A short introduction to arrangements of hyperplanes

survey

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Setup and notation

By a hyperplane arrangement we understand the set $\mathcal{A}$ of several hyperplanes of an ℓ - dimensional affine space V over a field K. If all the hyperplanes are linear, i.e., passing through a common point (called 0) , then $\mathcal{A}$ is central. If 0 is the only common point then $\mathcal{A}$ is essential. Often we will order $\mathcal{A}$ and then write $\mathcal{A} = \{H_1, \ldots, H_n\}$.

Any time when it is convenient, we fix a linear basis $(x_1, \ldots, x_\ell)$ of V^* and identify V with K^ℓ using the dual basis in V. Then for each hyperplane H of K^ℓ we fix a degree 1 polynomial $\alpha_H \in S = K[x_1, \ldots, x_\ell]$ such that H is the zero locus of α_H. This polynomial is uniquely defined up to multiplication by a nonzero element from K. If $\mathcal{A}$ is central all α_H are homogeneous. We will abbreviate α_{H_i} as α_i.

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Combinatorics of arrangements

For many invariants of arrangements hyperplanes themselves are not needed; these invariants are determined by the combinatorics of arrangements. There are two essentially equivalent combinatorial objects that $\mathcal{A}$ determines: a geometric lattice and a simple matroid. We will briefly discuss the former.

For an arrangement $\mathcal{A}$ its intersection lattice $L = L(\mathcal{A})$ consists of intersections of all subsets of hyperplanes from $\mathcal{A}$ (including V itself as the intersection of the empty set of hyperplanes). The partial order on L is the reverse inclusion of subspaces. In particular the unique minimal element of L is V and the unique maximal element is $\bigcap_{i=1}^{n} H_i$ (even if it is $\emptyset$). $\mathcal{A}$ itself becomes the set of all elements of L following the minimal element, called atoms.

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Geometric lattices

The poset L is far from arbitrary. Let us collect the following three properties of L. We are assuming for simplicity that $\mathcal{A}$ is central.

(i) it is atomic, i.e., its every element is the join (the least upper bound) of some atoms;
(ii) it is ranked, i.e., every nonrefinable flag (chain) $(V < X_1 < \cdots < X_r = X)$ from V to a fixed $X \in L$ has the same number of elements, namely the codimension of X; (in lattice theory, this number is called the rank of X and denoted by $\operatorname{rk} X$);

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Geometric lattices

(iii) for every $X, Y \in L$ the following semimodular inequality holds

$$\operatorname{rk} X + \operatorname{rk} Y \geq \operatorname{rk}(X \vee Y) + \operatorname{rk}(X \wedge Y)$$

where the symbols $\vee$ and $\wedge$ denote respectively the join and meet (i.e., the greatest lower bound).
Lattices satisfying the above properties are called geometric.
The rank $\operatorname{rk} L$ of a geometric lattice L is the maximal rank of its elements. Clearly $\operatorname{rk} L \leq \ell$ and $\operatorname{rk} L = \ell$ if and only if the arrangement is essential.

Möbius function

An important invariant of L (as of every poset) is its Möbius function. It is the function $\mu : L \times L \to \mathbb{Z}$ satisfying the conditions $\mu(X, X) = 1$, $\mu(X, Z) = 0$ unless $X \leq Z$, and

$$\sum_{Y \in L, X \leq Y \leq Z} \mu(X, Y) = 0$$

for every $X, Z \in L$, $X < Z$. We put $\mu(X) = \mu(V, X)$ for every $X \in L$.

Example

If H is an atom of L then $\mu(H) = 1$. If $X \in L$ is of rank 2 with precisely k atoms below it then $\mu(X) = -(k - 1)$.

For L the following generating function $\pi_L(t) = \sum_{X \in L} \mu(X)(-t)^{\operatorname{rk} X}$ is called the characteristic or Poincarè polynomial of L.

