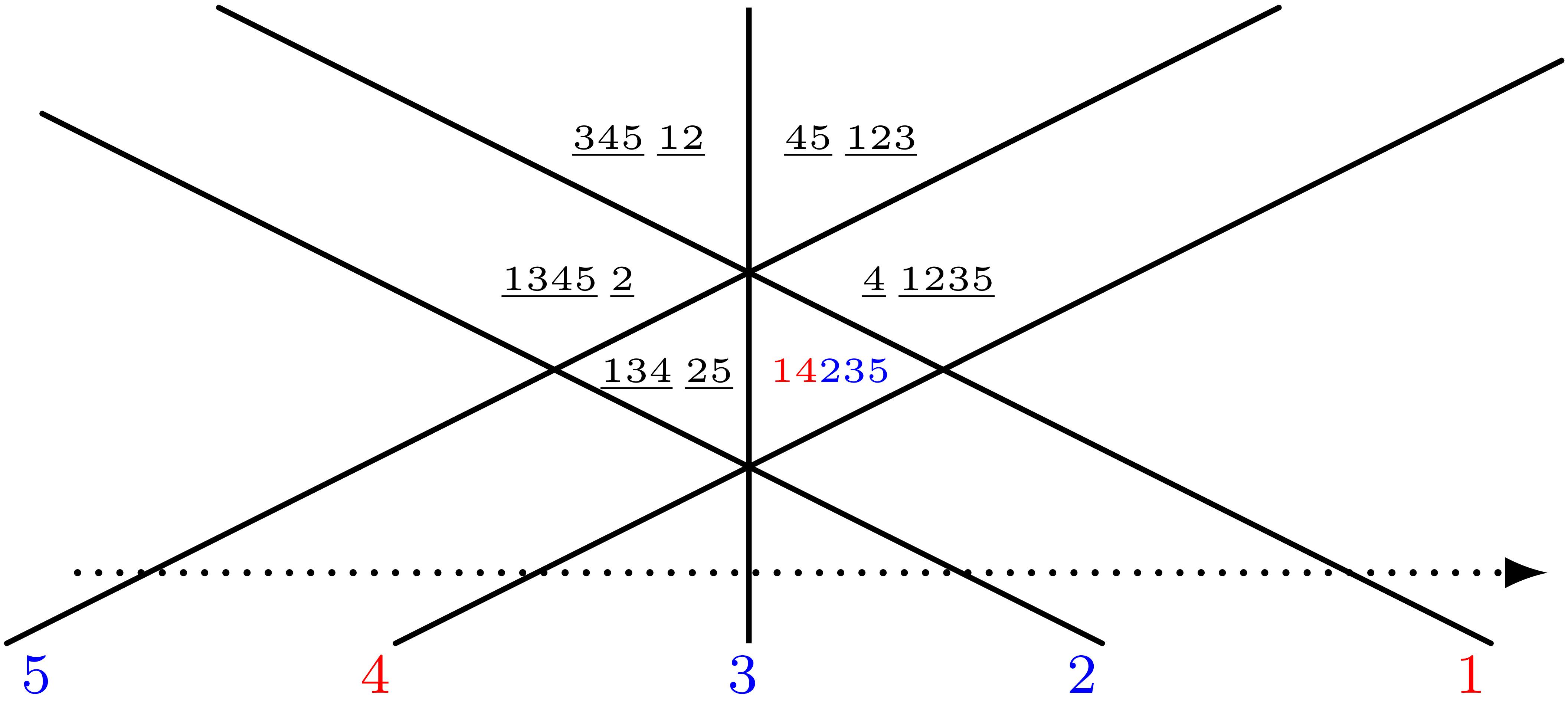
Minimal Stratification for Line Arrangements

Masahiko Yoshinaga (Kyoto U.)

Combinatorial and Geometric aspects of Hyperplane Arrangements
Centro di Ricerca Matematica "Ennio De Giorgi" Pisa, May 25, 2010.



OUR OBJECTS

$\mathcal{A} = \{H_1, \ldots, H_n\}, H_i \subset \mathbb{R}^2$ lines.

$M = M(\mathcal{A}) = \mathbb{C}^2 \setminus \bigcup_{i=1}^{n} H_i \otimes \mathbb{C}$, complement.

Motivations

How real and combinatorial structure of $\mathcal{A}$ are related to topology of $M(\mathcal{A})$?

- Topology: $\pi_1(M), H^1(M, \mathcal{L})$.
- Combinatorics: Incidences, chambers.

Contents

$$\begin{array}{ccc} \text{Minimal stratification} & \Longrightarrow & \text{presentation of } \pi_1 \\ \Downarrow & & \\ \text{Basis of OS-algebra} & \Longrightarrow & H^1(M, \mathcal{L}_\lambda) \end{array}$$

1 Minimal positive presentation

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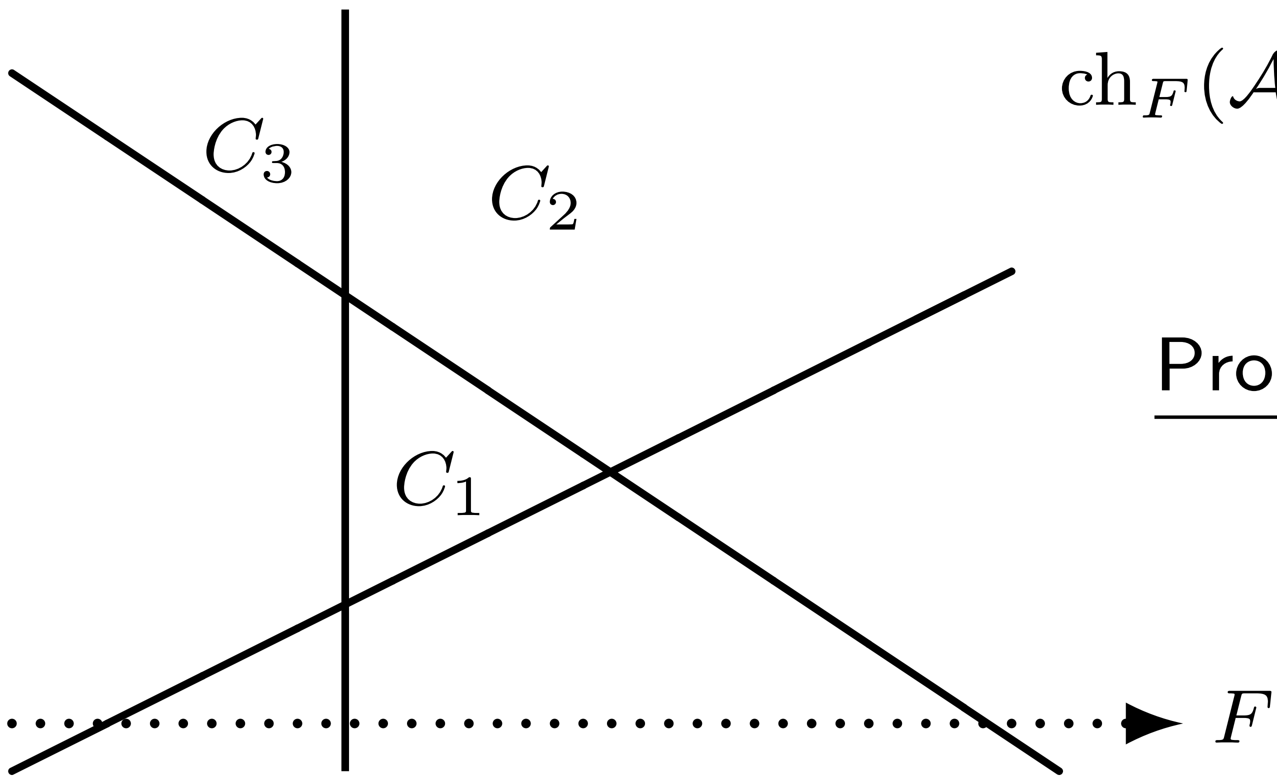
F: a generic oriented line near H_∞.

Def. $\mathrm{ch}_F(\mathcal{A}) := \{C : \text{ chamber} \mid C \cap F = \emptyset\}$

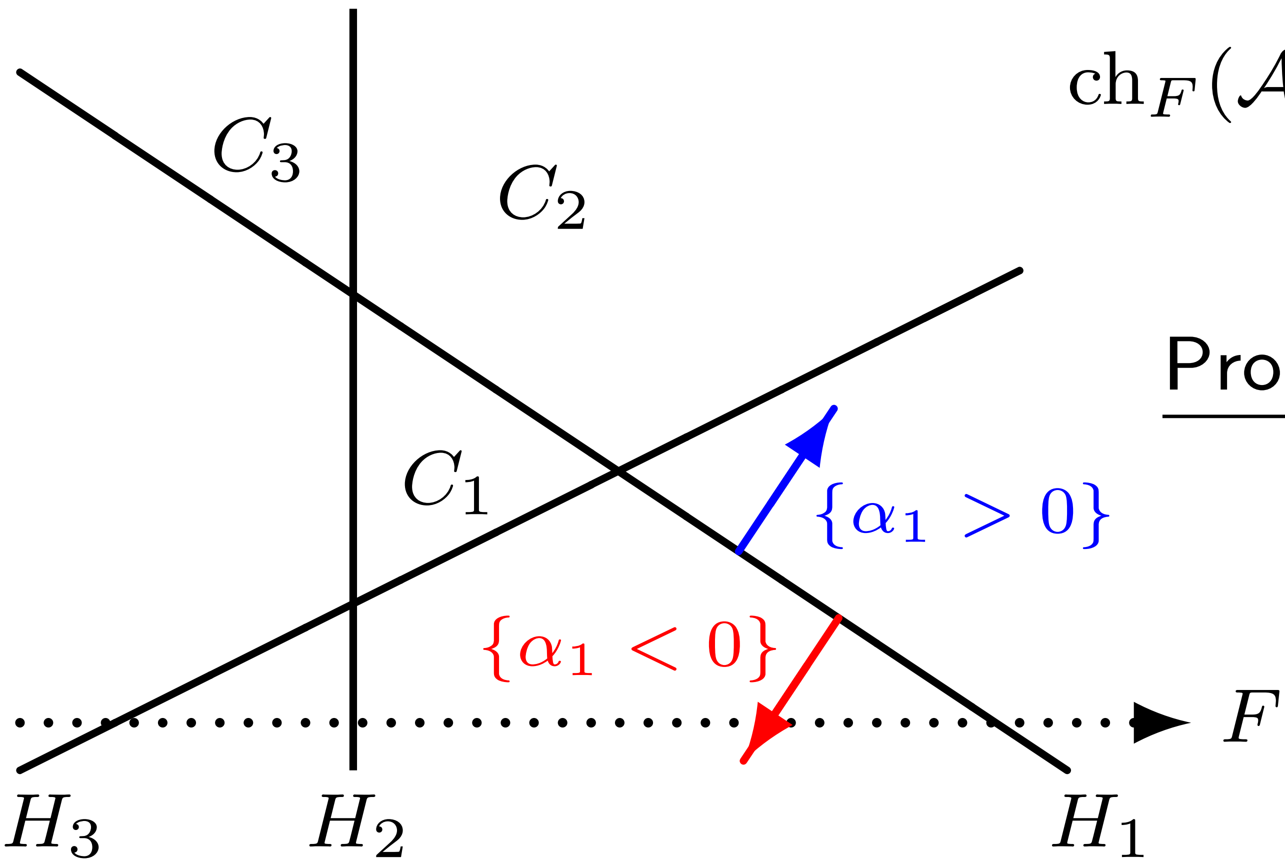


$\mathrm{ch}_F(\mathcal{A}) = \{C_1, C_2, C_3\}$

Prop. $|\mathrm{ch}_F(\mathcal{A})| = b_2(M)$

1 Minimal positive presentation

We also assume $H_1 \cap F > \cdots > H_n \cap F$, and defining equation α_i is compatible with F.

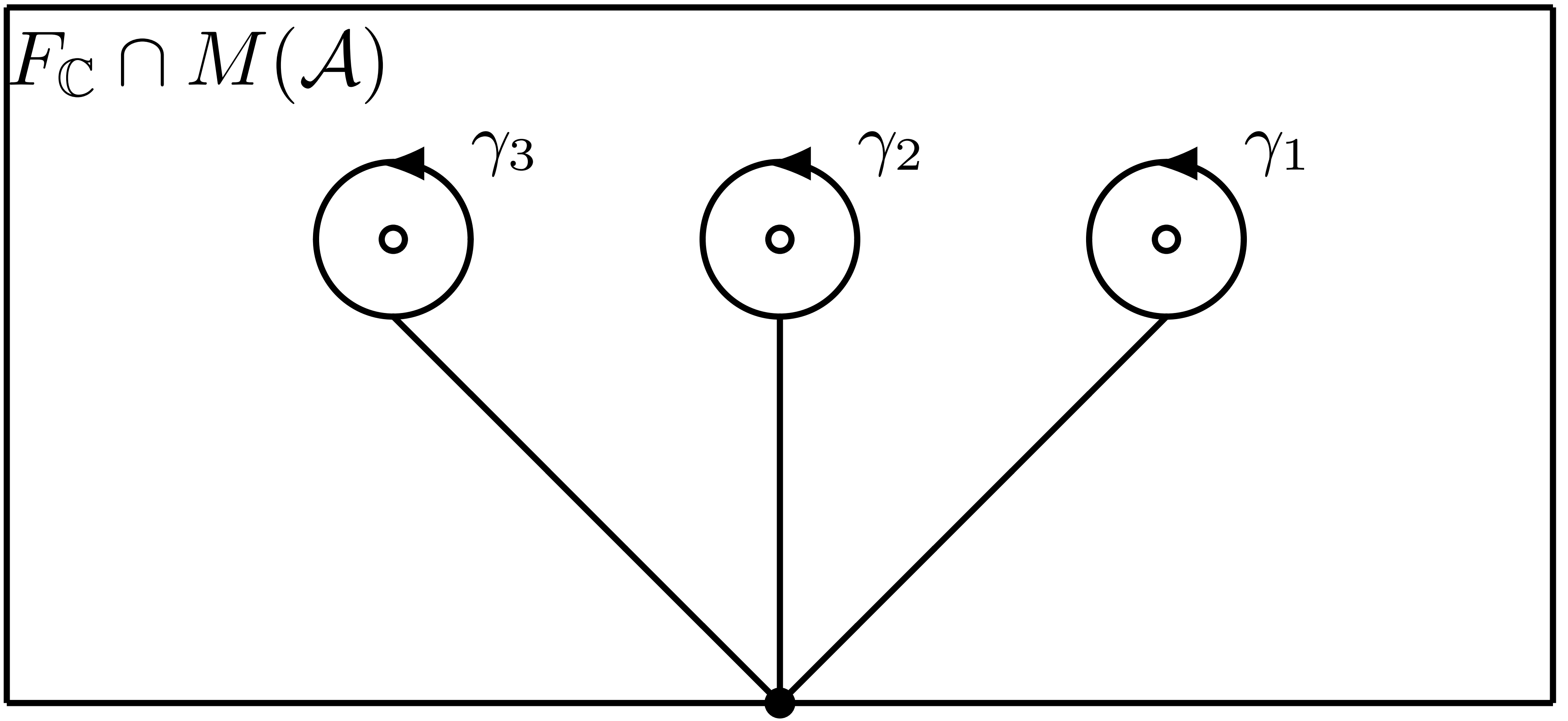


$\mathrm{ch}_F(\mathcal{A}) = \{C_1, C_2, C_3\}$

<u>Prop.</u> $|\mathrm{ch}_F(\mathcal{A})| = b_2(M)$

1 Minimal positive presentation

Generators: Meridians (with base $-\sqrt{-1}$).



1 Minimal positive presentation

Relations:

(1) Attach to $C \in \mathrm{ch}_F(\mathcal{A})$ a permutation $(i_1, i_2, \ldots, i_n)$ of $(1, \ldots, n)$ as

$$\underbrace{i_1 < \cdots < i_k}_{\text{Going through right side of } C}, \underbrace{i_{k+1} < \cdots < i_n}_{\text{Going through left side of } C}$$

(2) Associate to C a relation

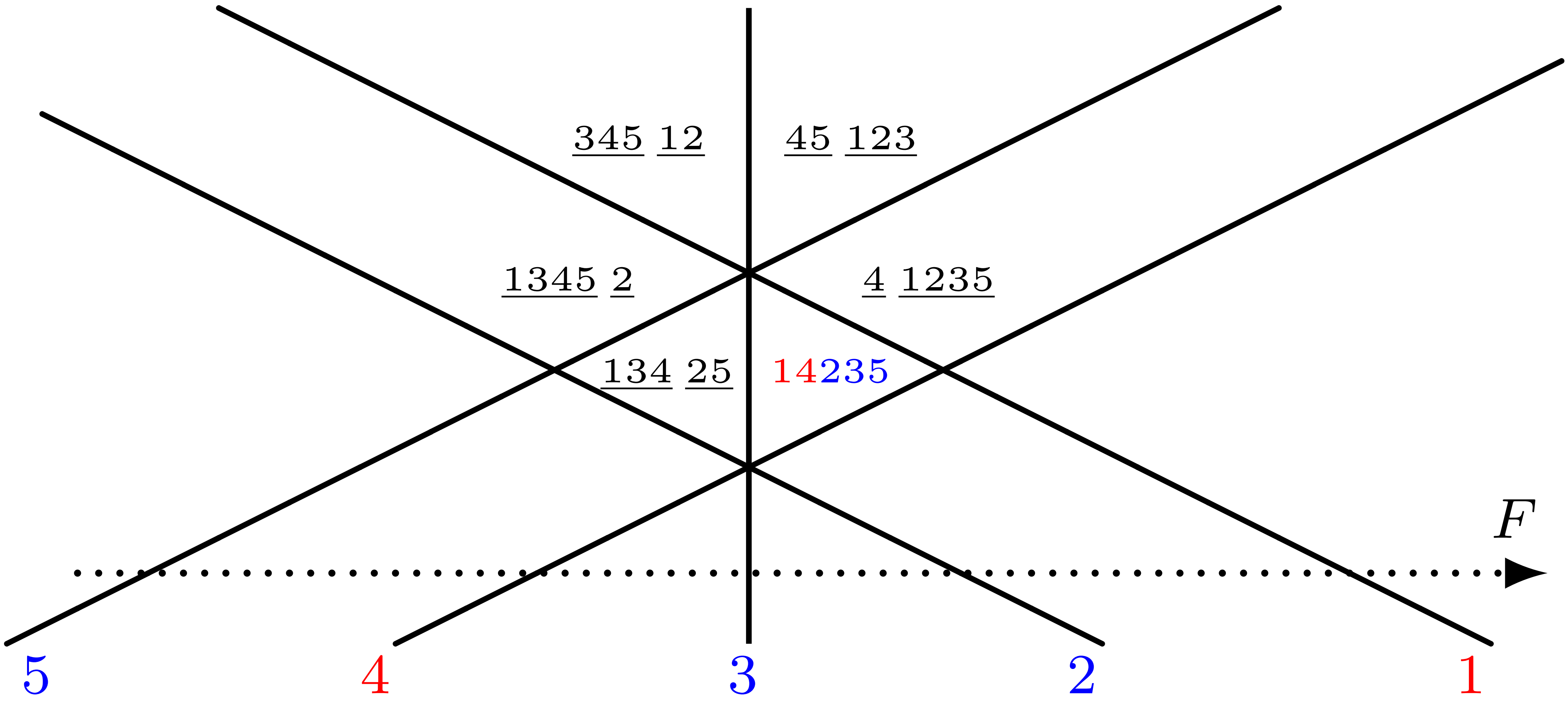
$$R(C) : \gamma_1\gamma_2 \ldots \gamma_n = \gamma_{i_1}\gamma_{i_2} \ldots \gamma_{i_n}.$$

Thm. $\pi_1(M) \cong \langle \gamma_1, \ldots, \gamma_n \mid R(C), C \in \mathrm{ch}_F(\mathcal{A}) \rangle$

