A universal algebra for (spaces of) knots

Ryan Budney
rybu@uvic.ca

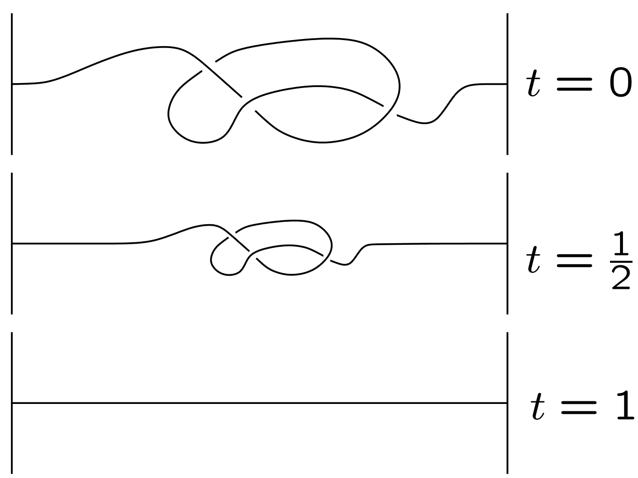
University of Victoria

Embedding spaces

Definition: For compact manifolds M and N let Emb(M, N) be the set of embeddings of M in N. Let $\mathcal{K}_{n,j} \equiv$ Emb$(\mathbf{R}^j, \mathbf{R}^n)$ be the set of embeddings of $\mathbf{R}^j$ in $\mathbf{R}^n$ which agrees with the inclusion $x \to (x, 0)$ outside of $\mathbf{I}^j \equiv [-1, 1]^j$.

Example: The C^0-uniform topology is the wrong topology on embedding spaces. Consider $F : [0, 1] \times \mathcal{K}_{n,j} \to \mathcal{K}_{n,j}$ where

$$F(t, f)(x) = \begin{cases} (1-t)f(\frac{1}{1-t} \cdot x) & t < 1 \\ (x, 0) & t = 1 \end{cases}$$



The ‘right’ topology on $\mathcal{K}_{n,j}$

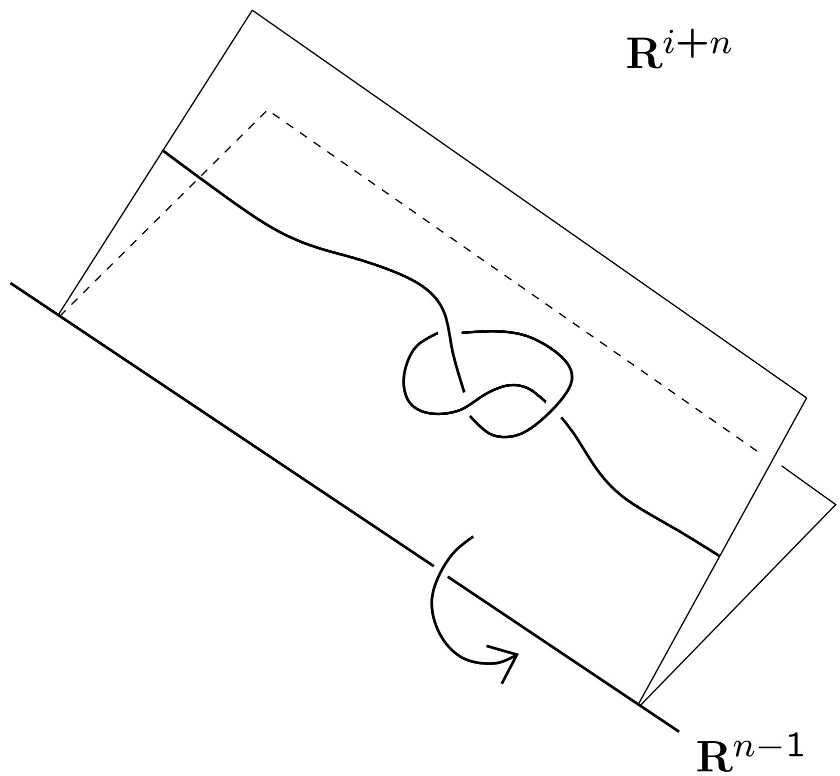
Definition: The C^k-metric on $\mathcal{K}_{n,j}$ is given by
$d_k(f,g) = max_{x \in \mathbf{R}^j} \left\{ \sqrt{\Sigma_{i=0}^{k} |D^i f_{(x)} - D^i g_{(x)}|^2} \right\}$

The topology on $\mathcal{K}_{n,j}$ is defined to be the one generated by all the C^k-metrics $k \in \{0, 1, 2, \cdots\}$.

The topology on Emb(M, N) is defined analogously, via charts.

Example of relevance

Proposition: Elements of $\pi_i\mathcal{K}_{n,1}$ produce 'spun' embeddings of S^{i+1}'s in $\mathbf{R}^{i+n}$.