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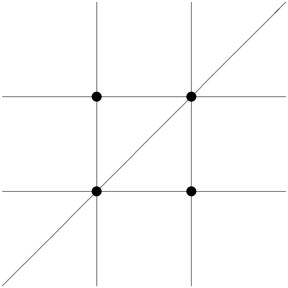
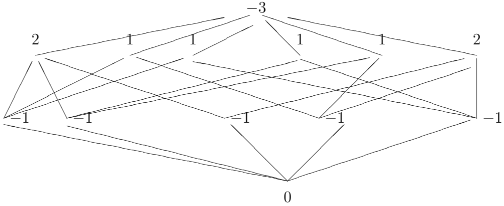
Example of L





$$1 + 6t + 8t^2 + 3t^3$$

Remark

The figure on the previous slide has many typos. A problem recommended to the beginners is to find all of them. Hint: look at the value of the M obius functions.

Homology of a of a poset

For an arbitrary poset, all its flags (i.e., the linearly ordered subsets) form an (abstract) simplicial complex called the order complex on the set of all its elements. The homotopy invariants of this complex are attributed to the poset itself. If the poset is a lattice then by its homotopy invariants one usually means those of its subposet with the largest and smallest elements deleted. If again the poset is a lattice than the order complex can be substituted by another (usually much smaller) simplicial complex Ξ. It is defined on the set of all atoms $\mathcal{A}$ and a subset $\sigma \subset \mathcal{A}$ is a simplex (of dimension $|\sigma| - 1$) if $\bigvee(\sigma)$ is not the greatest element. This complex is called atomic and it is homotopy equivalent to the order complex. Below we will need the pair (Δ, Ξ) of complexes where Δ is the simplex on all the atoms.

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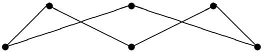
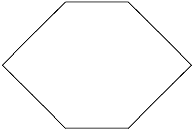
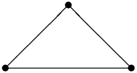
Complexes of a poset



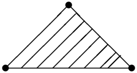


The order complex



Atomic complex

$\subset$



The pair of complexes

Exterior algebra

The lattice L determines the most important algebra associated with an arrangement. Let $\mathcal{A} = (H_1, \ldots, H_n)$ be an arrangement and k an arbitrary field (not necessarily equal to K). Let E be the exterior algebra over k with generators $e_1, \ldots, e_n$ in degree 1. Notice that the indices define a bijection from $\mathcal{A}$ to the generating set. Sometimes we will denote the generator corresponding to $H \in \mathcal{A}$ by e_H.

The algebra E is graded via $E = \oplus_{p=1}^{n} E^p$ where $E^1 = \oplus_{j=1}^{n} ke_j$ and $E^p = \bigwedge^p E^1$. The linear space E_j has the distinguished basis consisting of monomials $e_S = e_{i_1} \cdots e_{i_p}$ where $S = \{i_1, \ldots, i_p\}$ is running through all the subsets of $[n] = \{1, 2, \ldots, n\}$ of cardinality p and $i_1 < i_2 < \cdots < i_p$.

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DGA

The graded algebra E is a (commutative) DGA with respect to the differential ∂ of degree -1 uniquely defined by the conditions: linearity, $\partial e_i = 1$ for every $i = 1, \ldots, n$, and the graded Leibniz formula. Then for every $S \subset [n]$ of cardinality p

$$\partial e_S = \sum_{j=1}^{p} (-1)^{j-1} e_{S_j}$$

where S_j is the complement in S to its jth element.

For every $S \subset [n]$, put $\bigcap S = \bigcap_{i \in S} H_i$ and call S dependent if $\bigcap S \neq \emptyset$ and the set of linear polynomials $\{\alpha_i | i \in S\}$ is linearly dependent. Notice that being dependent is a combinatorial property - a set of atoms S is such if and only if $\operatorname{rk} \bigvee S < |S|$.

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OS algebra

Definition

Consider the ideal $I = I(\mathcal{A})$ of E generated by all e_S with $\bigcap S = \emptyset$ and all ∂e_S with S dependent. The algebra $A = A(\mathcal{A}) = E/I(\mathcal{A})$ is called the Orlik-Solomon (abbreviated as OS) algebra of $\mathcal{A}$. This algebra has been called also Brieskorn, BOS, and Arnold-Brieskorn.