1 Minimal positive presentation

Example

$$\pi_1(M) \cong \left\langle \gamma_1, \ldots, \gamma_5 \;\middle|\; \begin{array}{l} 12345 \\ = \color{red}{14}\color{blue}{235} = 13425 = 13452 \\ = 34512 = 45123 = 41235 \end{array} \right\rangle$$



1 Minimal positive presentation

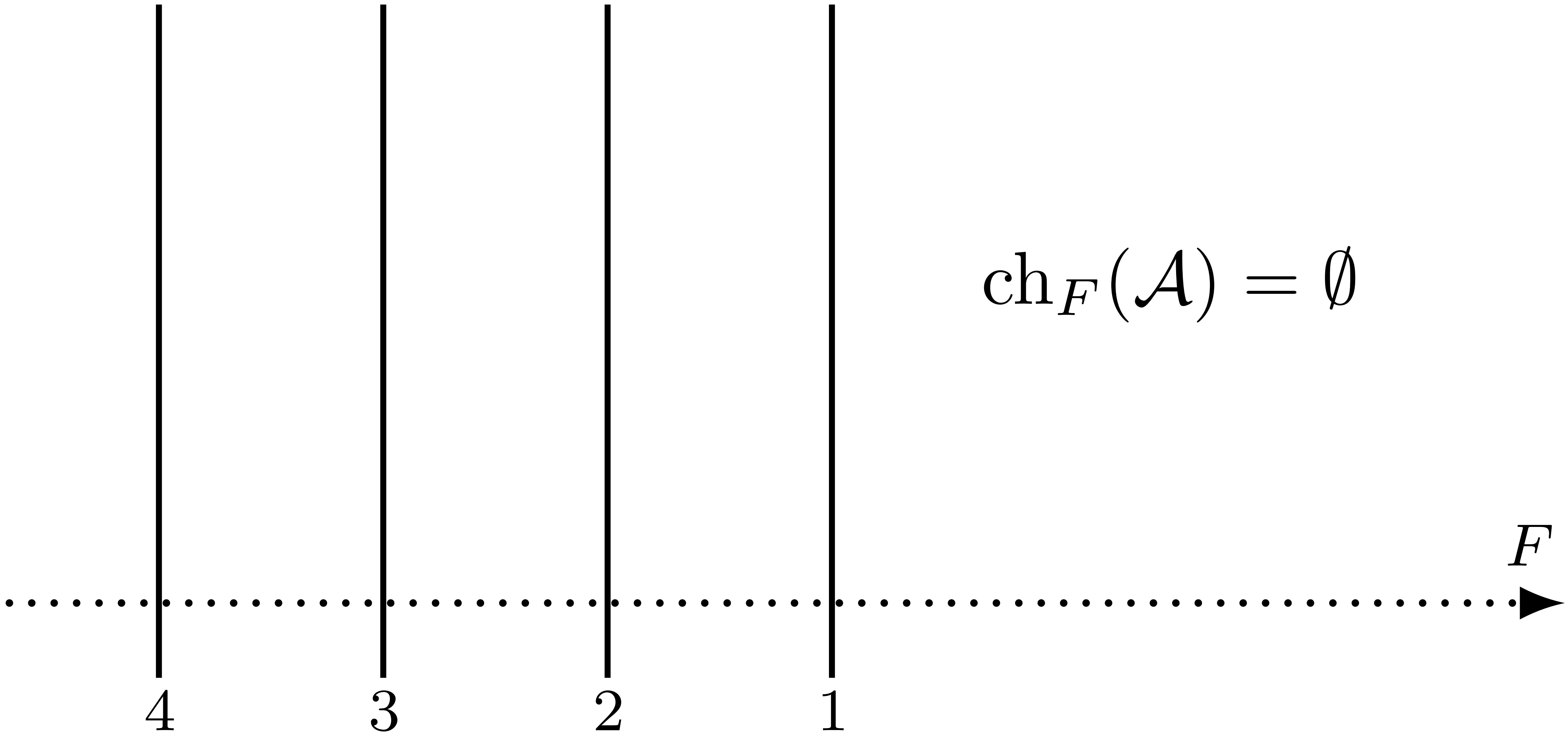
The correspondence "chamber$\rightarrow$relation" $C \longmapsto R(C)$ is natural in the following sense.

<u>Thm.</u> (Y. 2007) $\exists!$ continuous map (up to homotopy) $\sigma_C : (D^2, \partial D^2) \to (M, M \cap F_{\mathbb{C}})$ s.t.
(i) $\sigma_C(D^2) \pitchfork C = \{p_C\}$,
(ii) $\sigma_C(D^2) \pitchfork C' = \emptyset, (C' \in \mathrm{ch}_F \setminus \{C\})$.

We can read $R(C)$ from $\sigma_C|_{\partial D^2}$. We will give outline of another proof (in §2).

1 Minimal positive presentation

Example

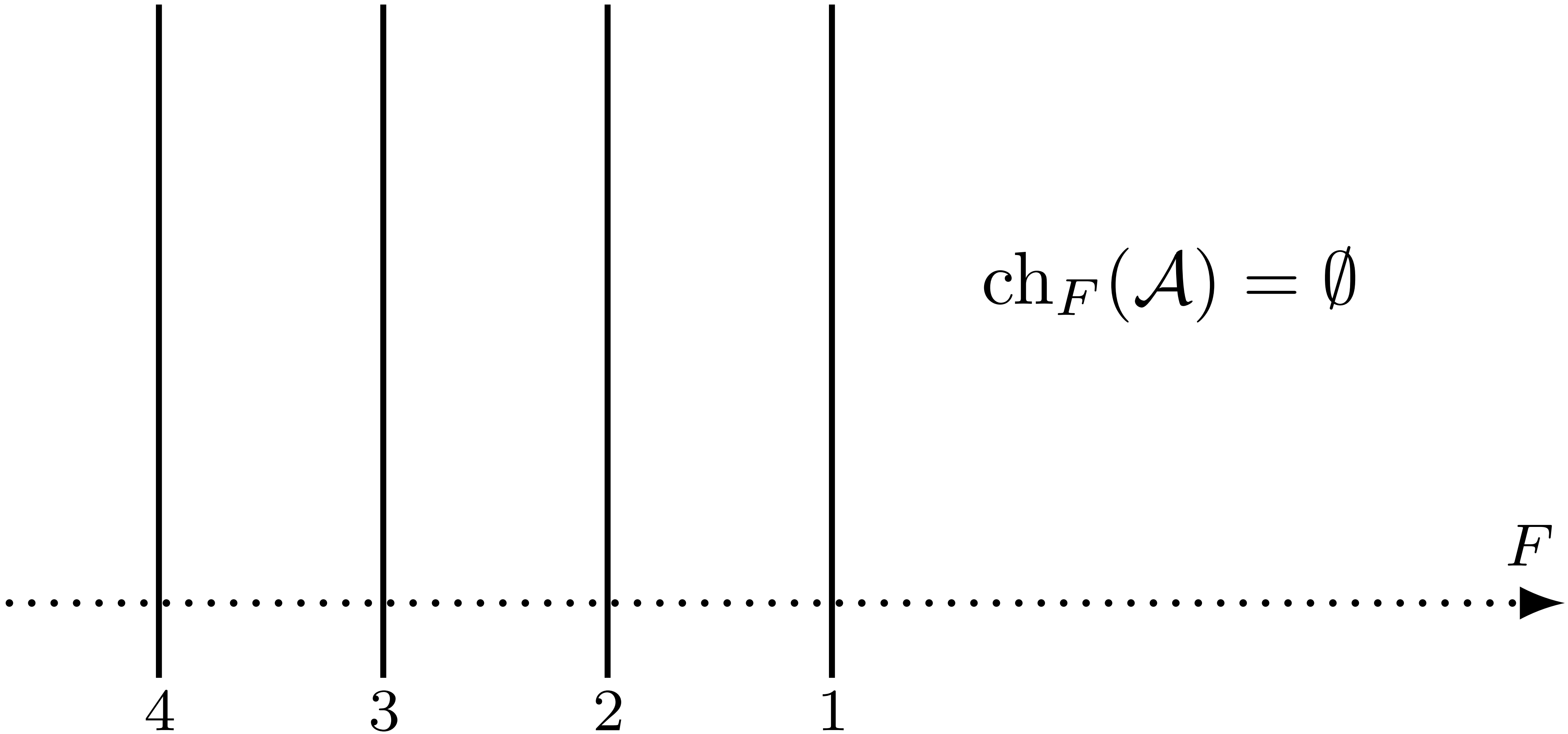


$$\mathrm{ch}_F(\mathcal{A}) = \emptyset$$

1 Minimal positive presentation

Example

$$\pi_1(M) \cong \langle \gamma_1, \gamma_2, \gamma_3, \gamma_4 \mid \text{no relations} \rangle$$



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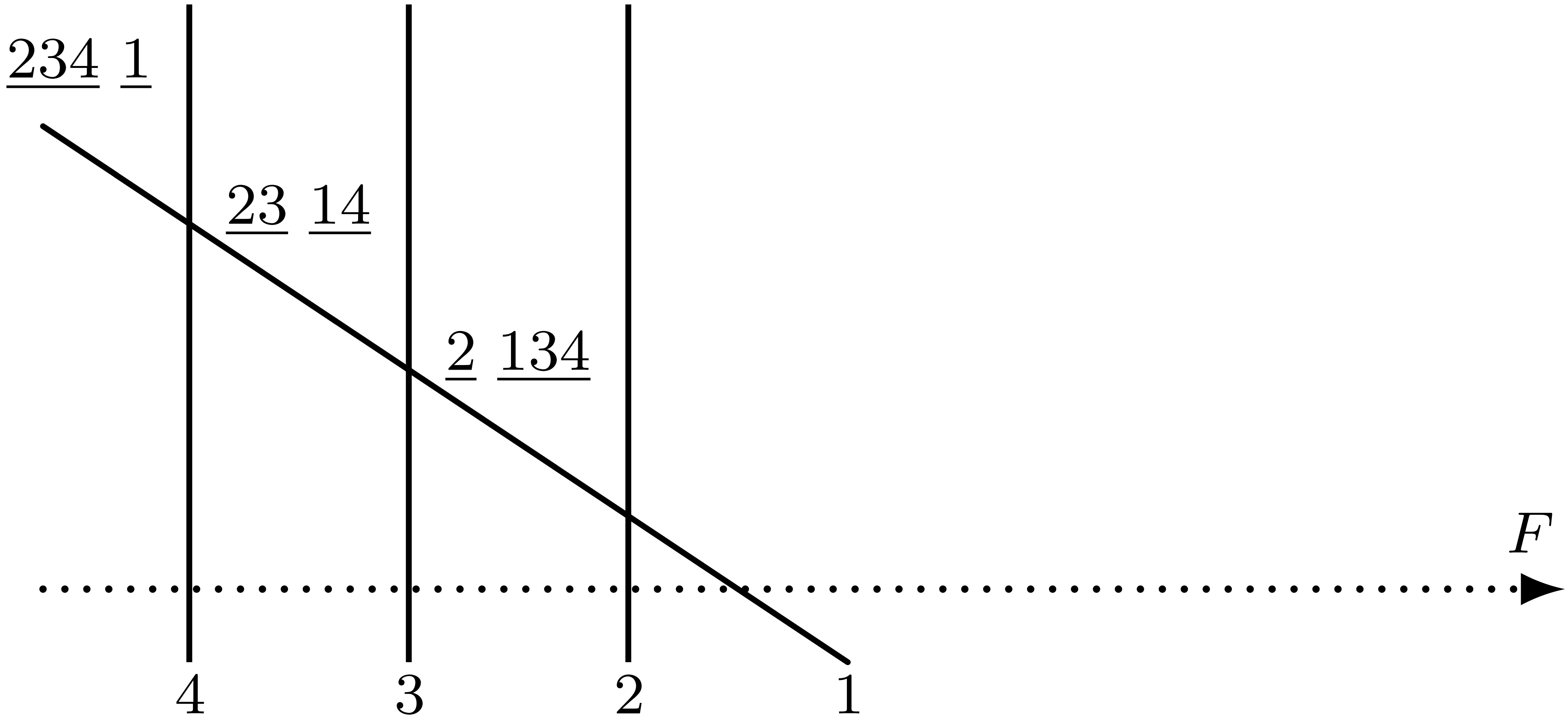
Example

$$\pi_1(M) \cong \left\langle \gamma_1, \gamma_2, \gamma_3, \gamma_4 \;\middle|\; \begin{array}{l} 1234 = 2134 \\ = 2314 = 2341 \end{array} \right\rangle$$



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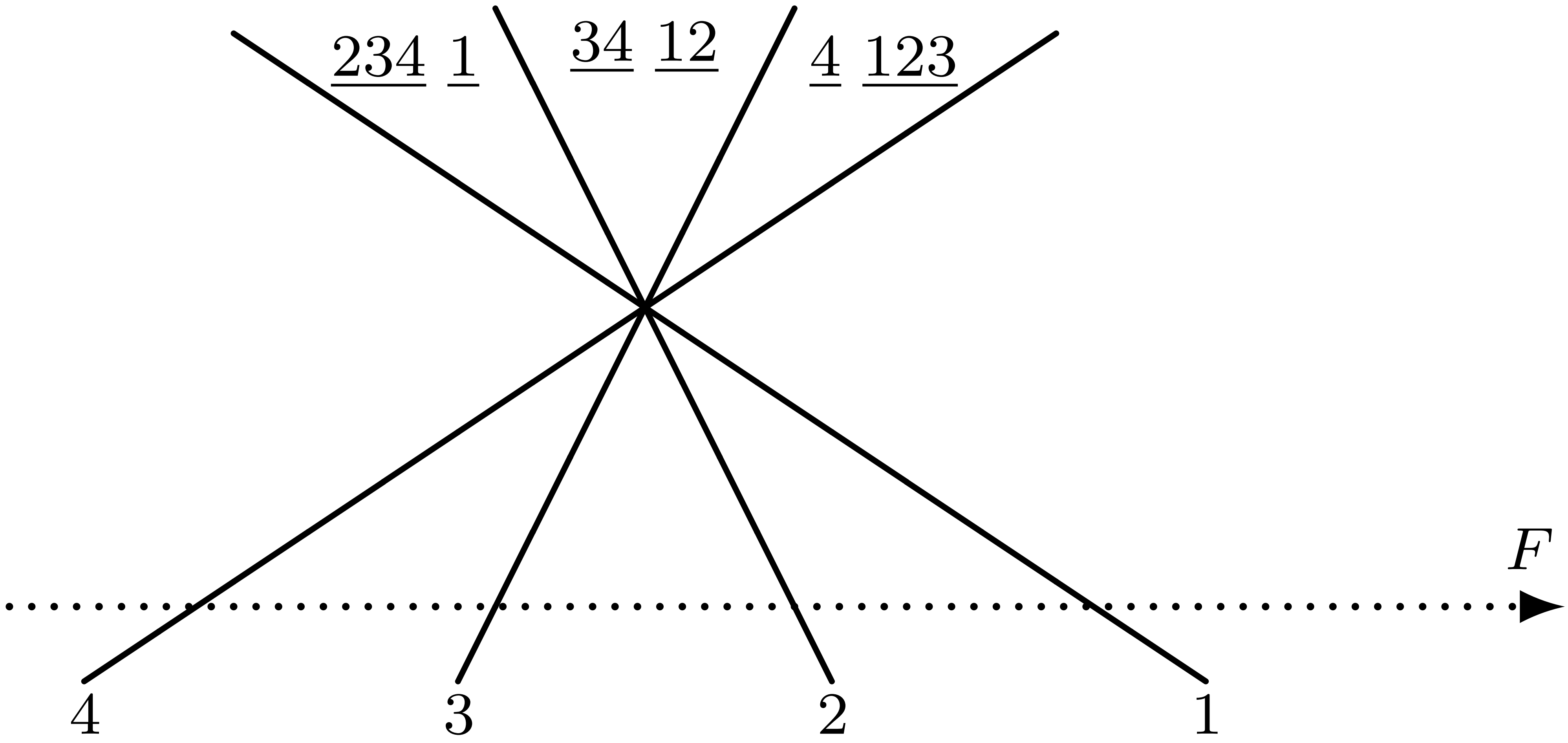
Example

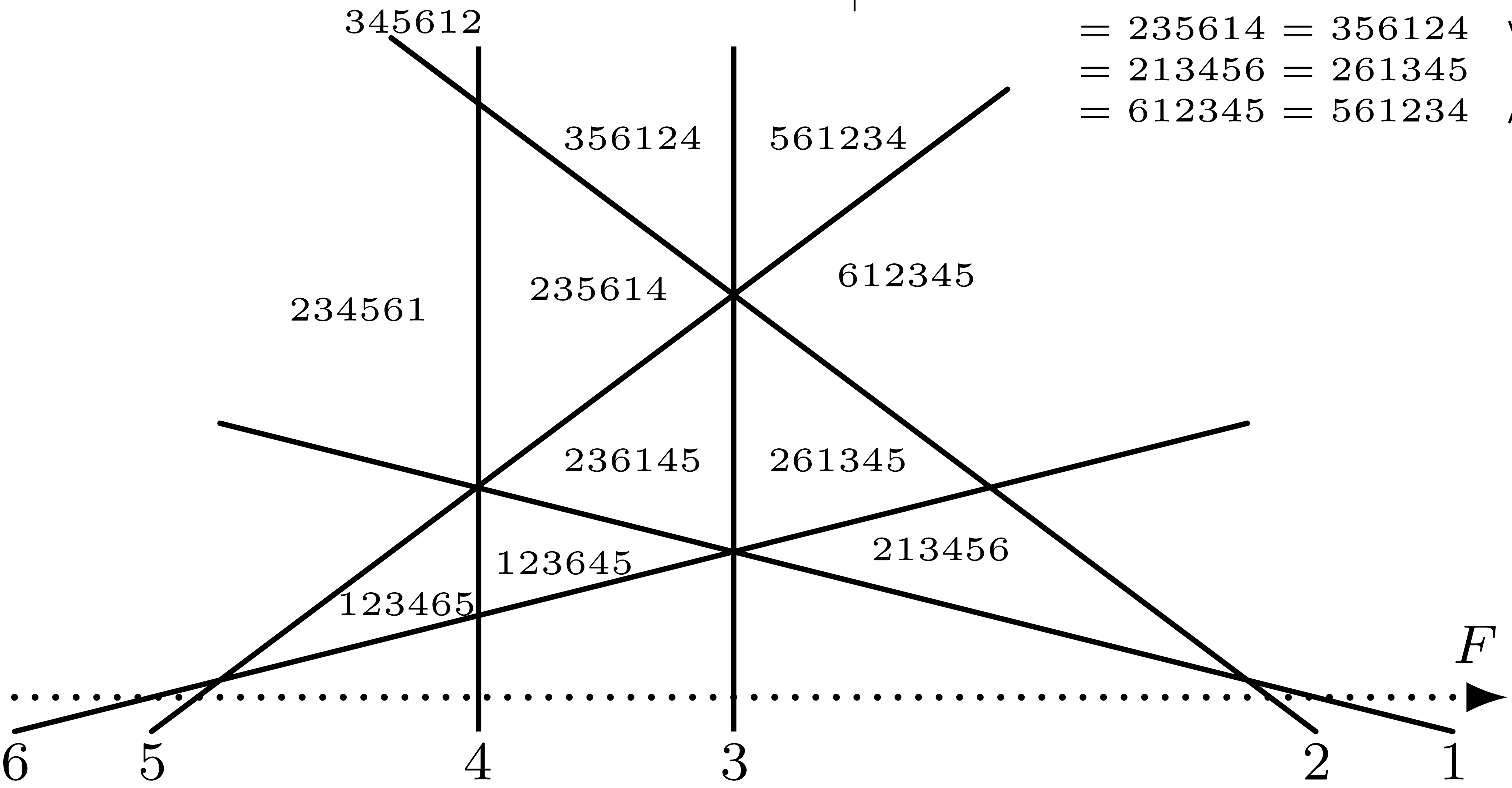
$$\pi_1(M) \cong \left\langle \gamma_1, \gamma_2, \gamma_3, \gamma_4 \;\middle|\; \begin{array}{l} 1234 = 2341 \\ = 3412 = 4123 \end{array} \right\rangle$$



1 Minimal positive presentation

Example

$$\pi_1(M) \cong \left\langle \gamma_1, \ldots, \gamma_6 \;\middle|\; \begin{array}{l} 123456 = 123465 \\ = 234561 = 345612 \\ = 123645 = 236145 \\ \qquad = 235614 = 356124 \\ \qquad = 213456 = 261345 \\ \qquad = 612345 = 561234 \end{array} \right\rangle$$



1 Minimal positive presentation

Remark A presentation of group

$$G = \langle \gamma_1, \ldots, \gamma_n \mid R_1, \ldots, R_b \rangle$$

is called *minimal* if $n = b_1(G)$ and $b = b_2(G)$.

