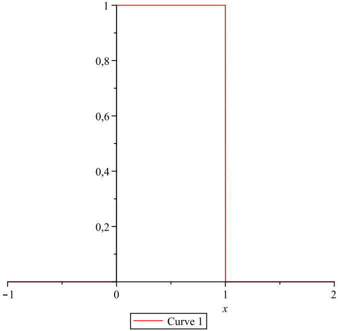
$$((Todd(d/dt))^N Box_N)(t)$$

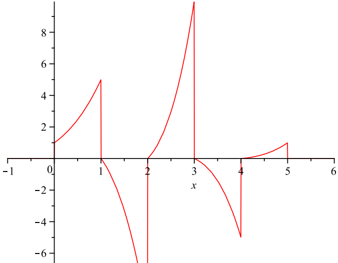
Then

$$((Todd(d/dt))^N Box_N)|_{\mathbb{Z}} = \delta_0$$

(limits from the right)

Examples





We have used essentially the relation that $\mathcal{P}_N(n)$, the number of integral points in the the standard simplex can be obtained from the volume $vol(S_N(t))$ by applying the Todd operator.
Consider $M := P_{N-1}(\mathbb{C})$ realized by

$$\{(z_1, z_2, \ldots, z_N); \sum_i |z_i|^2 = t\}/e^{i\theta}$$

with symplectic form $\Omega_t = tc$. Here c is the Fubini-Study canonical 2-form on M, with $\int c^{N-1} = 1$. We compute

$$vol(M_t) := \int_M e^{tc} = \frac{t^{N-1}}{(N-1)!}$$

Let $t = n$ an integral value,
Let $\mathcal{L}_n$ be the line bundle on M with holomorphic sections polynomials on degree n: Thus $H^0(M, \mathcal{L}_n)$ has basis $z_1^{n_1} \ldots z_N^{n_N}$ with $n_i \geq 0; \sum n_i = n$. That is the number of integral points in $S_{N-1}(n)$.
If we apply the Todd operator

$$Todd(\frac{d}{dt}) \int_M e^{tc}$$

we obtain

$$\int_M e^{tc} (\frac{c}{1 - e^{-c}})^N$$

For $t = n$, we then obtain

$$Todd(\frac{d}{dt}) \int_M e^{tc}|_{t=n} = \int_M chern(L_n) Todd(M).$$

The Riemann Roch theorem asserts that this is the dimension of $H^0(M, L_n)$ (no higher cohomology).

The Mother Formula

CONCLUSION:

The relation

$$((Todd(\frac{d}{dt}))^N Box_N)|_{\mathbb{Z}} = \delta_0$$

is the mother formula:

Children

•: Inversion formula for semi-discrete convolution

•

Riemann-Roch theorem for $P_{N-1}(\mathbb{C})$.

•

Multiplicities formulae: last talk.