Clearly the ideal I is homogeneous whence A is a graded algebra; we write $A = \oplus_p A^p$ where A^p is the component of degree p. In particular the linear spaces E^1 and A^1 are isomorphic and we will identify them.

Notice that for any nonempty $S \subset [n]$ and $i \in S$ one has $e_i \partial e_S = \pm e_S$ whence I contains e_S for every dependent set S. This implies that A is generated as a linear space by the images of e_S such that $\bigcap S \neq \emptyset$ and S is independent.

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Homological interpretation

The Orlik - Solomon algebra is not only determined by L but also it has an interpretation in terms of the homology of L. Let us first define the relative atomic complex $\nabla = \nabla(L)$. It is the chain complex over k with a linear basis consisting of all $\sigma \subset \mathcal{A}$; $\deg \sigma = |\sigma|$. The differential is defined by

$$d(\sigma) = \sum_{i \mid \bigvee(\sigma \setminus \{H_i\}) = \bigvee(\sigma)} (-1)^i \sigma \setminus \{H_i\}.$$

The chain complex ∇ has grading by elements of L, i.e., $\nabla = \oplus_{X \in L} \nabla_X$ where ∇_X is the subcomplex generated by all σ with $\bigvee \sigma = X$.

Notice that ∇_X is the relative chain complex for the pair $(\Delta, \Xi)_X$ for the lattice $L_X = \{Y \in L | Y \leq X\}$. In particular the cohomology of ∇ is the direct sum of the shifted by 1 local cohomology of L.

Also notice that if σ is an independent subset of $\mathcal{A}$ then it is a cycle in ∇. Denote by $[\sigma]$ its homology class.

Multiplication

Define

$$\sigma \cdot \tau = \begin{cases} 0, & \text{if } \mathrm{rk}(\bigvee(\sigma \cup \tau)) \neq \mathrm{rk}(\bigvee(\sigma)) + \mathrm{rk}(\bigvee(\tau)), \\ \epsilon(\sigma, \tau)\sigma \cup \tau, & \text{otherwise.} \end{cases} \tag{1}$$

where $\epsilon(\sigma, \tau)$ is the sign of the permutation of $\sigma \cup \tau$ putting all elements of τ after elements of σ and preserving fixed orders inside these sets (the *shuffle* of σ and τ).

Theorem

The multiplication defined above converts ∇ to a (commutative) DGA graded by L. The correspondence $e_i \mapsto [\{i\}]$ generates an isomorphism $A \to H_*(\nabla; k)$ of graded algebras.

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Corollary

The following statements follow from the previous theorem.

Corollary

(i) The algebra A is graded by L, i.e., $A = \oplus_{X \in L} A_X$ where A_X is a graded linear subspace of A (in fact homogeneous) generated by e_S with $\bigvee S = X$ and $A_X A_Y \subset A_{X \vee Y}$.

(ii)The Hilbert series $H(A, t) = \pi_L(t)$.

This corollary uses not only the theorem but also certain property of homology of geometric lattices (namely the Folkman theorem).

Cohomology of M

From know on we assume $K = k = \mathbb{C}$.

Theorem (Arnold, Brieskorn, Orlik-Solomon)

Let $\mathcal{A}$ be an arrangement in $\mathbb{C}^\ell$ and M its complement.
(i) The de Rham homomorphism for M restricts to an isomorphism of the graded algebras $\mathcal{F}$ and $H^*(M, \mathbb{C})$ where $\mathcal{F}$ is the subalgebra of the algebra of closed holomorphic forms on M generated by all the forms $\frac{d\alpha_H}{\alpha_H}$ ($H \in \mathcal{A}$);
(ii) Let $[\omega]$ be the cohomology class of a form ω. Then the correspondence $[\frac{d\alpha_H}{\alpha_H}] \mapsto e_H$ defines a graded algebra isomorphism $H^*(M, \mathbb{C}) \simeq A$.

Remark. The theorem still holds if one defines all three algebras over $\mathbb{Z}$. In particular the cohomology of M is torsion free.

Poincaré polynomial

Corollary

The Poincarè polynomial of M coincides with the characteristic polynomial $\pi_L(t)$ of L.