- Randell, Falk: Minimal presentation for $\pi_1(M(\mathcal{A}))$.
- Dimca, Papadima, Suciu, Randell: minimality of $M(\mathcal{A})$.
- Generally, the relations have conjugations.

- Eliyahu, Garber, Teicher, “Conjugation-free geometric presentatioin.”
- Homogeneously presented monoids have solvable word problem. However our monoids are rarely embedded in the group.

1 Minimal positive presentation

Corollary. Let

$$\pi_1(M(\mathcal{A})) = \langle \gamma_1, \ldots, \gamma_n \mid R_1, \ldots, R_b \rangle$$

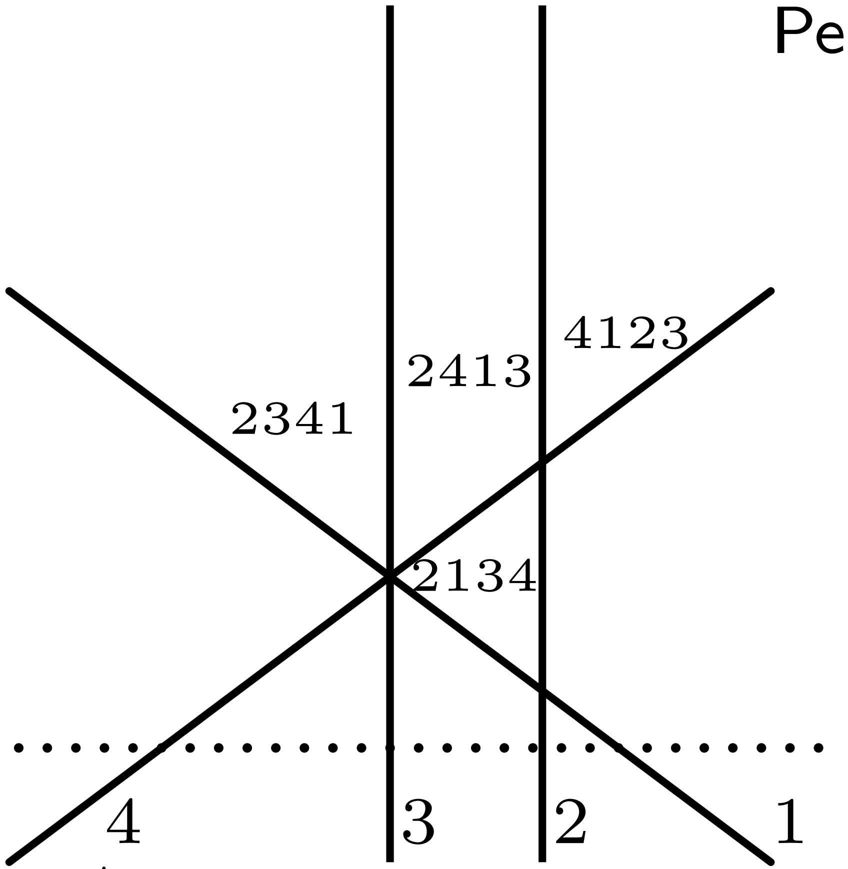
be a minimal presentation. Then there are $(\frac{n(n-1)}{2} - b)$-words $R_{b+1}, \ldots, R_{\frac{n(n-1)}{2}}$ such that

$$\mathbb{Z}^n \cong \langle \gamma_1, \ldots, \gamma_n \mid R_1, \ldots, R_{\frac{n(n-1)}{2}} \rangle.$$

Remark. Not all minimally presented groups have this property, e.g., $G = \left\langle \gamma_1, \gamma_2 \mid [\gamma_1, \gamma_2]^2 \right\rangle$.

1 Minimal positive presentation

(Proof of Cor.)

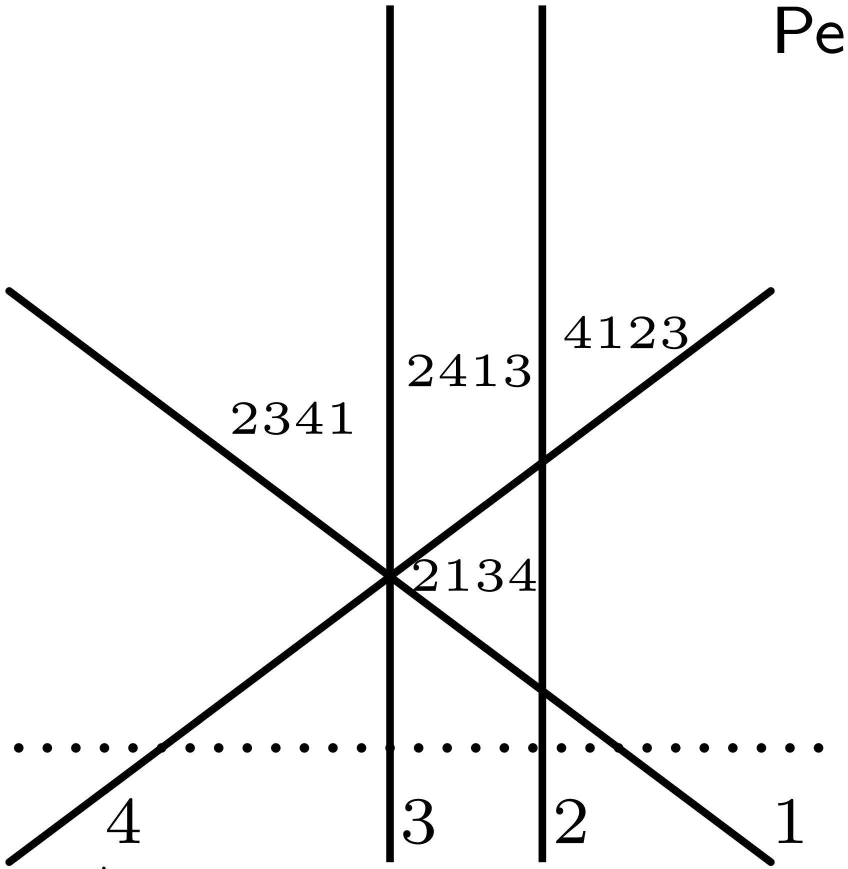


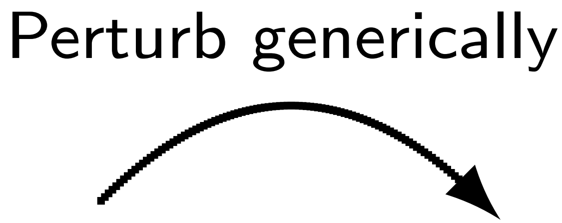
Perturb generically

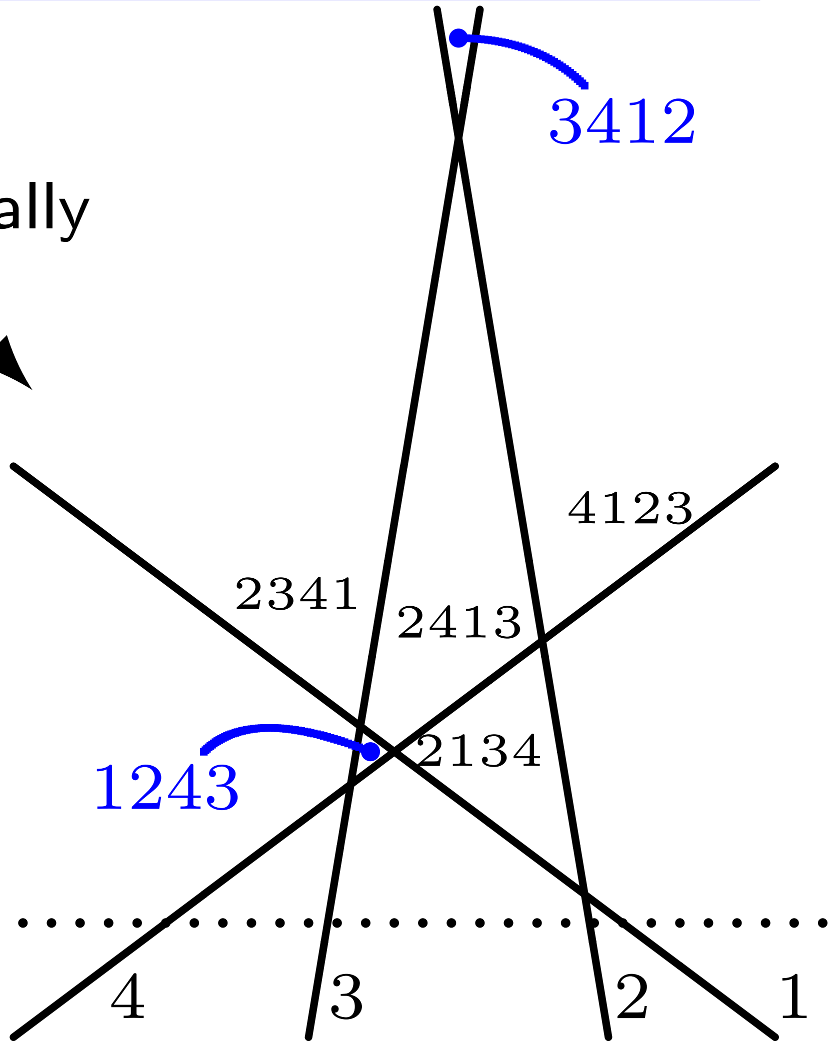
$$\left\langle \gamma_i \;\middle|\; \begin{array}{l} 1234 \\ = 2134 = 2341 \\ = 2413 = 4123 \end{array} \right\rangle$$

1 Minimal positive presentation

(Proof of Cor.)







$$\left\langle \gamma_i \,\middle|\, \begin{array}{l} 1234 \\ = 2134 = 2341 \\ = 2413 = 4123 \end{array} \right\rangle \to \left\langle \gamma_i \,\middle|\, \begin{array}{l} 1234 \\ = 2134 = 2341 = 1243 \\ = 2413 = 4123 = 3412 \end{array} \right\rangle = \mathbb{Z}^4.$$

(Q.E.D.)

2 Minimal Stratification

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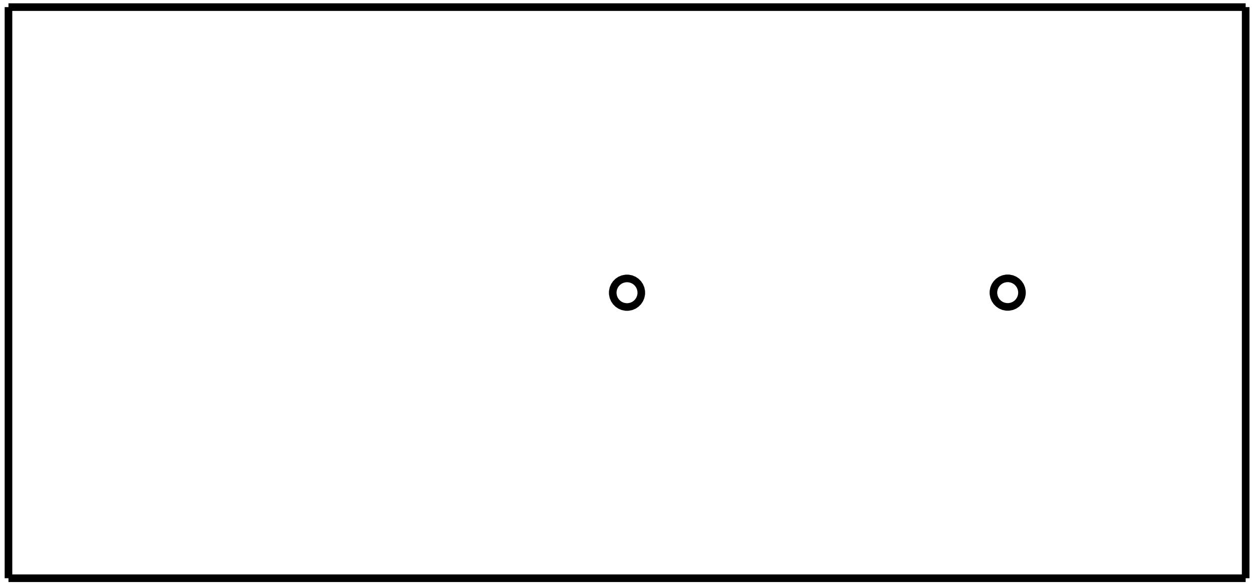
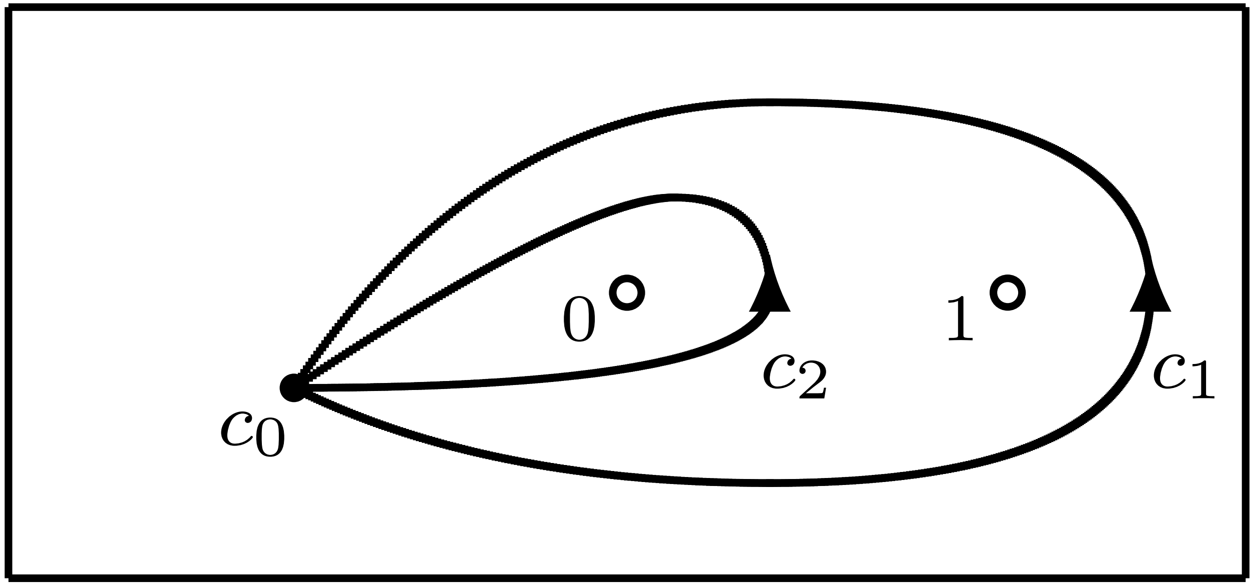


$$\mathcal{A} = \{0, 1\}$$

$$M(\mathcal{A}) = \mathbb{C} \setminus \{0, 1\}$$

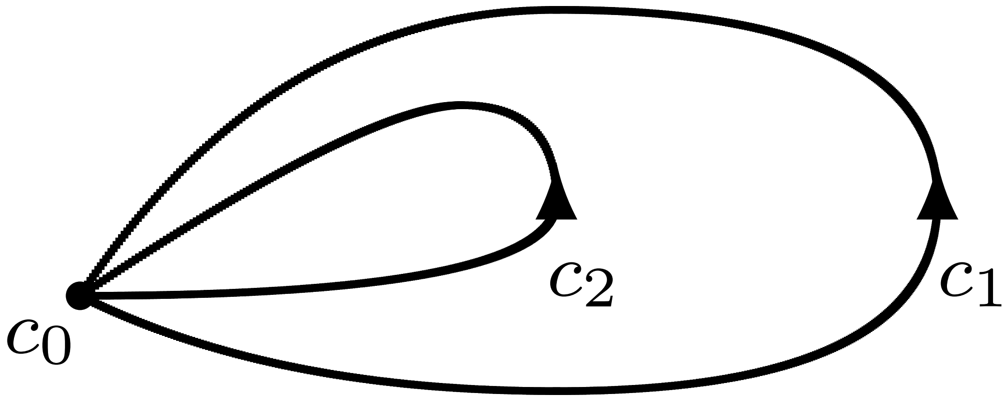
2 Minimal Stratification



$\mathcal{A} = \{0, 1\}$

$M(\mathcal{A}) = \mathbb{C} \setminus \{0, 1\}$

$\wr\wr$ Homotopy equivalence

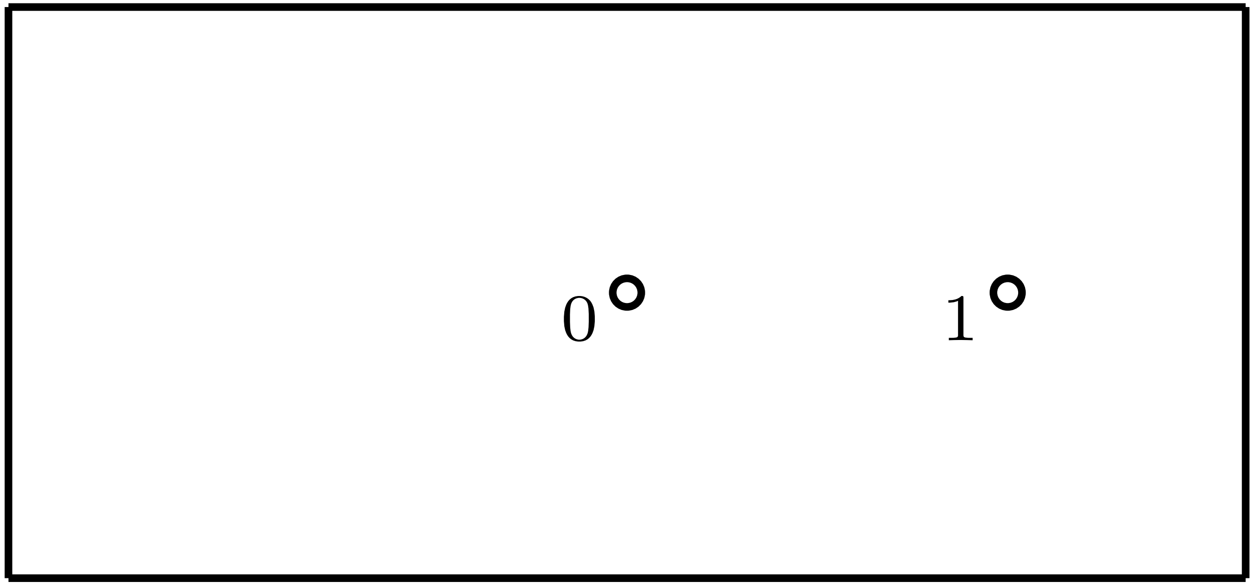


$\pi_1(M) = \langle c_1, c_2 \rangle$

2 Minimal Stratification

Instead of looking at homotopy equivalent CW complex, we consider stratification with contractible strata.

2 Minimal Stratification



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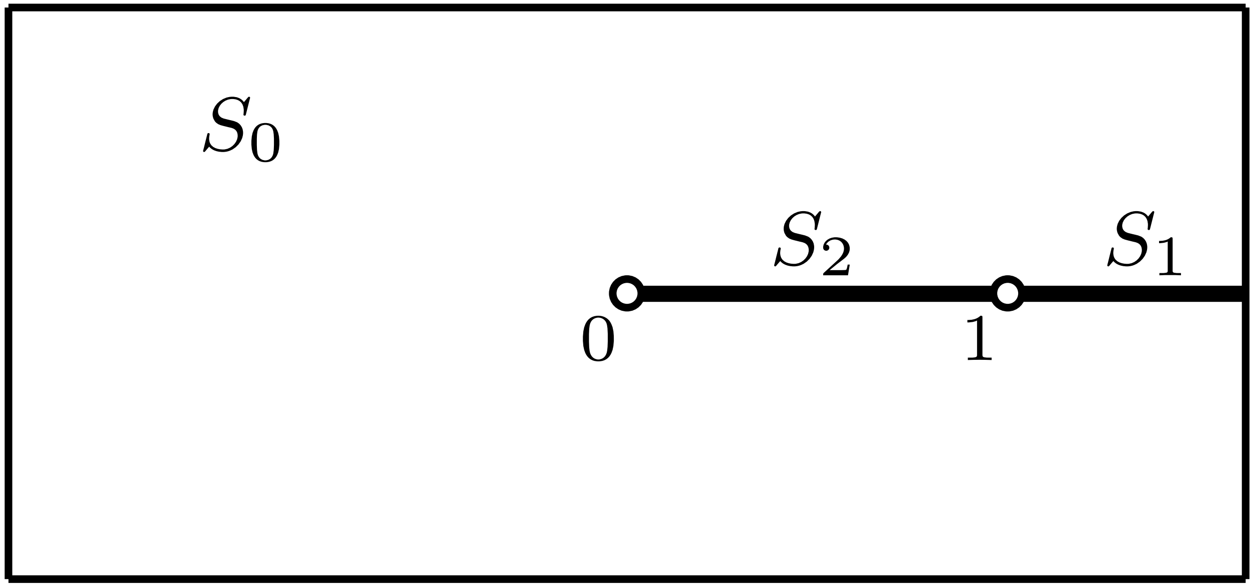
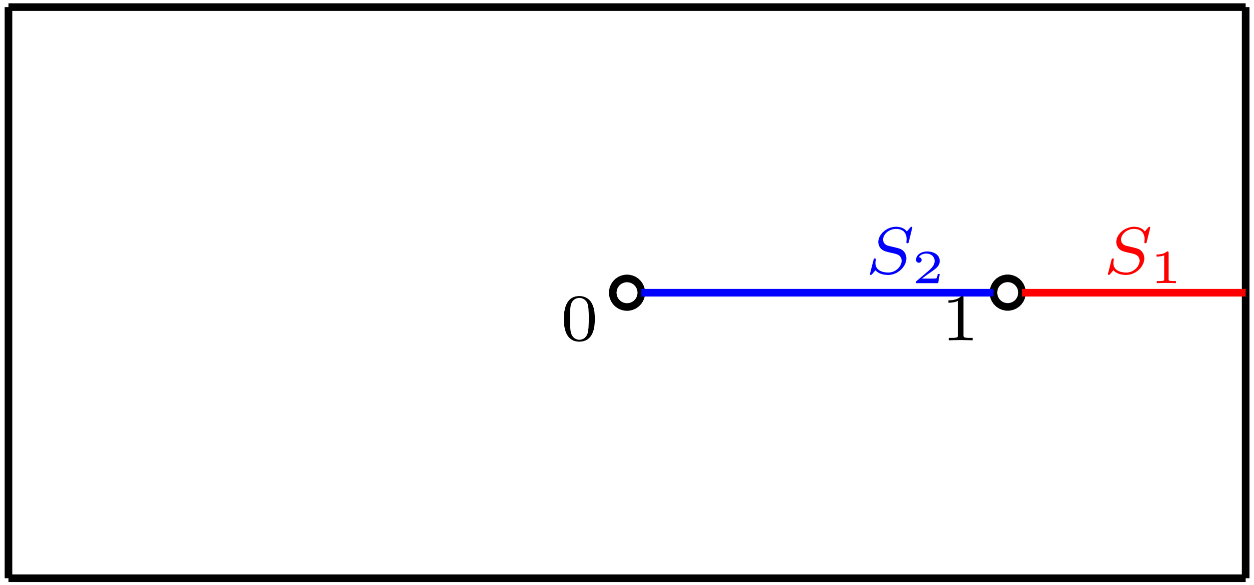
|| Stratifying by contractible sets



$$S_1 = (1, \infty)$$
$$S_2 = (0, 1)$$
$$S_0 = \mathbb{C} \setminus \mathbb{R}_{\geq 0}$$

$$M = S_0 \sqcup S_1 \sqcup S_2$$

2 Minimal Stratification



$\mathcal{A} = \{0, 1\}$

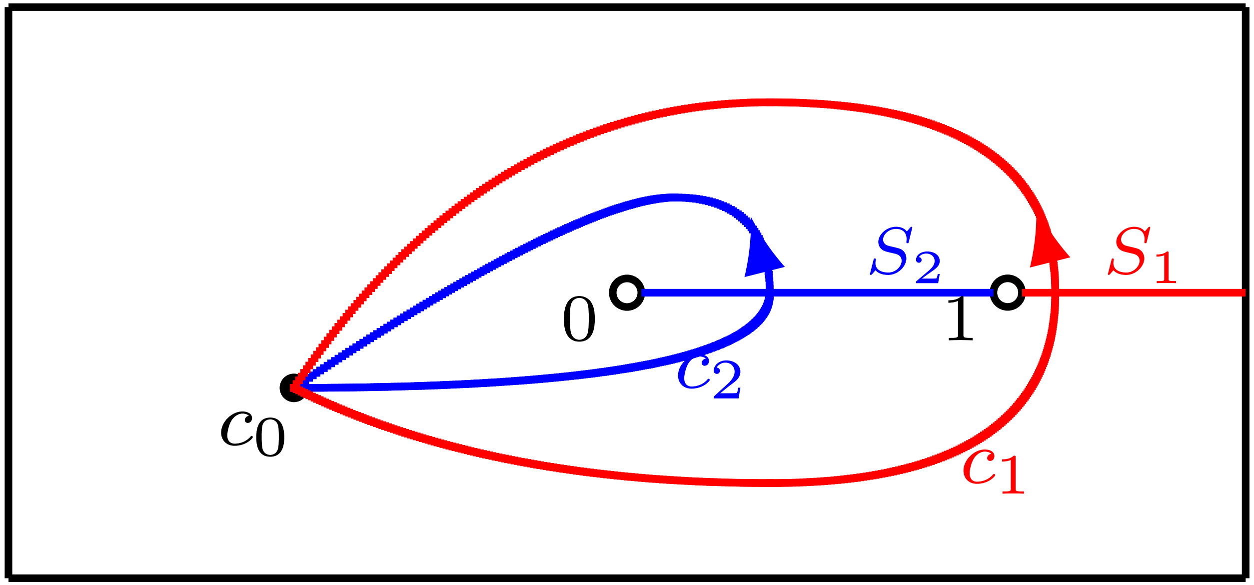
$M(\mathcal{A}) = \mathbb{C} \setminus \{0, 1\}$

$S_2 = (0, 1)$

$S_1 = (1, \infty)$

How to recover π_1 from stratification?

2 Minimal Stratification



$\mathcal{A} = \{0, 1\}$

$M(\mathcal{A}) = \mathbb{C} \setminus \{0, 1\}$

$S_2 = (0, 1)$

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Since strata are contractible,
S_1 and S_2 determine transversal generators
c_1 and c_2 uniquely (up to homotopy).

How to recover π_1 from stratification?

2 Minimal Stratification

Example. Σ_2: surface of genus $g = 2$.

Computing $\pi_1(\Sigma_2)$ from stratification.

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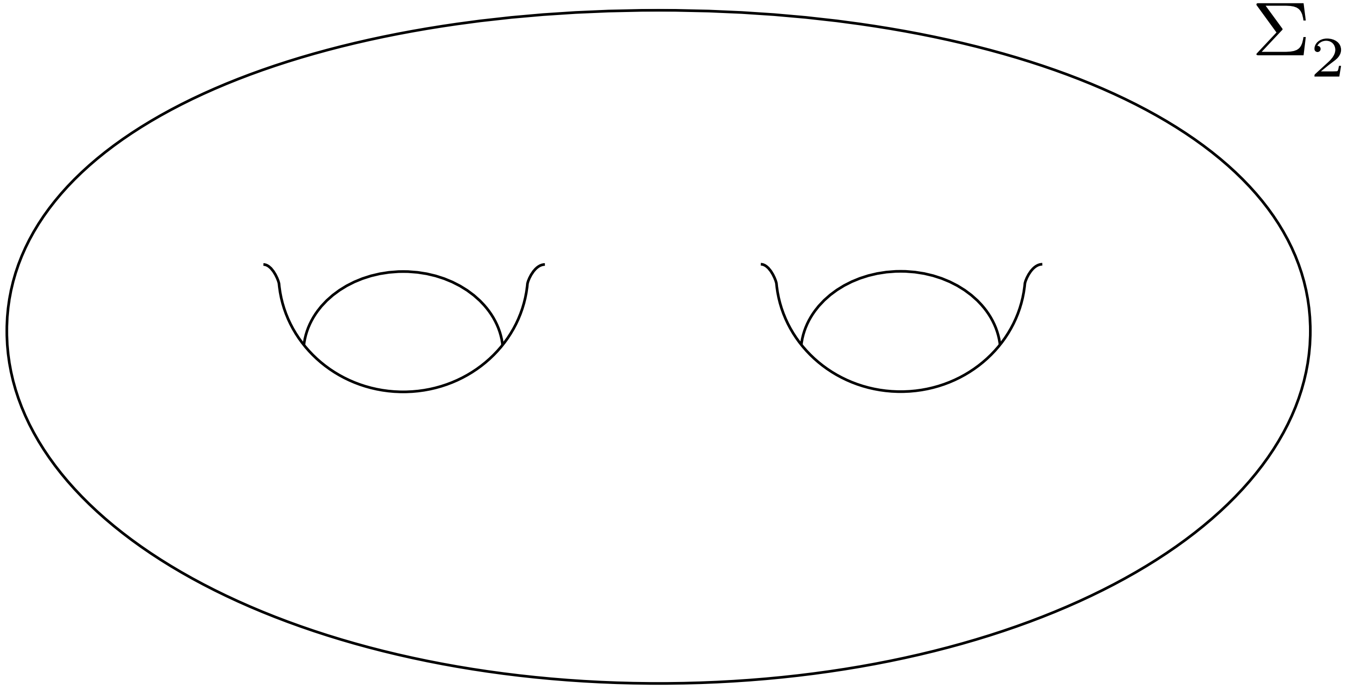

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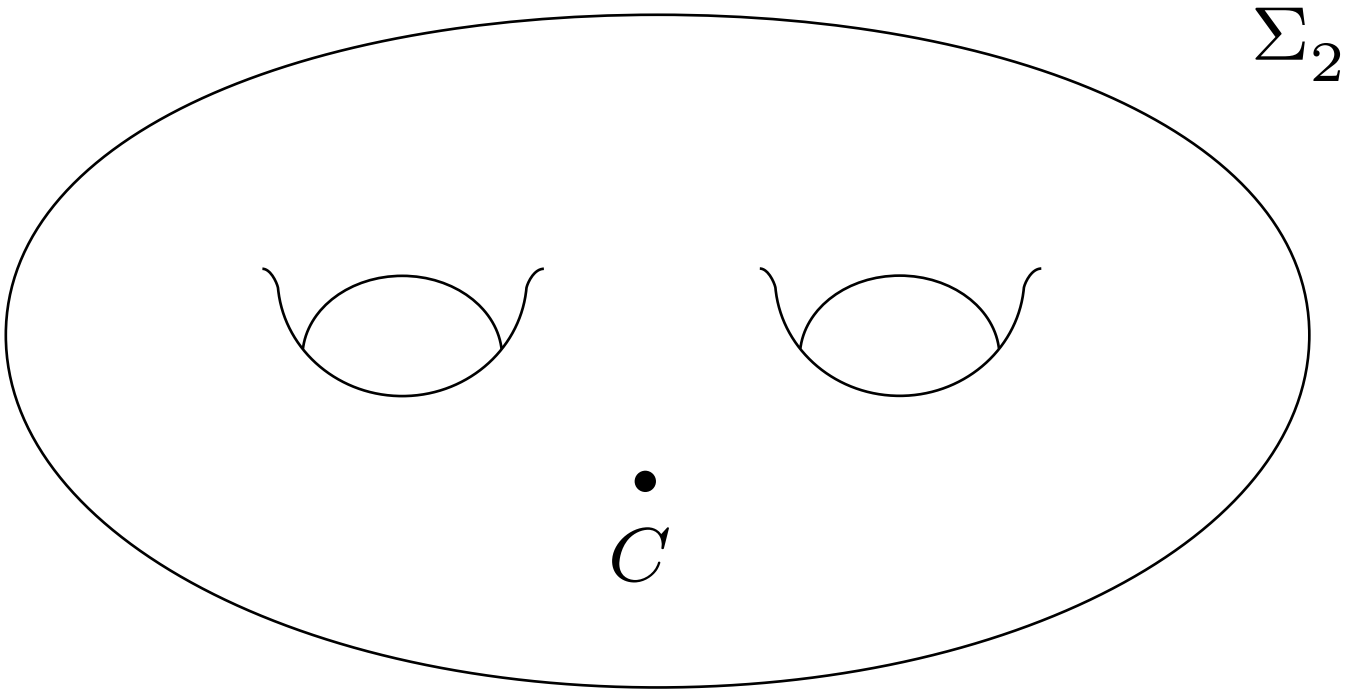

C: 0-dim stratum.

2 Minimal Stratification

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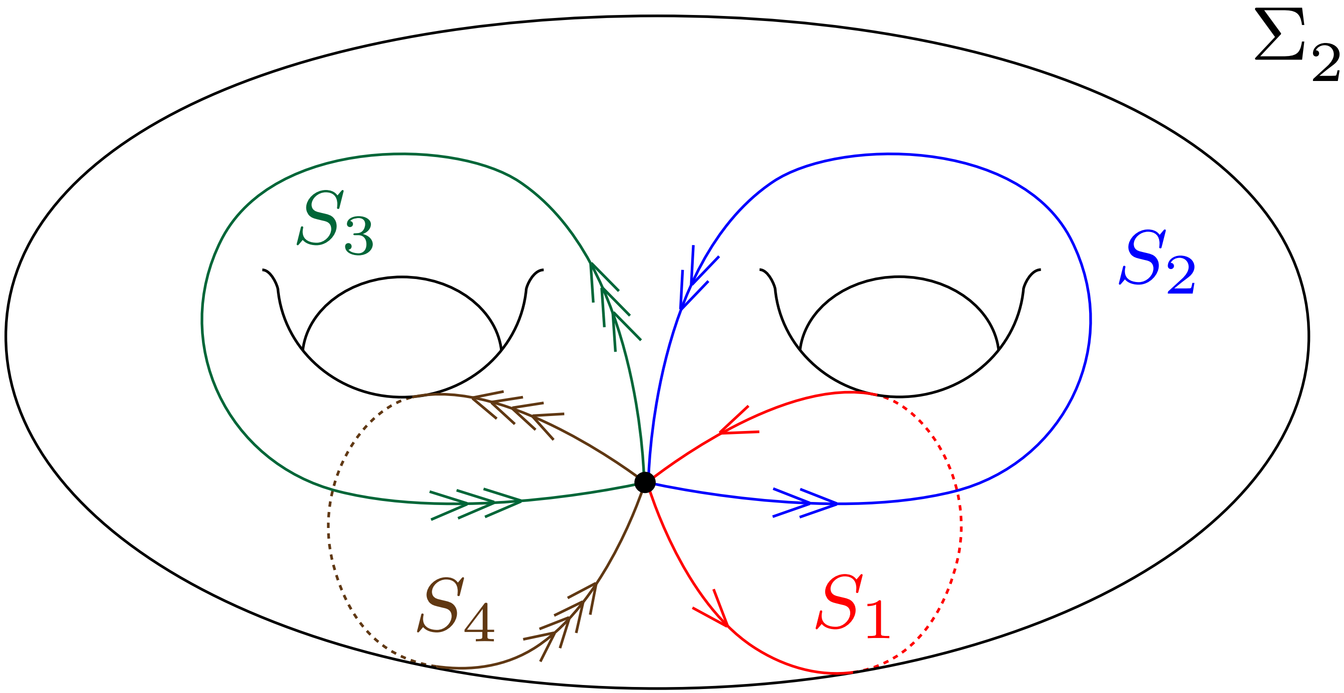

S_i: 1-dim strata.

2 Minimal Stratification

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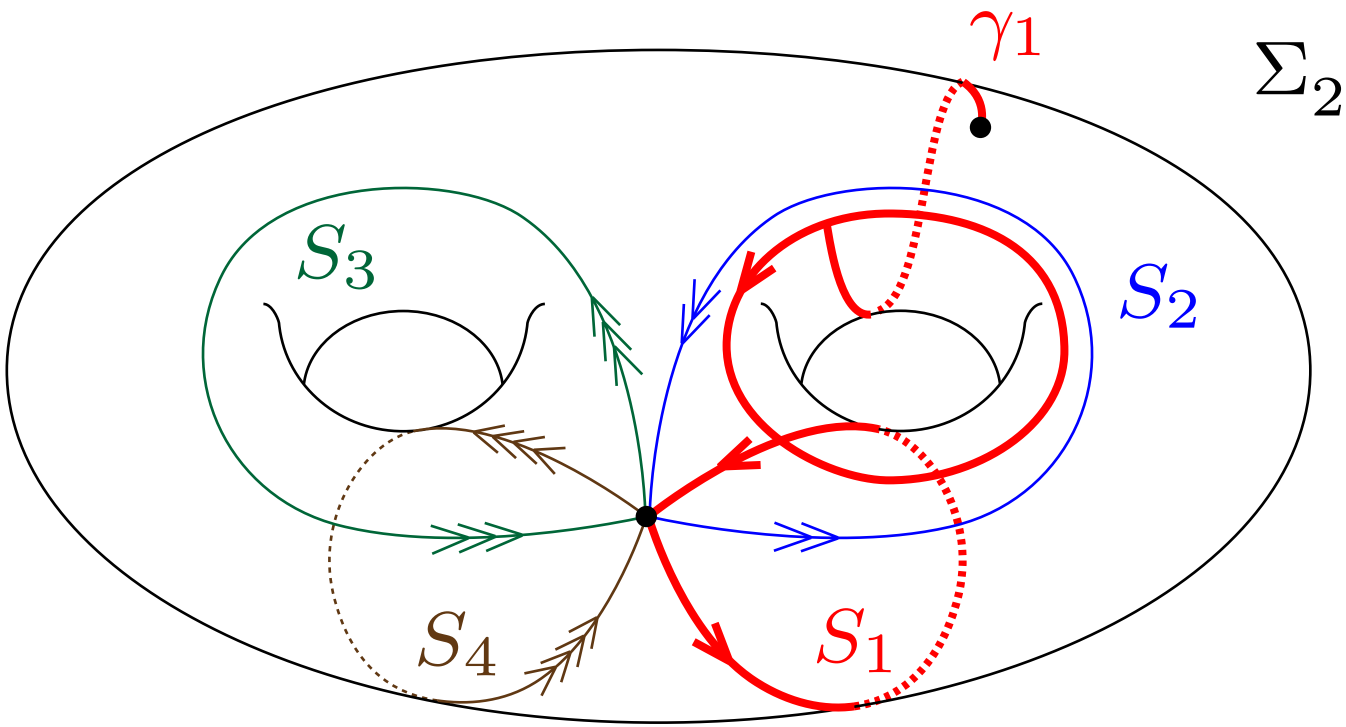