Corollary

Space M is formal, i.e., the DGA of differential holomorphic forms on it is quasi-isomorphic to its cohomology algebra.

Braid arrangements

Example. Fix a positive integer n and consider the arrangement $\mathcal{A}_{n-1}$ in $\mathbb{C}^n$ given by linear functionals $x_i - x_j,\ 1 \leq i < j \leq n$. In fact $\mathcal{A}_{n-1}$ consists of all reflecting hyperplanes of the Coxeter group of type A_{n-1}. The complement M_n of this arrangement can be identified with the configuration space of n distinct ordered points in $\mathbb{C}$. Considering loops in this space makes it pretty clear that $\pi_1(M_n)$ is the pure braid group on n strings.

The natural way to study M_n is to project it to M_{n-1} ignoring the last coordinate of points in $\mathbb{C}^n$. This defines a fiber bundle projection that is the restriction to M_n of the projection $\mathbb{C}^n \to \mathbb{C}^n/X$ where X is a coordinate line (M_n is linearly fibered). The fiber of the projection is $\mathbb{C}$ without $n-1$ points. Repeating this process one obtains a sequence of such projections with decreasing n that ends at projecting to a point.

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Fiber-type arrangements

Generalizing that construction one obtains a recursive definition of a fiber-type arrangement.
For $X \in L$ we put $\mathcal{A}_X = \{H/X | H \in \mathcal{A}, H \supset X\}$ and consider it as an arrangement in V/X (cf. L_X).

Definition

An arrangement $\mathcal{A}$ in V is fiber-type if there is a line $X \in L(\mathcal{A})$ for which A_X is fiber-type and $M(\mathcal{A})$ is linearly fibered over $M(\mathcal{A}_X)$. Also arrangement of one hyperplane is fiber-type.

Using the sequence of consecutive fiber bundles it is possible to prove for every fiber-type arrangement that
(1) M is a $K[\pi, 1]$-space;
(2) $\pi_1(M)$ is a semidirect product of free groups.

Supersolvable lattices

It turns out that being fiber-type is a combinatorial property of an arrangement.

Let L here be an arbitrary geometric lattice. Then $X \in L$ is modular if

$$\mathrm{rk}\, X + \mathrm{rk}\, Y = \mathrm{rk}(X \vee Y) + \mathrm{rk}(X \wedge Y)$$

for every $Y \in L$. The lattice L is supersolvable if it contains a maximal flag of modular elements. If for an arrangement $\mathcal{A}$ its lattice is supersolvable we say $\mathcal{A}$ is.

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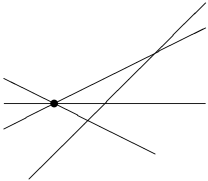
Examples

(1) All central arrangements of lines are supersolvable.

(2) A non-central arrangement $\mathcal{A}$ of lines is supersolvable if and only if there is a point $P \in L(\mathcal{A})$ such that for any other point $Q \in L(\mathcal{A})$ the line $PQ \in \mathcal{A}$.

S/S and not S/S



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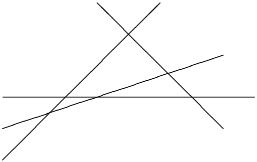

not S/S

(3) A Coxeter arrangement, i.e., the arrangement of all reflecting hyperplanes of a Coxeter groups, may be supersolvable or not. For instance, types A_n and B_n are supersolvable for all n, but type D_n is not for $n \geq 4$.

Remark. In spite of the last comment about D_n for every reflection arrangement $\mathcal{A}$, i.e., the arrangement of all reflecting hyperplanes of a finite reflection group, $M(\mathcal{A})$ is $K[\pi, 1]$.

Modules of derivations

We consider central arrangements in $V \simeq \mathbb{C}^\ell$. Recall that $S = \mathbb{C}[x_1, \ldots, x_\ell]$ (or, invariantly, the symmetric algebra of the dual space V^*) .

Definition

A derivation of S is a $\mathbb{C}$-linear map $\theta : S \to S$ satisfying the Leibniz condition

$$\theta(fg) = \theta(f)g + g\theta(f)$$

for every $f, g \in S$.