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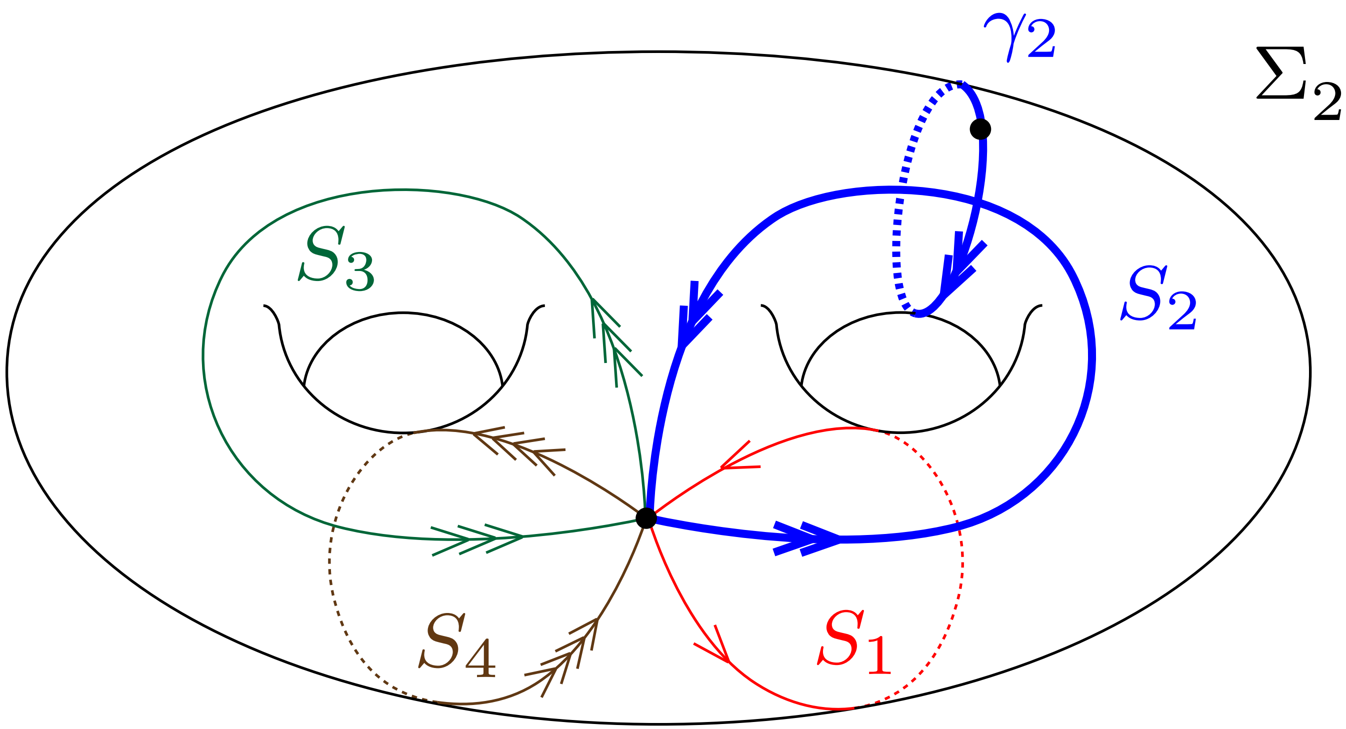

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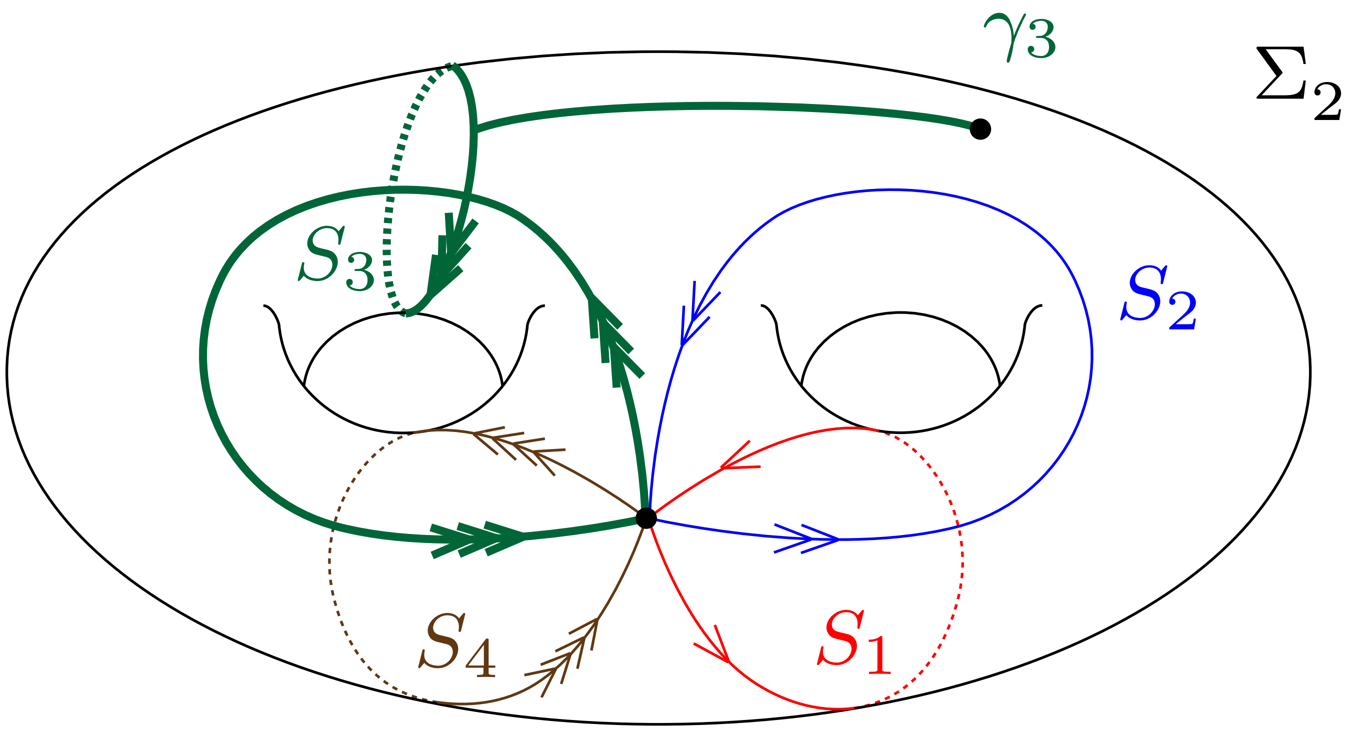

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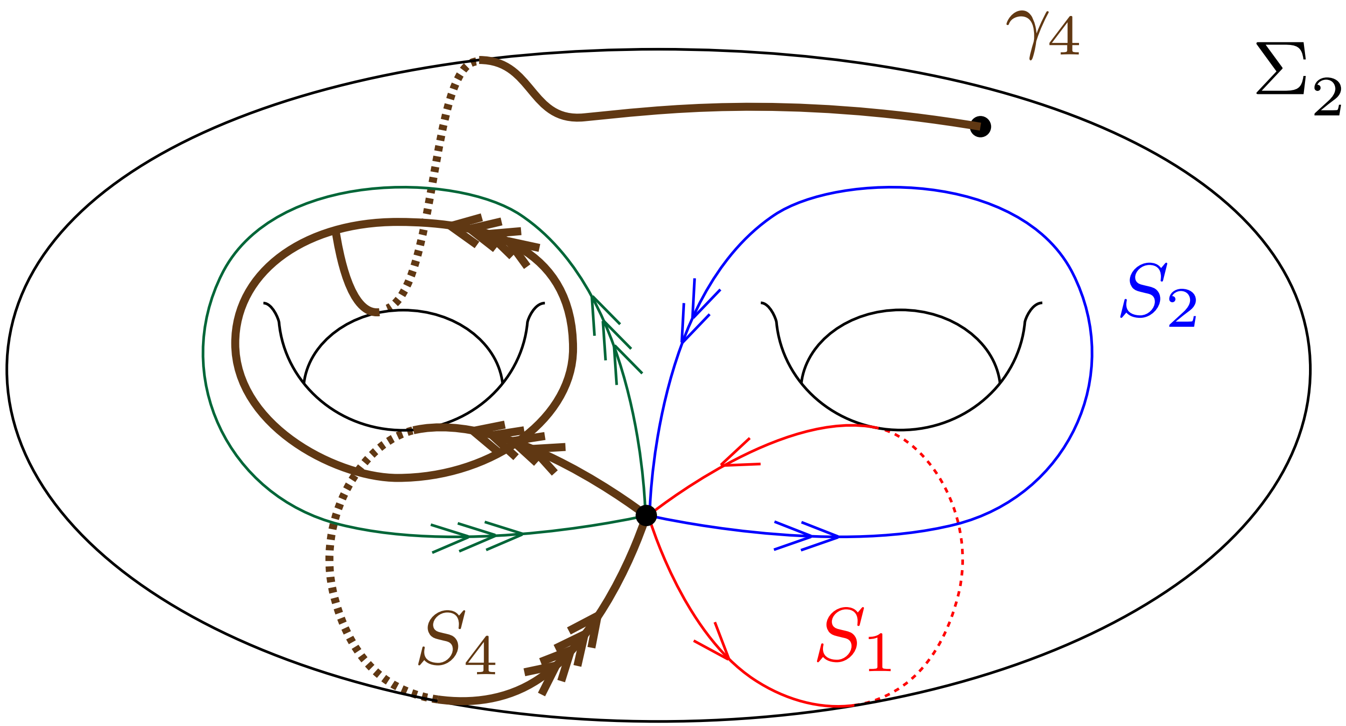


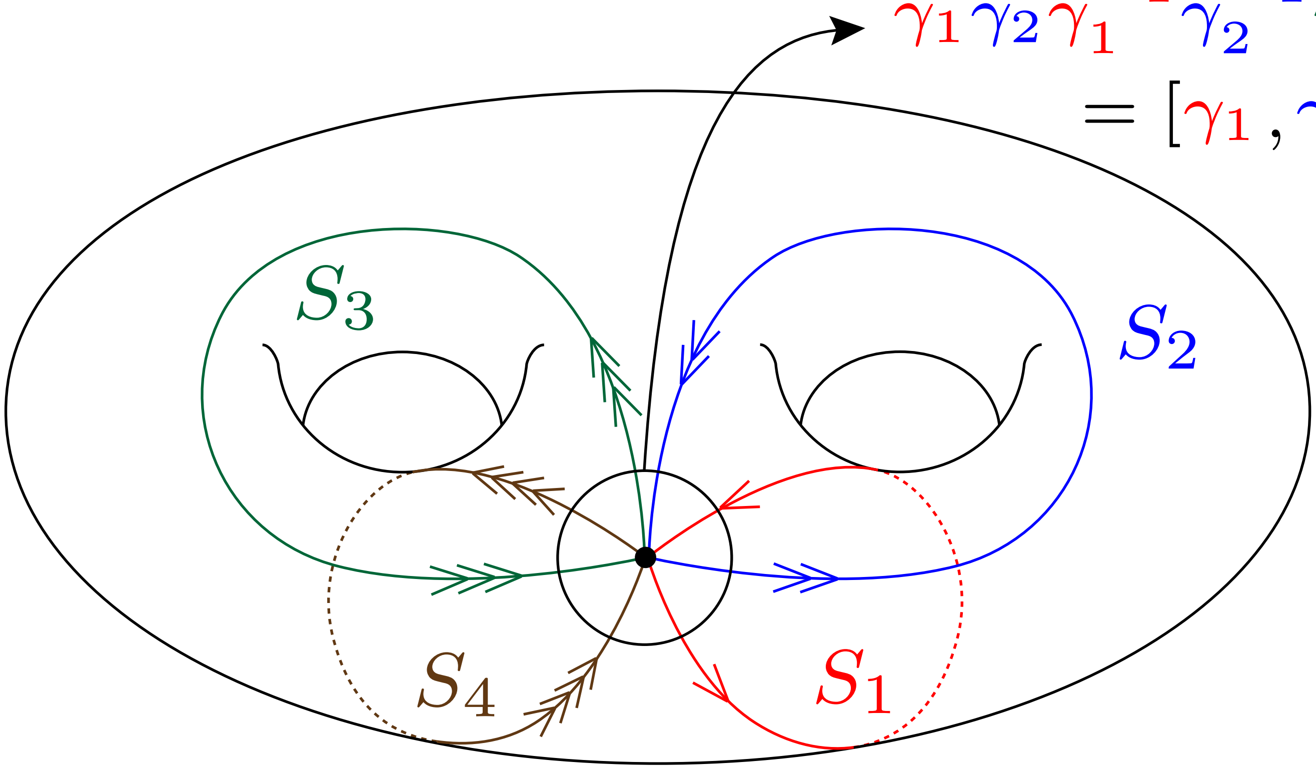
Where is the relations?

2 Minimal Stratification

Example. Σ_2: surface of genus $g = 2$.

Computing $\pi_1(\Sigma_2)$ from stratification.

$$\gamma_1\gamma_2\gamma_1^{-1}\gamma_2^{-1}\gamma_3\gamma_4\gamma_3^{-1}\gamma_4^{-1} = [\gamma_1, \gamma_2][\gamma_3, \gamma_4] = 1$$



Look at $\mathrm{codim} = 2$ stratum.

2 Minimal Stratification

To compute $\pi_1(M(\mathcal{A}))$, we will follow the Strategy:

1. Stratify $M = \bigsqcup_\lambda S_\lambda$ by contractible strata.
2. Generator $\longleftrightarrow \mathrm{codim} = 1$ strata.
3. Relation $\longleftrightarrow \mathrm{codim} = 2$ strata.

2 Minimal Stratification

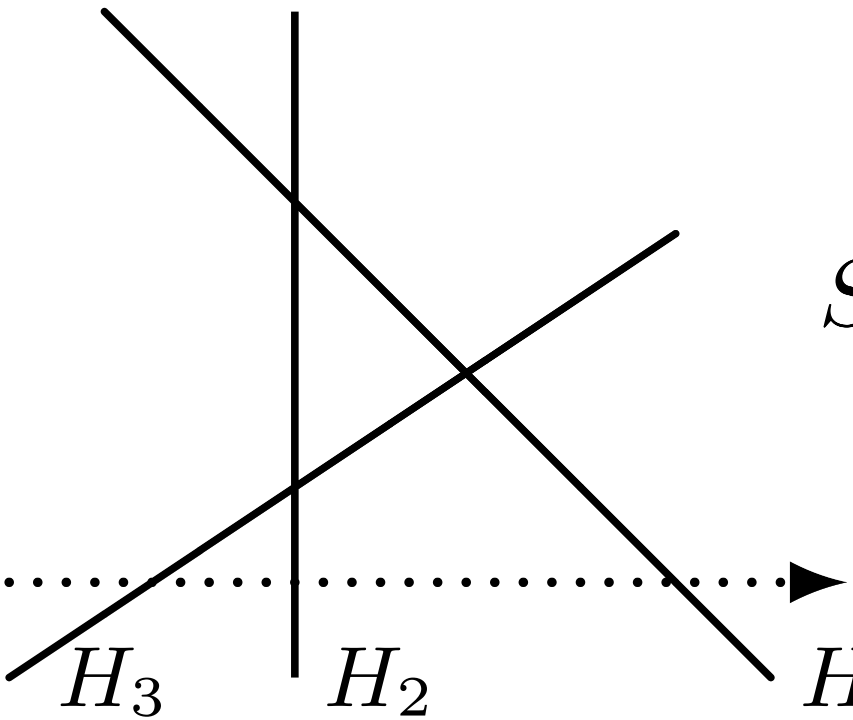
Stratify by using the following objects.

$$S_1 := \{z \in M \mid \alpha_1(z) \in \mathbb{R}_{>0}\},$$

$$S_2 := \left\{z \in M \,\middle|\, \frac{\alpha_2(z)}{\alpha_1(z)} \in \mathbb{R}_{<0}\right\},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$S_n := \left\{z \in M \,\middle|\, \frac{\alpha_n(z)}{\alpha_{n-1}(z)} \in \mathbb{R}_{<0}\right\}.$$



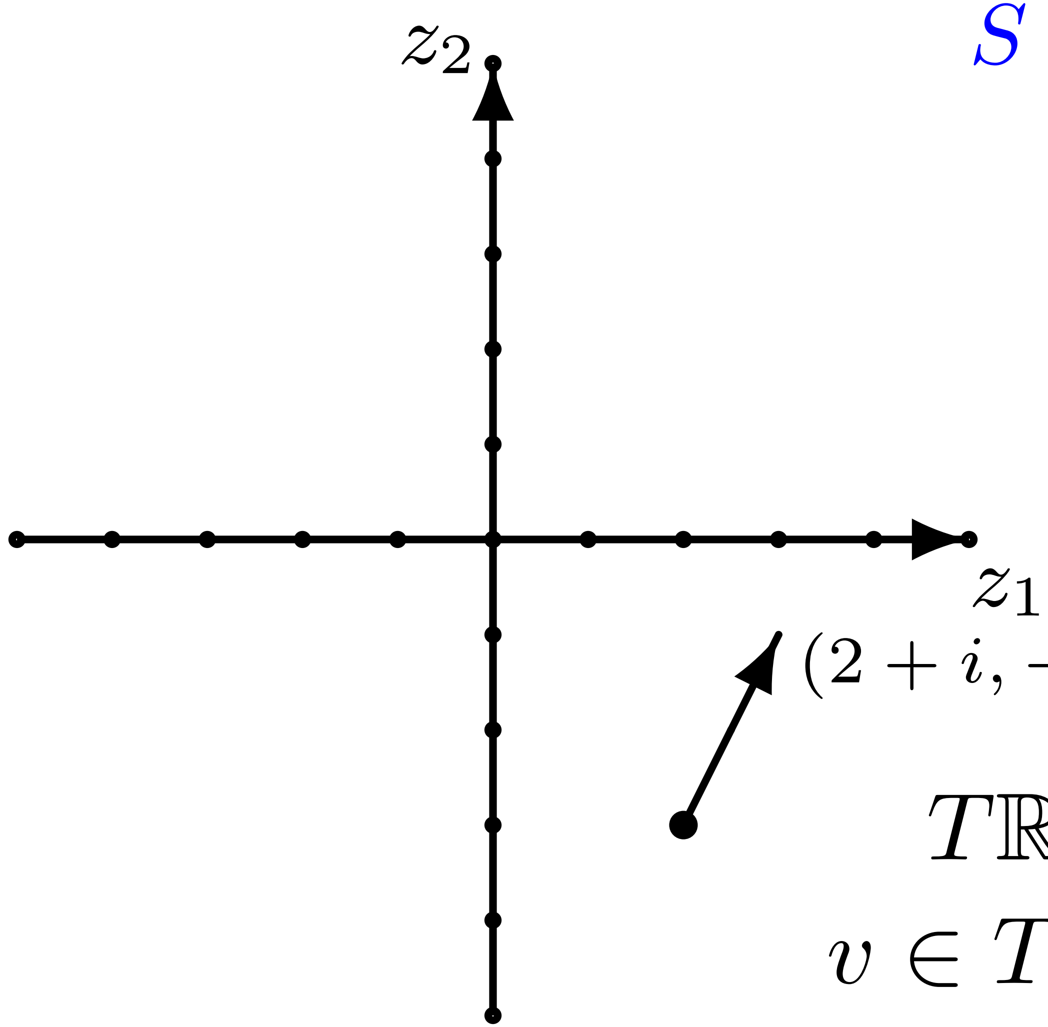
$H_1 = \{\alpha_1 = 0\}$

What shape? How sit in $\mathbb{C}^2$?

2 Minimal Stratification

Consider $S = \{(z_1, z_2) \in \mathbb{C}^2 \mid \frac{z_2}{z_1} \in \mathbb{R}_{>0}\}$

$S \cong \mathbb{C}^* \times \mathbb{R}_{>0}$



$(2 + i, -3 + 2i) \notin S$

$$\begin{array}{ccc} T\mathbb{R}^2 & \cong & \mathbb{C}^2 \\ v \in T_x\mathbb{R}^2 & \mapsto & x + \sqrt{-1}v \end{array}$$

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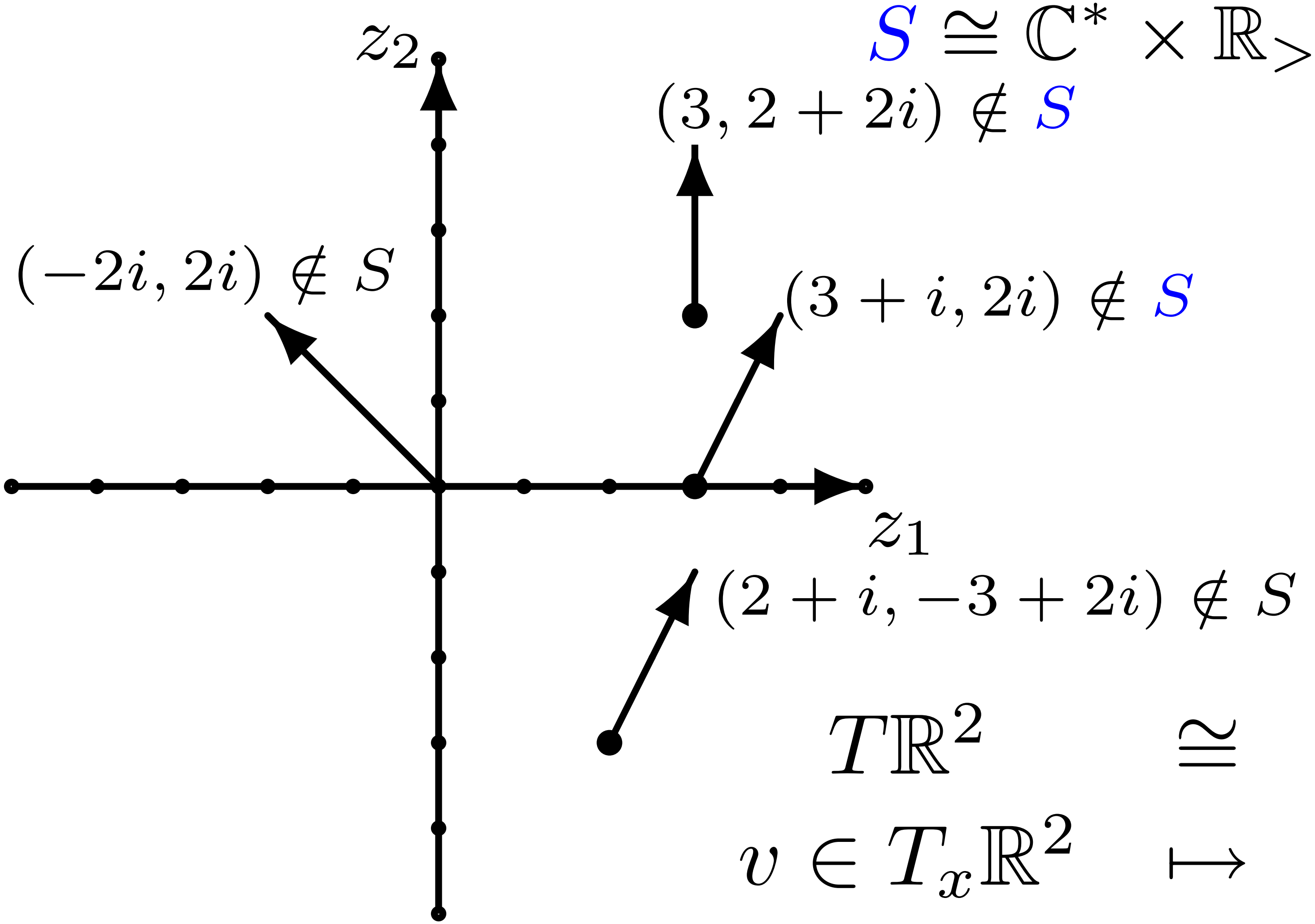

$$
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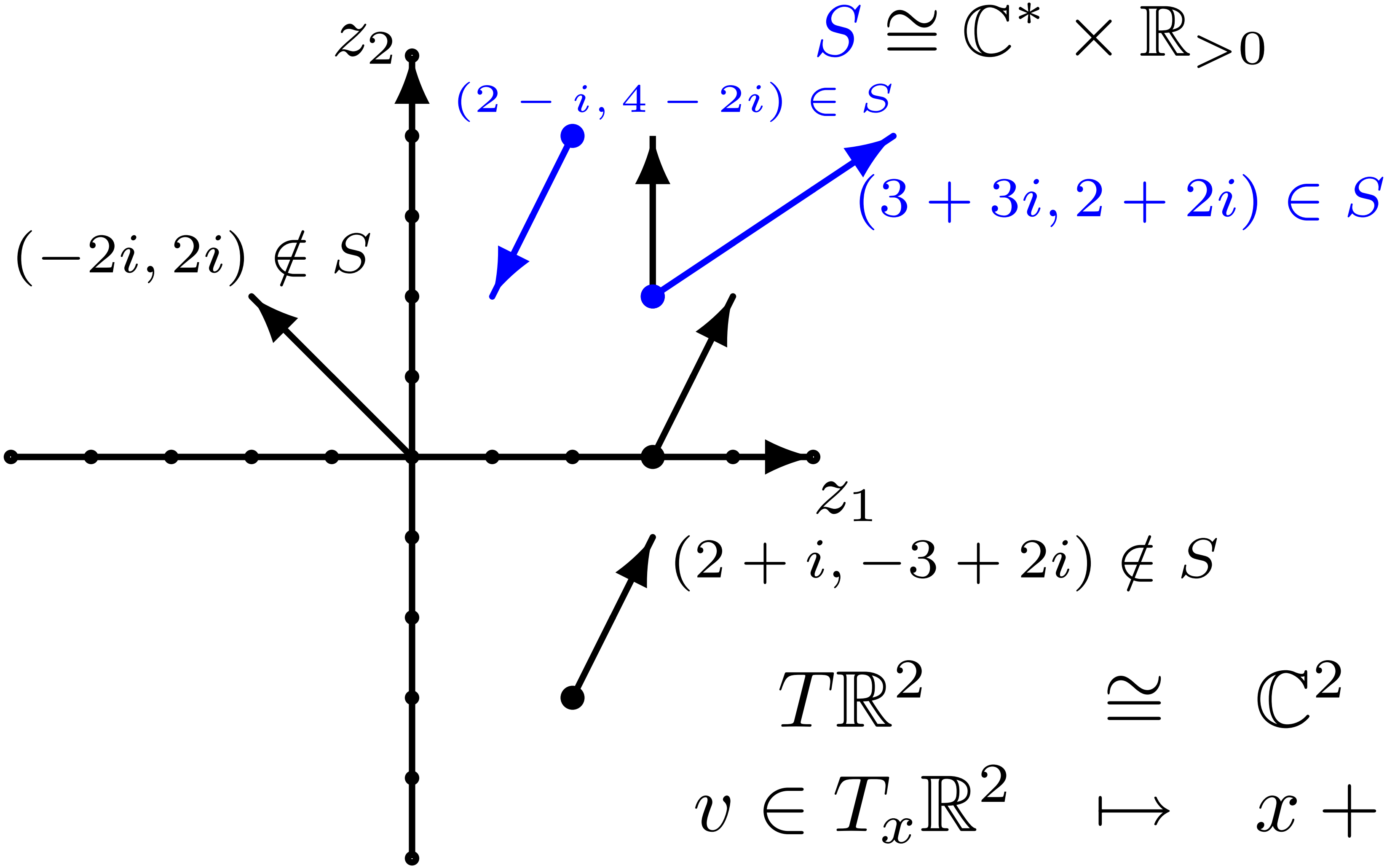

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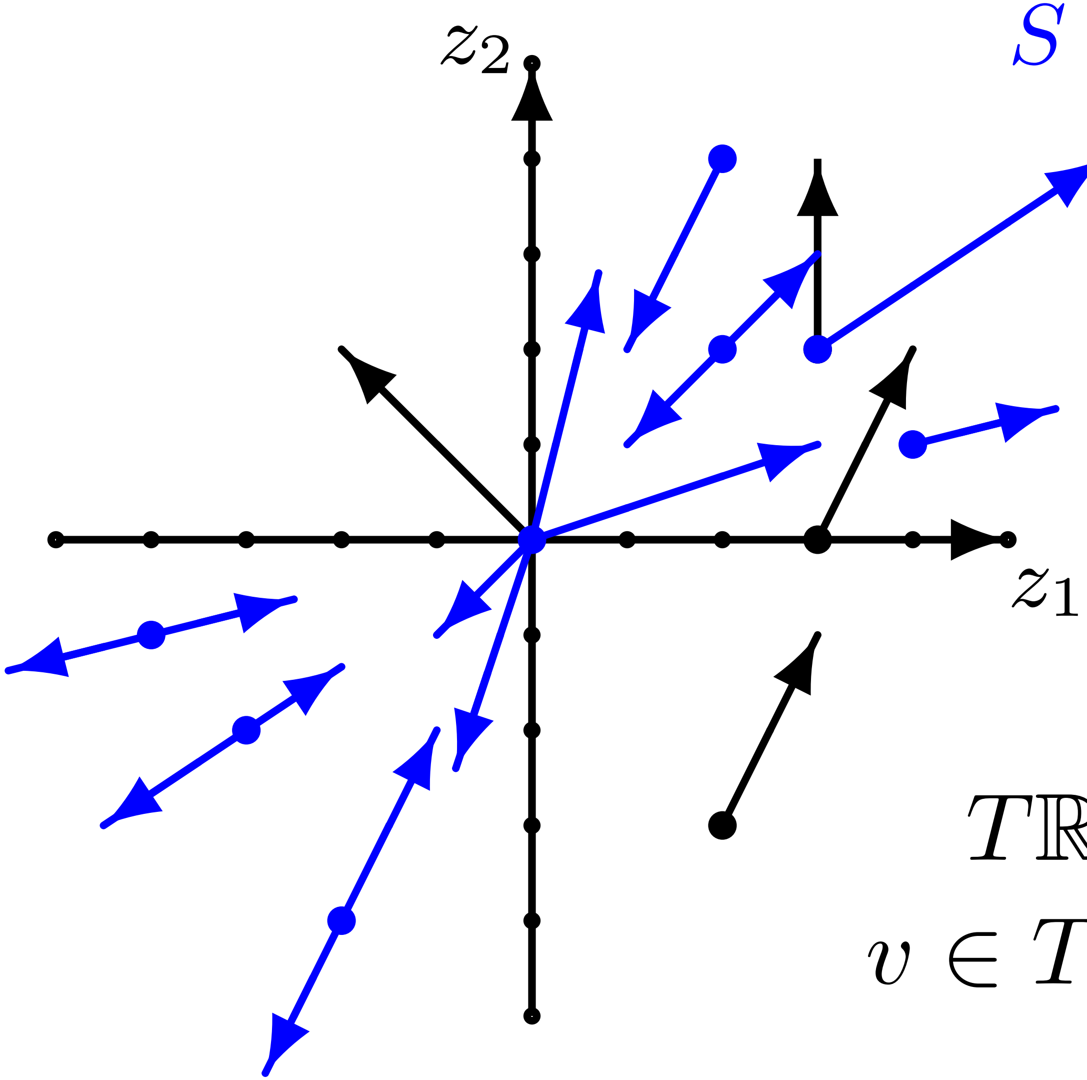

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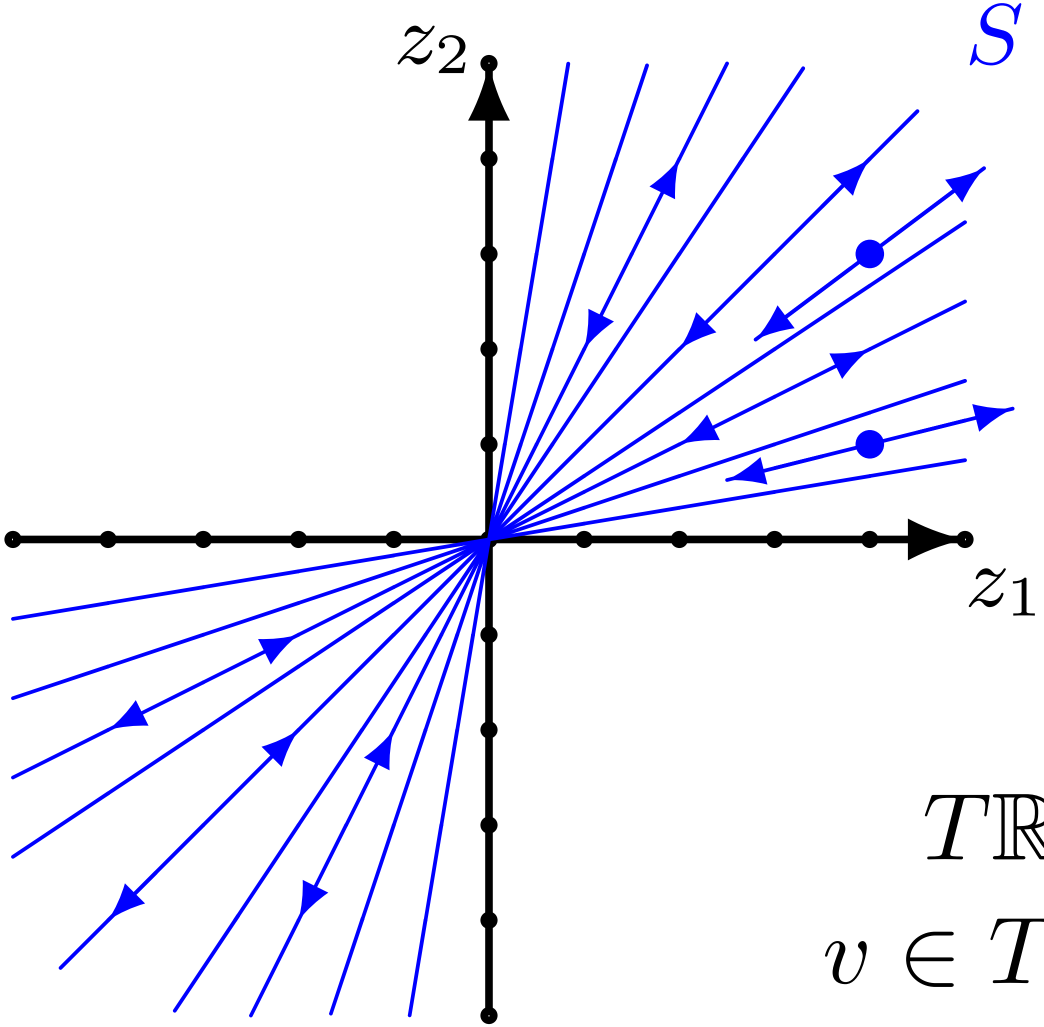

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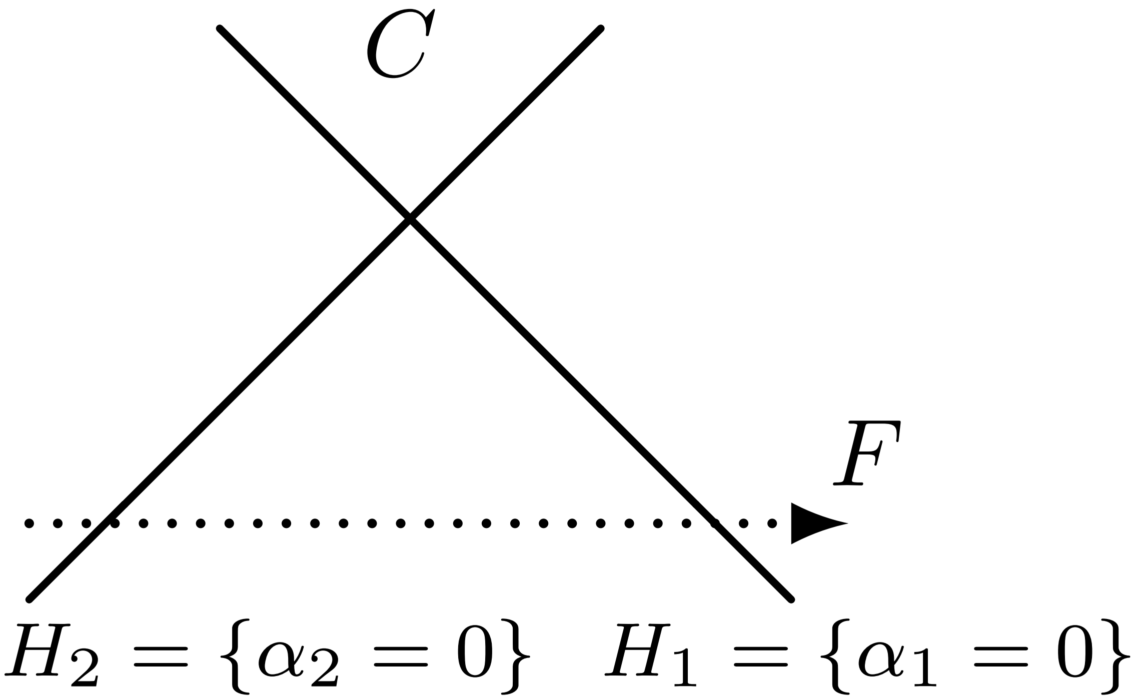
2 Minimal Stratification

Example.

We shall outline the proof with:



$$(\mathrm{ch}_F(\mathcal{A}) = \{C\})$$

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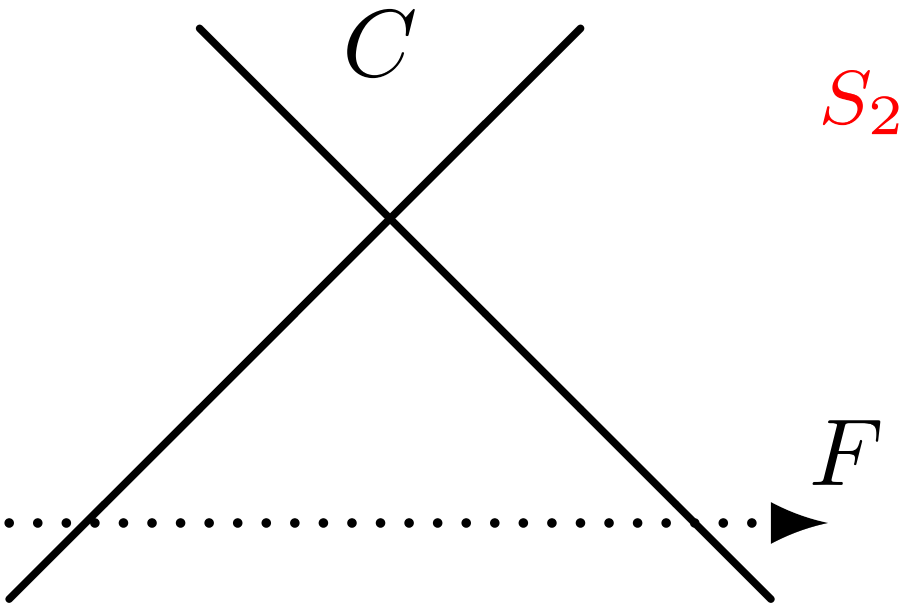
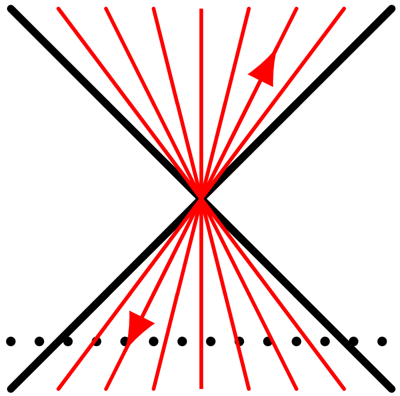
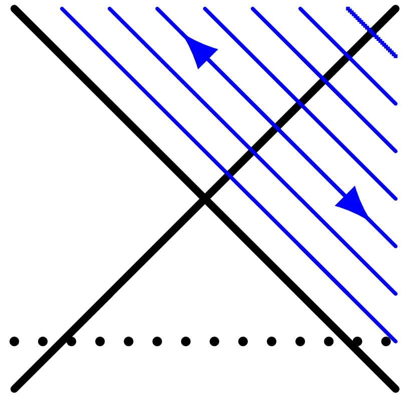