The set $\mathrm{Der}(S)$ of all derivations is naturally an S-module. This is a free module of rank ℓ with a basis consisting of partial derivatives $D_i = \frac{\partial}{\partial x_i}$, $(i = 1, \ldots, \ell)$.

Module $D(\mathcal{A})$

The following S-module reflects more about the arrangement. Let $\mathcal{A}$ be a central arrangement and $Q = \prod_{H \in \mathcal{A}} \alpha_H$ its defining polynomial.

Definition

The module of $\mathcal{A}$-derivations is

$$D(\mathcal{A}) = \{\theta \in \mathrm{Der}(S) | \theta(Q) \in QS\}.$$

$D(\mathcal{A})$ is a graded submodule of $\mathrm{Der}(S)$ which is not necessarily free though. For every $\theta \in D(\mathcal{A})$ we still have $\theta = \sum_i \theta_i D_i$ with uniquely defined $\theta_i \in S$ but in general $D_i \notin D(\mathcal{A})$. θ is homogeneous if $\deg \theta_i$ does not depend on i and then this degree is called the degree of θ.

Free arrangements

Definition

A central arrangement $\mathcal{A}$ is free if the S-module $D(\mathcal{A})$ is free.

$D(\mathcal{A})$ is free if and only if it generated by ℓ homogeneous generators. Another (Saito's) criterion says $D(\mathcal{A})$ is free if and only if it contains a system of ℓ homogeneous linearly independent over S derivations with the sum of there degrees equal $n = |\mathcal{A}|$.

Examples.

(1) Every central arrangement of lines is free.

(2) Consider the arrangement in $\mathbb{C}^3$ given by $Q = xyz(x + y + z)$. Then it can be seen that $D(\mathcal{A})$ does not contain two linearly independent derivations of degree less or equal one (one of such is the Euler derivation $\theta_E = \sum_i x_i D_i$). Since $n = 4$ Saito's criterion cannot be satisfied and $\mathcal{A}$ is not free.

Free arrangements

Definition

A central arrangement $\mathcal{A}$ is free if the S-module $D(\mathcal{A})$ is free.

$D(\mathcal{A})$ is free if and only if it generated by ℓ homogeneous generators. Another (Saito's) criterion says $D(\mathcal{A})$ is free if and only if it contains a system of ℓ homogeneous linearly independent over S derivations with the sum of there degrees equal $n = |\mathcal{A}|$.

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Free arrangements and combinatorics

The following results (by Terao) about free arrangements shows that the freeness is related to combinatorics.

Theorem

(i) Every supersolvable arrangement is free.
(ii) If an arrangement $\mathcal{A}$ is free then the characteristic polynomial

$$\pi_{L(\mathcal{A})}(t) = \prod_{i=1}^{\ell}(1 + b_i t)$$

where b_i are the degrees of the homogeneous generators of $D(\mathcal{A})$.

Terao Conjecture. The property of arrangement being free is combinatorial, i.e., it is determined by $L(\mathcal{A})$.
There are many partial results supported the conjecture.

Multiarrangements

A multiarrangement is a pair $(\mathcal{A}, m)$ where $\mathcal{A}$ is a central arrangement and $m : \mathcal{A} \to \mathbb{Z}_{\geq 0}$ is a multiplicity function.

There are several instances in arrangement theory where a multiplicity function appears naturally. Here are two examples.

(i) Suppose $\mathcal{A}$ is a central arrangement and $H_0 \in \mathcal{A}$. The restriction of $\mathcal{A}$ to H_0 is the arrangement $\mathcal{A}_{H_0} = \{H \cap H_0 \mid H \in \mathcal{A} \setminus \{H_0\}\}$. For every $\bar{H}$ from the restriction there is the natural mutiplicity

$$m(\bar{H}) = |\{H \in \mathcal{A} \mid H \cap H_0 = \bar{H}\}|.$$

(ii) Let $\mathcal{A}$ be a reflection arrangement of a finite reflection group G. In the theory of invariants of G the following multiplicity is often used: $m(H) = o_H - 1$ where $H \in \mathcal{A}$ and o_H is the order in G of the reflection at H.