$H_2 = \{\alpha_2 = 0\}$ $H_1 = \{\alpha_1 = 0\}$

$(\mathrm{ch}_F(\mathcal{A}) = \{C\})$



$S_2 = \{\frac{\alpha_2}{\alpha_1} \in \mathbb{R}_{<0}\}$



$S_1 = \{\alpha_1 \in \mathbb{R}_{>0}\}$

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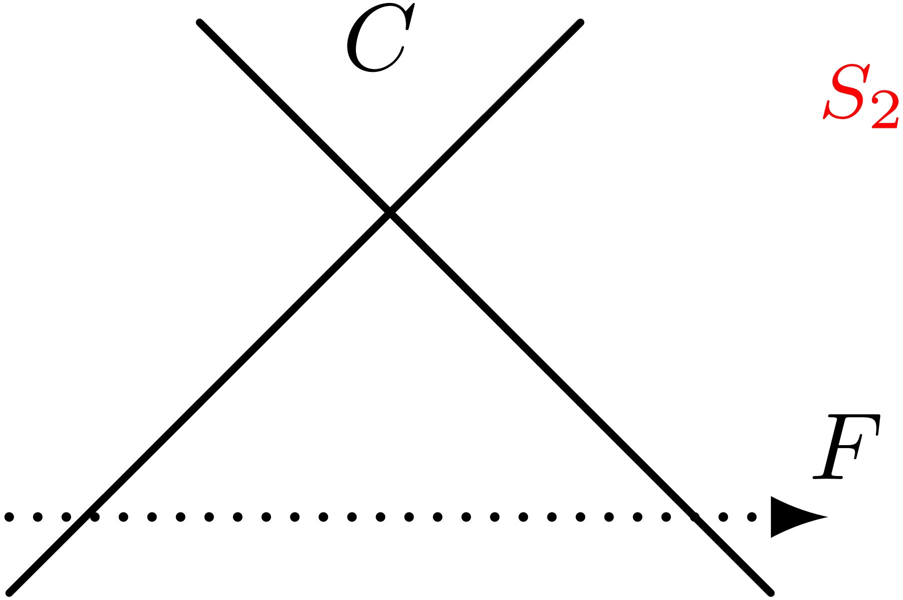
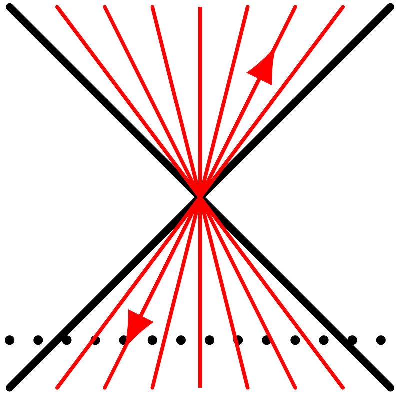
We shall outline the proof with:

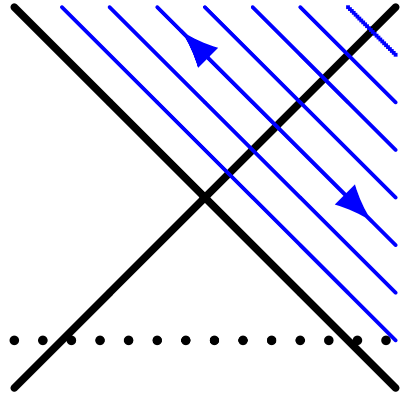


$H_2 = \{\alpha_2 = 0\}$ $\quad$ $H_1 = \{\alpha_1 = 0\}$

$(\mathrm{ch}_F(\mathcal{A}) = \{C\})$

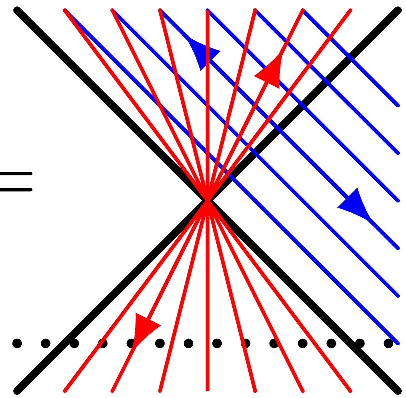


$S_2 = \{\frac{\alpha_2}{\alpha_1} \in \mathbb{R}_{<0}\}$



$S_1 = \{\alpha_1 \in \mathbb{R}_{>0}\}$

$S_1 \cap S_2 =$



$= C$

2 Minimal Stratification

Lemma.

(i) $S_i^\circ := S_i \setminus \bigcup_{C \in \mathrm{ch}_F} C$, then

$$S_i^\circ \cong (\mathbb{C}^* \setminus \mathbb{R}_{>0}) \times \mathbb{R}_{>0} \cong \mathbb{R}^3 \text{ (Homeo.)}$$

(ii) $U := M(\mathcal{A}) \setminus \bigcup_i S_i$, then $U \cong \mathbb{R}^4$.

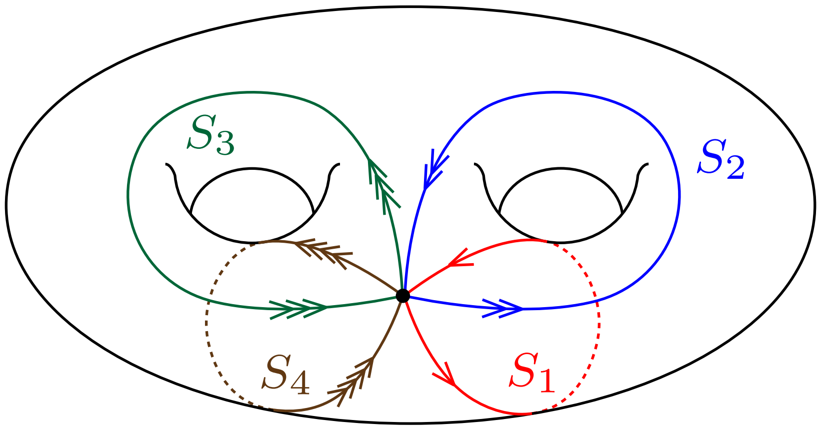
(iii) Giving a stratification:

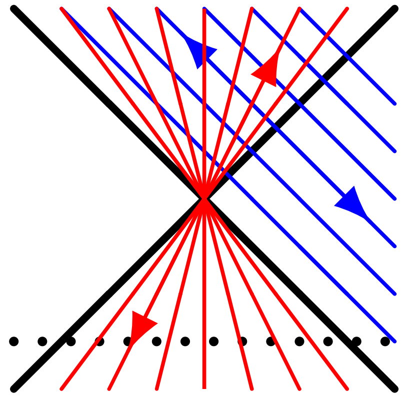
$$M(\mathcal{A}) = U \sqcup \bigsqcup_{i=1}^{n} S_i^\circ \sqcup \bigsqcup_{C \in \mathrm{ch}_F} C.$$

2 Minimal Stratification

To compute $\pi_1(M(\mathcal{A}))$, we will follow the Strategy:

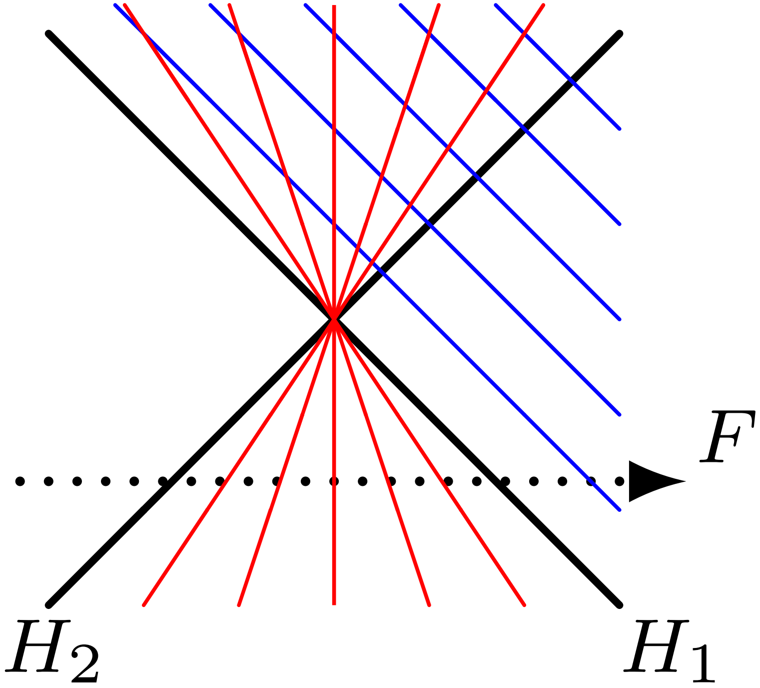
1. Stratify $M = \bigsqcup_\lambda S_\lambda$ by contractible strata.
2. Generator $\longleftrightarrow \mathrm{codim} = 1$ strata.
3. Relation $\longleftrightarrow \mathrm{codim} = 2$ strata.





$M(\mathcal{A}) = U \sqcup S_1^\circ \cup S_2^\circ \cup C \triangleright$

2 Minimal Stratification

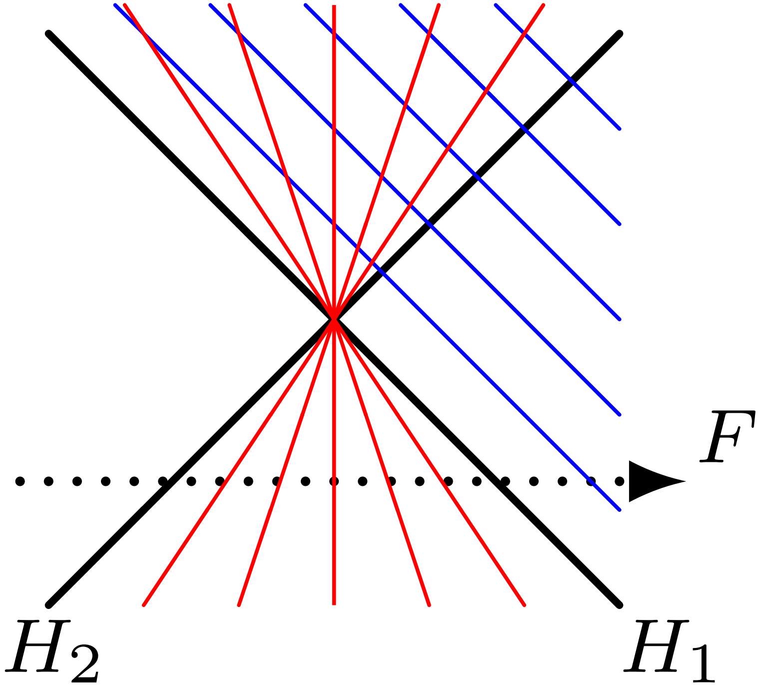


To look at transversal generators,

$$S_i \cap (F \otimes \mathbb{C}) = ?$$

$$S_1 = \{\alpha_1 \in \mathbb{R}_{>0}\},$$
$$S_2 = \{\frac{\alpha_2}{\alpha_1} \in \mathbb{R}_{<0}\}.$$

2 Minimal Stratification



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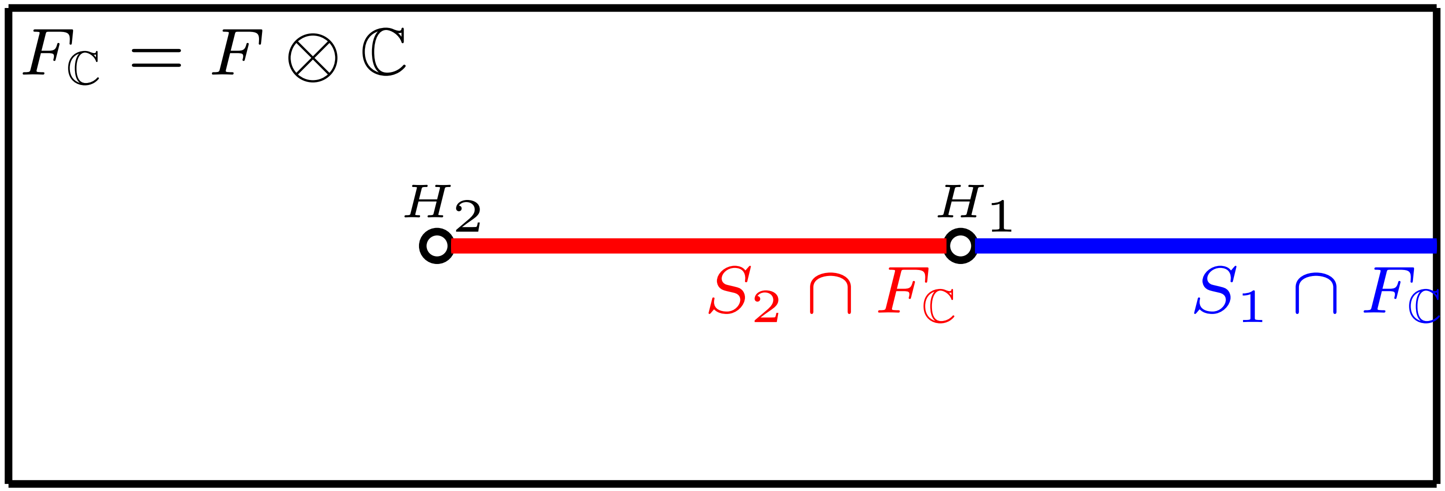
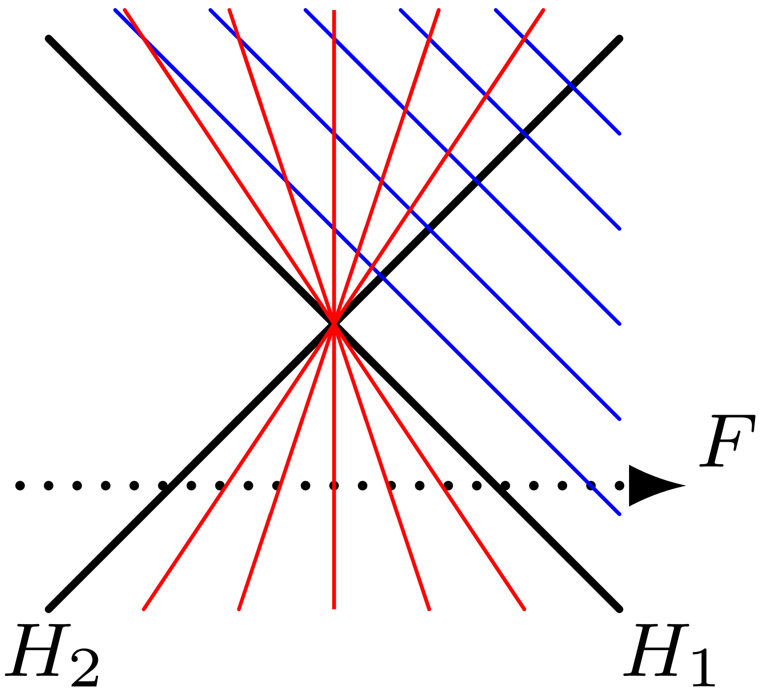
$=$



2 Minimal Stratification



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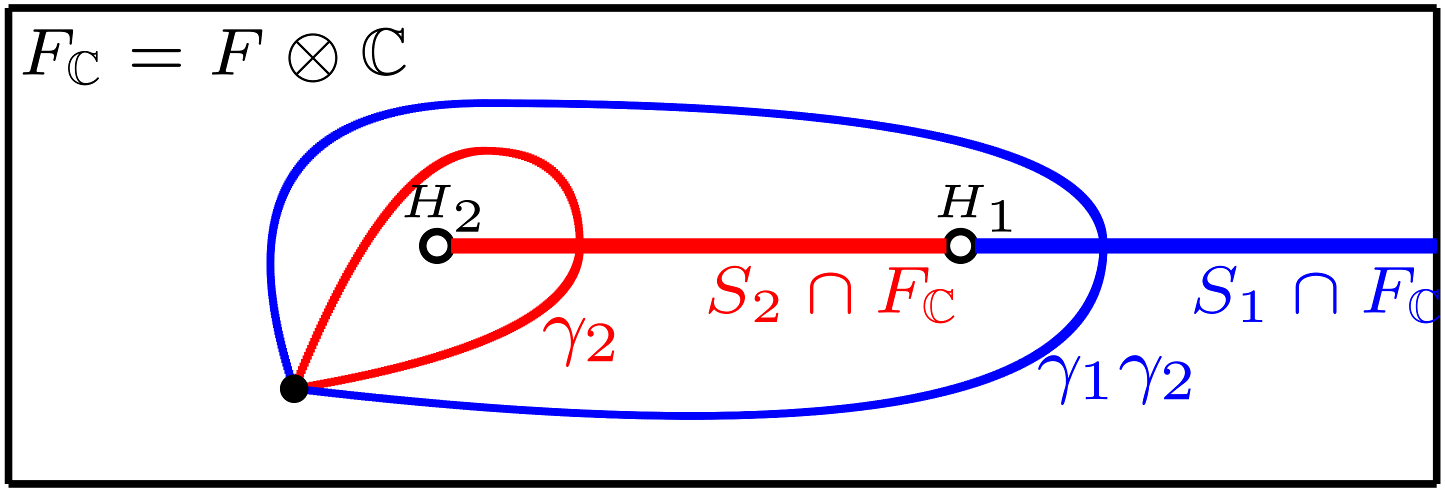
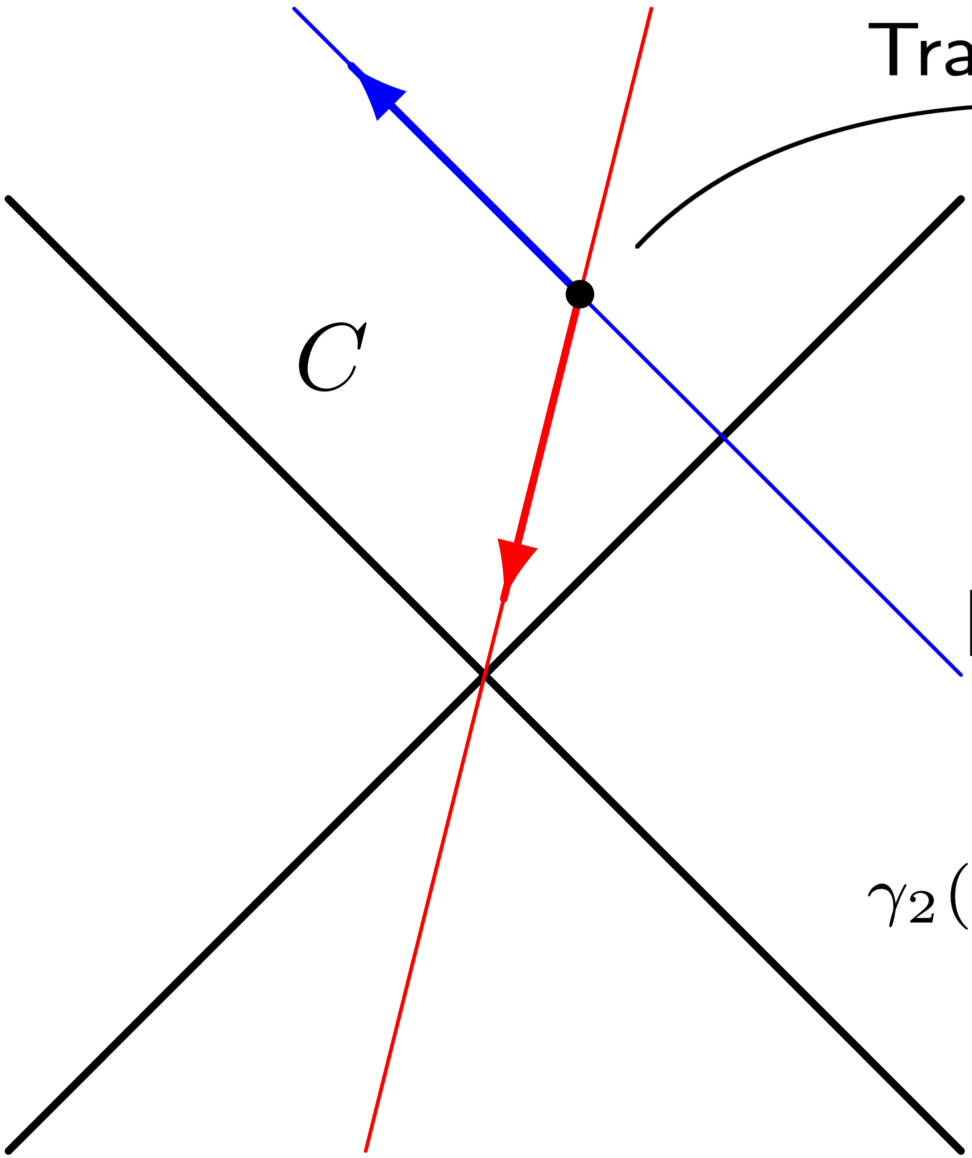
$$=$$

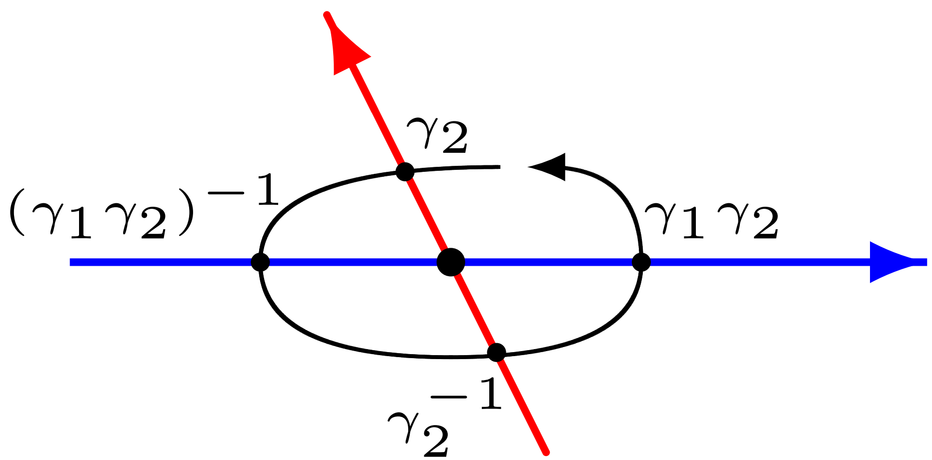


The generator corresponding to S_i is $\gamma_i\gamma_{i+1}\ldots\gamma_n$.

2 Minimal Stratification



Transversal slice to C.



Relation:

look at $\mathrm{codim} = 2$.

$$\gamma_2(\gamma_1\gamma_2)^{-1}\gamma_2^{-1}(\gamma_1\gamma_2)$$
$$= \gamma_1^{-1}\gamma_2^{-1}\gamma_1\gamma_2$$

$\Longrightarrow \gamma_1\gamma_2 = \gamma_2\gamma_1$. (Q.E.D.)

3 Chamber basis of OS-algebra

3 Chamber basis of OS-algebra

$\omega_i := \frac{1}{2\pi\sqrt{-1}} \frac{d\alpha_i}{\alpha_i} (\in H^1(M, \mathbb{Z}))$.

$\Omega^\bullet$: meromorphic differential forms.

$\cup$

$$\begin{aligned} A^\bullet &:= \mathbb{C}\langle \omega_i \mid i = 1, \ldots, n \rangle, \\ &= A^0 \oplus A^1 \oplus A^2. \end{aligned}$$

Thm. (Brieskorn) $A^\bullet \xrightarrow{\simeq} H^\bullet(M, \mathbb{C})$.

We are connecting differential forms and chambers.

3 Chamber basis of OS-algebra

Def. Borel-Moore homology $H_{\bullet}^{BM}(M, \mathbb{Z})$ is a homology of (infinite) locally finite chains.

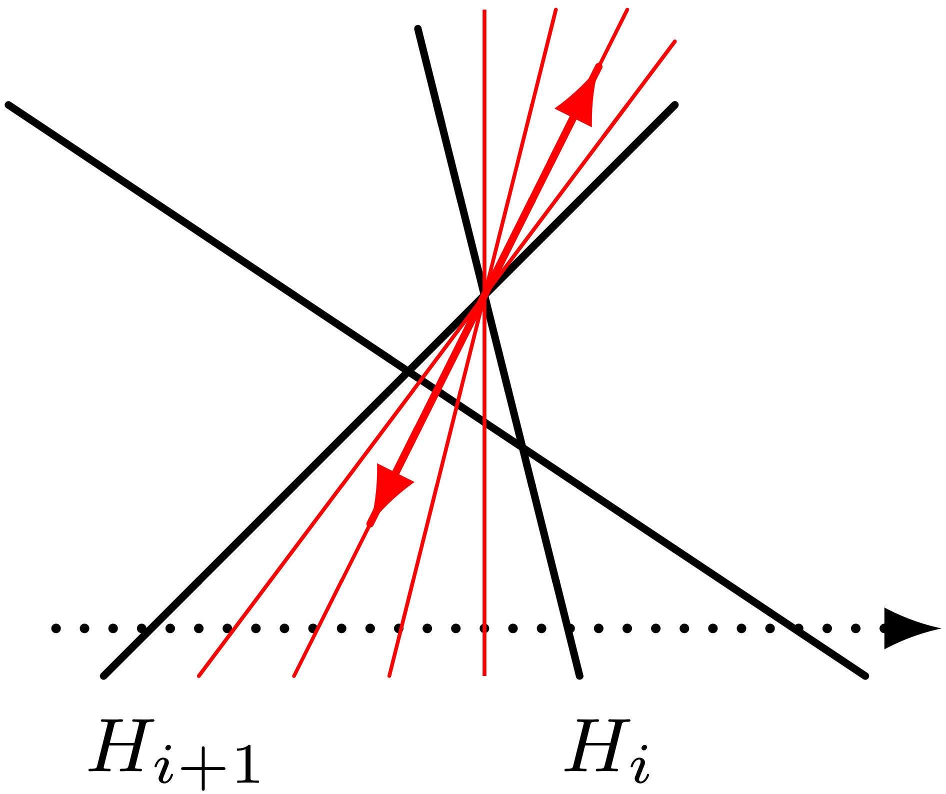
Since $M = M(\mathcal{A})$ is an oriented 4-manifold, we have natural isomorphisms:

$$H_4^{BM}(M, \mathbb{Z}) \cong H^0(M, \mathbb{Z})(\cong \mathbb{Z}),$$
$$H_3^{BM}(M, \mathbb{Z}) \cong H^1(M, \mathbb{Z})(\cong \mathbb{Z}^n),$$
$$H_2^{BM}(M, \mathbb{Z}) \cong H^2(M, \mathbb{Z}).$$

3 Chamber basis of OS-algebra

Prop. The isomorphism $H_3^{BM} \cong H^1$ is given by:

$$H_3^{BM}(M, \mathbb{Z}) \cong H^1(M, \mathbb{Z})$$
$$[S_1] \longmapsto \omega_1$$
$$[S_{i+1}] \longmapsto \omega_{i+1} - \omega_i$$



$$S_{i+1} = \{\tfrac{\alpha_{i+1}}{\alpha_i} \in \mathbb{R}_{<0}\}$$

3 Chamber basis of OS-algebra

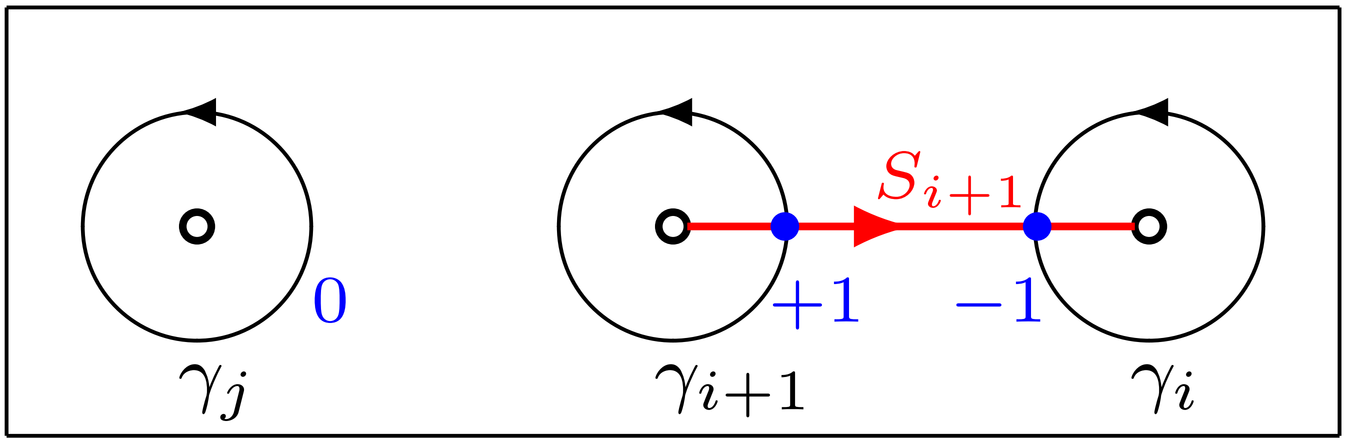
(Proof.) Use the fact:

$$\omega_{i+1} - \omega_i \in H^1(M) \xrightarrow{\cong} H_3^{BM}(M) \ni [S_{i+1}]$$

$H^1(M)$ and $H_1(M)$: dual (Cap product or integration)

$H_3^{BM}(M)$ and $H_1(M)$: dual (Intersection)

$$H_1(M)$$



$$\int_{\gamma_j} (\omega_{i+1} - \omega_i) = \begin{cases} -1 & j = i, \\ +1 & j = i+1, \\ 0 & \text{else.} \end{cases}$$

$\therefore S_i = \omega_{i+1} - \omega_i$. (Q.E.D.)

3 Chamber basis of OS-algebra

The isomorphism $H^1 \xrightarrow{\simeq} H_3^{BM}$ is extended as:

$$\begin{array}{ccc} H^2(M) & \xrightarrow{\simeq} & H_2^{BM}(M) \\ (\omega_{i+1} - \omega_i) \wedge (\omega_{j+1} - \omega_j) & \longmapsto & [S_i \cap S_j] \end{array}$$

We have

$$H^2(M) \cong H_2^{BM}(M) \cong \bigoplus_{C \in \mathrm{ch}_F} \mathbb{C} \cdot [C].$$

3 Chamber basis of OS-algebra

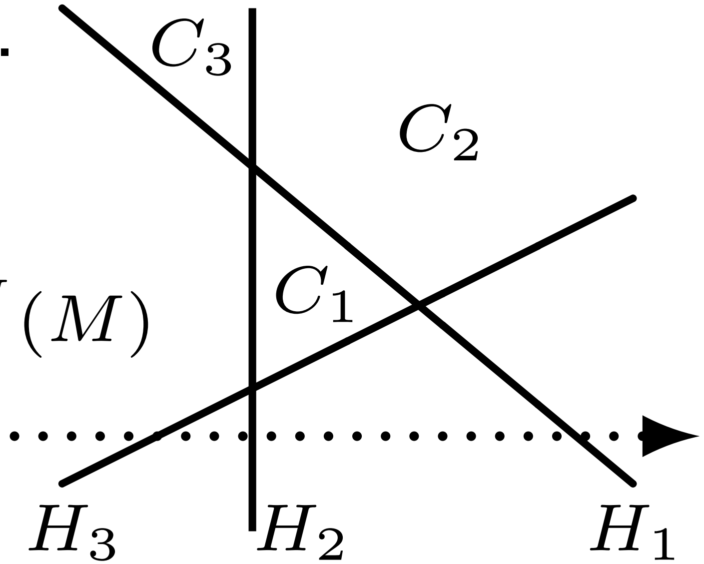
Def. Let $\mathrm{ch}_F(\mathcal{A}) = \{C_1, \ldots, C_b\}$.
We call
$1 \in H^0(M)$,
$[S_1] = \omega_1, [S_{i+1}] = \omega_{i+1} - \omega_i \in H^1(M)$, and
$[C_1], \ldots, [C_b] \in H^2(M)$,
the *chamber basis* of $H^\bullet(M) = A^\bullet$.

$$A^0 = \mathbb{C} \cdot 1,\ A^1 = \bigoplus_{i=1}^{n} \mathbb{C} \cdot [S_i],\ A^2 = \bigoplus_{p=1}^{b} \mathbb{C} \cdot [C_p].$$

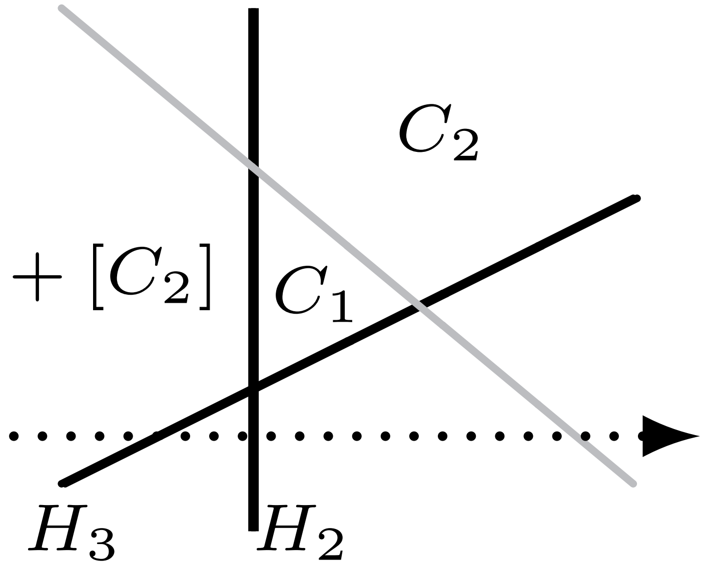
3 Chamber basis of OS-algebra

Example.

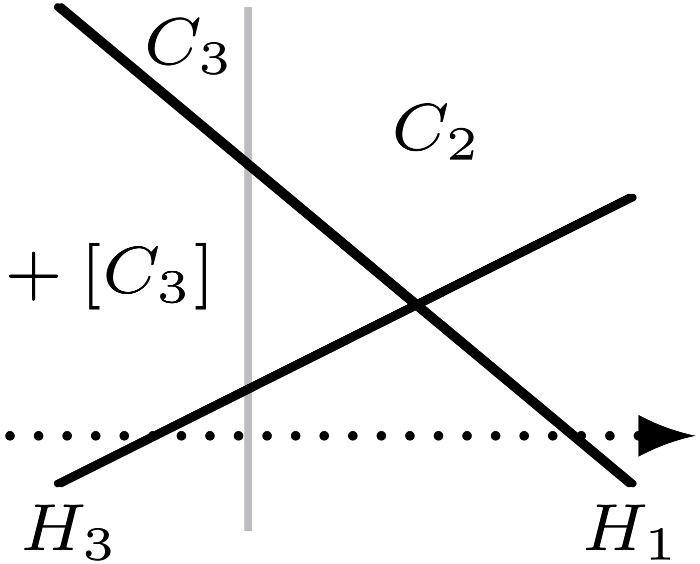
$H^2(M) \xrightarrow{\simeq} H_2^{BM}(M)$



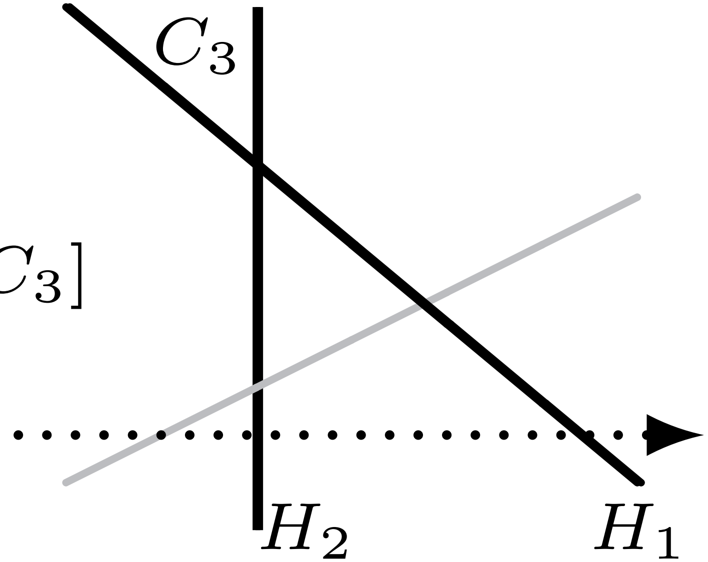
$\omega_3 \wedge \omega_2 = [C_1] + [C_2]$



$\omega_3 \wedge \omega_1 = [C_2] + [C_3]$



$\omega_2 \wedge \omega_1 = [C_3]$



3 Chamber basis of OS-algebra

Remark. Ko-ki Ito and I recently constructed explicit basis of $H_*^{BM}(M(\mathcal{A}), \mathbb{Z})$ for any complexified real arrangements in $\mathbb{R}^\ell$.

4 Chamber basis and $H^1(M, \mathcal{L}_\lambda)$

4 Chamber basis and $H^1(M, \mathcal{L}_\lambda)$

Applying chamber basis to local system cohomology group.

$\mathcal{A} = \{H_1, \ldots, H_n\}$, $M = M(\mathcal{A})$, γ_i, ω_i as above.

Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$,

Def. $\mathcal{L}_\lambda$: the rank one local system defined by

$$\pi_1(M) \ni \gamma_i \longmapsto e^{2\pi\sqrt{-1}\lambda_i} \in \mathbb{C}^*.$$

Question: $H^1(M, \mathcal{L}_\lambda) = ?$

4 Chamber basis and $H^1(M, \mathcal{L}_\lambda)$

<u>Thm</u> (Esnault-Schechtman-Viehweg). Set $\omega_\lambda = \sum_{i=1}^n \lambda_i d\log\alpha_i$. If λ_i are enough small, then

$$H^*(M, \mathcal{L}_\lambda) \cong H^*(A^\bullet, \omega_\lambda \wedge).$$

<u>Question</u>: In general, can one recover $H^1(M, \mathcal{L}_\lambda)$ from $(A^\bullet, \omega_\lambda \wedge)$?

Yes, using chamber basis.

4 Chamber basis and $H^1(M, \mathcal{L}_\lambda)$

Looking at $(A^\bullet, \omega_\lambda \wedge)$ via chamber basis.

$$A^0 \to A^1 : \quad 1 \mapsto \omega_\lambda = \sum_{i=1}^{n} \eta_i \cdot [S_i],$$

$$A^1 \to A^2 : \quad [S_i] \mapsto \omega_\lambda \wedge [S_i] = \sum_{p=1}^{b} \rho_i^p \cdot [C_p].$$

To obtain $H^1(M, \mathcal{L}_\lambda)$, we need "deform" the coefficients η_i, ρ_i^p.

Def. $\Delta(x) := e^{x/2} - e^{-x/2} = 2\sinh(x/2)$.

4 Chamber basis and $H^1(M, \mathcal{L}_\lambda)$

Thm. Let us define

$$\nabla : A^0 \to A^1, \quad 1 \mapsto \sum_{i=1}^{n} \Delta(\eta_i) \cdot [S_i],$$

$$\nabla : A^1 \to A^2, \quad [S_i] \mapsto \sum_{p=1}^{b} \Delta(\rho_i^p) \cdot [C_p],$$

then $(A^\bullet, \nabla)$ is a complex for $\forall \lambda \in \mathbb{C}^n$, and

$$H^*(A^\bullet, \nabla) \cong H^*(M, \mathcal{L}_\lambda).$$

4 Chamber basis and $H^1(M, \mathcal{L}_\lambda)$

Thm. Define

$$\nabla : A^0 \to A^1, \quad 1 \mapsto \sum_{i=1}^{n} \Delta(\eta_i) \cdot [S_i],$$
$$\nabla : A^1 \to A^2, \quad [S_i] \mapsto \sum_{p=1}^{b} \Delta(\rho_i^p) \cdot [C_p],$$

then $(A^\bullet, \nabla)$ is a complex, and

$$H^*(A^\bullet, \nabla) \cong H^*(M, \mathcal{L}_\lambda).$$

Remark. This does not mean "$H^1(M, \mathcal{L}_\lambda)$ is combinatorial".

5 References

http://www.math.kyoto-u.ac.jp/∼mhyo/index.html

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