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The notion of being free can be extended to multiarrangements. In a recent development Yoshinaga obtained a new criterion for the freeness of $\mathcal{A}$ using the multiarrangements $\mathcal{A}_H$. Using this criterion one can prove the Terao conjecture for arrangements with at most 11 hyperplanes.

Fundamental group of M

We will consider central arrangements only. Using as before the Lefschetz hyperplane section theorem it suffices to consider arrangements of planes in $\mathbb{C}^3$.

The important result about $\pi_1(M)$ is negative - this group is not determined by the lattice L. The example (by G. Rybnikov in 1994) consists of two arrangements of 13 hyperplanes each with $\operatorname{rk} L = 4$. It is still not very well understood, in particular no general group invariant is known that distinguishes π_1 for the two arrangemens in the example.

π_1 is generated by a set $\{z_1, z_2, \ldots, z_n\}$ which is in correspondence with $\mathcal{A}$. There are several known presentations of $\pi_1 = \pi_1(M)$ using this set. None of them is sufficiently simple or instructive to be described in the talk. We give several examples.

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Examples

Examples. (i) First we define a generic arrangements. $\mathcal{A}$ in $\mathbb{C}^3$ is is generic if every subset of it of rank 2 is independent. For a generic arrangement $\pi_1(M) = \mathbb{Z}^n$. Moreover $\pi_p(M) = 0$ for $1 < p < \ell$.

(ii) Let $\mathcal{A}$ be a Coxeter arrangement corresponding to a Coxeter group G. Then π_1 is the pure Artin group corersponding to G. (Moreover M is $K[\pi, 1]$.)

(iii) Let $\mathcal{A}$ be given by the polynomial $Q = x(x-y)(x+y)(2x-y+z)$. Then π_1 is given by presentation

$$\big\langle x_1, x_2, x_3, x_4 | x_1 x_2 x_4 = x_4 x_1 x_2 = x_2 x_4 x_1,$$
$$[x_1, x_3] = [x_2, x_3] = [x_4, x_3] = 1 \big\rangle.$$

It gives $\pi_1 \simeq F_2 \times \mathbb{Z}^2$ where F_2 is the free group on two generators.

Open problems

In spite of known presentations the following questions about π_1 are open in general.

(1) Is it torsion-free?

(2) Is it residually nilpotent?

(3) Is it residually finite?

(4) Find a group invariant that distinguishes two groups in the Rybnikov example.

Quadratic and Koszul algebras

A distinguished class of graded algebras is formed by Koszul algebras. One of many equivalent definitions of this class is as follows. A graded connected K-algebra is Koszul if the minimal free graded resolution of its trivial module K is linear, i.e., the matrices of all mappings in it have all their entries of degree one. If an algebra is Koszul then it is generated in degree one and the ideal of relations among generators is generated in degree two. An algebra with these two properties is a quadratic algebra.

A quadratic algebra $A = \oplus A_p$ ($A_0 = K$) can be represented as $A = T(A_1)/J$ where $T(A_1)$ is the tensor algebra on the space A_1 in degree one and J is the graded ideal of relations. Then the quadratic algebra $A^! = T(A_1^*)/J^*$ where A_1^* is the dual linear space of A_1 and J^* is the annihilator of J is called the quadratic dual ("shriek") of A.

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Properties of Koszul algebras

The following implications are very well-known.

(1) A quadratic algebra A is Koszul if and only if $A^!$ is Koszul.

(2) If A is Koszul then the following relation between the Hilbert series holds

$$H(A,t)\cdot H(A^!,-t)=1.$$

The converse of (2) is false in general.

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Koszul and quadratic OS algebras

Recall that an OS-algebra A is graded commutative and generated in degree one.

(1) A is not necessarily quadratic. There are several necessary conditions on L but no nice equivalent condition is known.

(2) On the other hand, if $\mathcal{A}$ is supersolvable then A is Koszul whence also quadratic. Moreover $\mathcal{A}$ is supresolvable if and only if the defining ideal of A has a quadratic Gröbner basis.

(3) A being Koszul is equivalent to a topological property of the complement M. A space with this property is called a rational $K[\pi, 1]$ and can be defined using the rational model of the space.

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One corollary of A being Koszul (equiv., M being a rational $K[\pi, 1]$) is a connection between $G = \pi_1(M)$ and A.

More precisely assume A is quadratic and let $\Gamma_1 = G \supset \Gamma_2 \supset \cdots \supset \Gamma_p \supset \cdots$ be the lower central series of G (i.e., $\Gamma_p = [G, \Gamma_{p-1}]$). The Abelian group $G^* = (\oplus_p \Gamma_P/\Gamma_{p-1}) \otimes \mathbb{Q}$ has the natural structure of a graded Lie algebra induced by taken commutators in G. If U is the universal enveloping algebra of G^* then $U \simeq A^!$ and the Hilbert series of U is

$$H(U, t) = \prod_{p \geq 1} (1 - t^n)^{-\phi_p}$$

where $\phi_p = \operatorname{rk} \Gamma_p/\Gamma_{p+1}$.
Now if A is Koszul then

$$\pi(L, -t) = H(A, -t) = H(U, t)^{-1} = \prod_{p \geq 1} (1 - t^n)^{\phi_p}.$$

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Open problems

The interesting open problems in that circle of questions are as follows.

Which of the following implications can be inverted in the realm of OS algebras:

The defining ideal of A has a quadratic Gröbner basis $\Longrightarrow$ A is Koszul $\Longrightarrow$ $H(A,t)\cdot H(A^!,-t)=1$.

Motion planning

One of applications of arrangement theory is to topological robotics.

Let X be a topological space, thought of as the configuration space of a mechanical system. Given two points $A, B \in X$, one wants to connect them by a path in X; this path represents a continuous motion of the system from one configuration to the other. A solution to this motion planning problem is a rule (algorithm) that takes $(A, B) \in X \times X$ as an input and produces a path from A to B as an output.

Let PX denote the space of all continuous paths $\gamma : [0, 1] \to X$, equipped with the compact-open topology, and let $f : PX \to X \times X$ be the map assigning the end points to a path: $f(\gamma) = (\gamma(0), \gamma(1))$. The map f is a fibration whose fiber is the based loop space ΩX. The motion planning problem consists of finding a section s of this fibration.

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Topological complexity

The section s cannot be continuous, unless X is contractible. M.Farber has defined $\mathbf{TC}(X)$, the topological complexity of X, as the smallest number k such that $X \times X$ can be covered by open sets $U_1, \ldots, U_k$, so that for every $i = 1, \ldots, k$ there exists a continuous section $s_i : U_i \to PX, f \circ s_i = \mathrm{id}$.

Farber's topological complexity has various properties allowing one to obtain several lower and upper bounds for it in terms of other invariants. However precise computation of $\mathbf{TC}(X)$ for concrete X is often a challenging problem.

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Properties of $\mathbf{TC}(X)$.

1. $\mathbf{TC}(X)$ is a homotopy invariant of X.
2. If X is r-connected then

$$\mathbf{TC}(X) < \frac{2\dim X + 1}{r+1} + 1.$$

3. $\mathbf{TC}(X)$ is greater than the zero-divisor-cup-length of the ring $H^*(X;k)$ for every field k.

TC(M)

Arrangement complements (as well as configuration spaces of points in $\mathbb{R}^n$) can be very naturally viewed as configuration spaces of mechanical systems. For instance, for a braid arrangement the complement M appears for the system of several robots on a large plane. The complement of an arbitrary arrangement in $\mathbb{C}^\ell$ would appear if a robot has 2ℓ parameters and the hyperplanes represent linear obstructions.

The known results about **TC**(M) are as follows:
(1) If $\mathcal{A}$ is an arrangement of n hyperplanes in general position in $\mathbb{C}^\ell$ then $\mathbf{TC}(M(\mathcal{A})) = \min\{n+1, 2\ell+1\}$.

(2) Let $\mathcal{A}$ be a Coxeter arrangement of classical types (A,B, or D). Then $\mathbf{TC}(M(\mathcal{A})) = 2\,\mathrm{rk}(\mathcal{A})$.

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