Theorem: (B '07) Spinning produces an isomorphism

$$\mathbf{Z} \simeq \pi_2\mathcal{K}_{4,1} \to \pi_0\mathrm{Emb}(S^3, \mathbf{R}^6) \simeq \mathbf{Z}$$

Embedding and the cubes operad

Theorem: (B '07) The connect-sum operation $\mathcal{K}_{3,1} \times \mathcal{K}_{3,1} \to \mathcal{K}_{3,1}$ extends to an action of the operad of 2-cubes on $\mathcal{K}_{3,1}$. Moreover, $\mathcal{K}_{3,1}$ is free with respect to this action,

$$\mathcal{K}_{3,1} \simeq \mathcal{C}_2(\mathcal{P} \sqcup \{*\})$$

Here $\mathcal{P} \subset \mathcal{K}_{3,1}$ is subspace consisting of knots that are prime.

Corollary:

$$\Omega B\mathcal{K}_{3,1} \simeq \Omega^2\Sigma^2(\mathcal{P} \sqcup \{*\})$$

$$\simeq \Omega^2 \bigvee_{[f]\in\pi_0\mathcal{P}} (S^2 \vee \Sigma^2\mathcal{K}_{3,1}(f))$$

Theorem: (B, Cohen '09) $H_*(\mathcal{K}_{3,1}; \mathbf{Q})$ is a free Poisson algebra.

Little cubes operad

A (single) little n-cube is an embedding $L : \mathbf{I}^n \to \mathbf{I}^n$ such that $L = l_1 \times \cdots \times l_n$ where $l_i : \mathbf{I} \to \mathbf{I}$ has the form $l_i(t) = a_i t + b_i$ with $a_i > 0$. $\mathbf{I} = [-1, 1]$.

A j-tuple $(L_1, L_2, \cdots, L_j)$ is 'j little n-cubes' if

1) L_i is a little n-cube for all $1 \leq i \leq j$.

2) The interior of the images of L_i and L_k are disjoint provided $i \neq k$.

The space of j little n-cubes is denoted $\mathcal{C}_n(j)$.

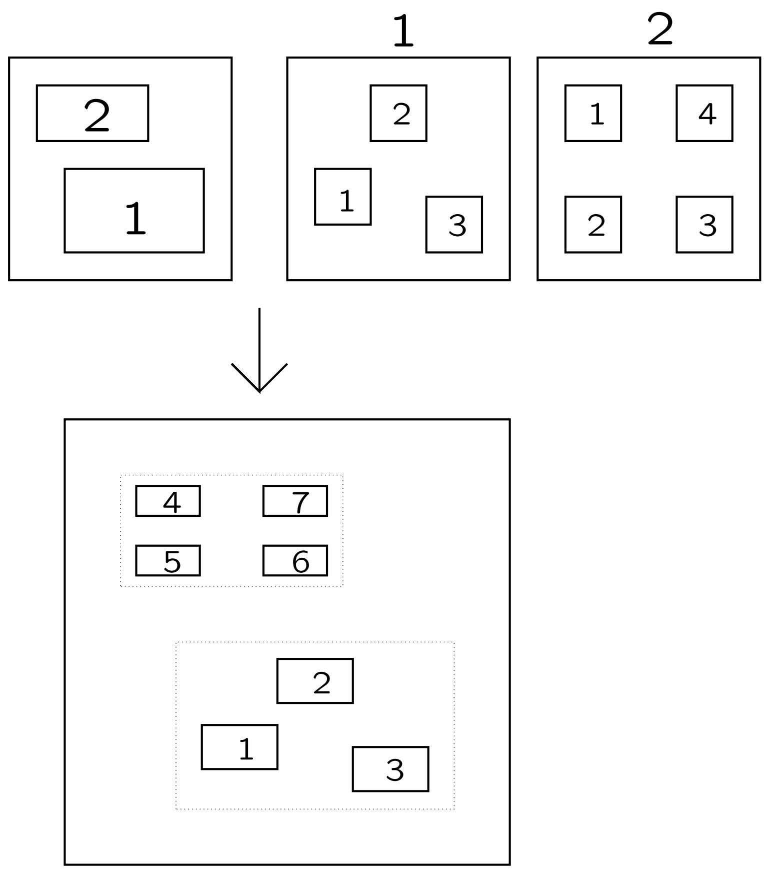
$\mathcal{C}_n := \{\mathcal{C}_n(0), \mathcal{C}_n(1), \mathcal{C}_n(2), \cdots\}$ is the operad of little n-cubes, the operad structure being given by the composition maps

$$\mathcal{C}_n(j) \times \Big(\mathcal{C}_n(k_1) \times \cdots \times \mathcal{C}_n(k_j)\Big) \to \mathcal{C}_n(k_1 + \cdots + k_j)$$

Composition of little cubes

$$\mathcal{C}_n(j) \times \Big(\mathcal{C}_n(k_1) \times \cdots \times \mathcal{C}_n(k_j)\Big) \to \mathcal{C}_n(k_1 + \cdots + k_j)$$

Example:



$$\mathcal{C}_2(2) \times (\mathcal{C}_2(3) \times \mathcal{C}_2(4)) \to \mathcal{C}_2(7)$$

Σ-operad action on a space X

$$\kappa_i : \mathcal{C}_n(i) \times X^i \to X \quad i \in \{1, 2, 3, \cdots\}$$

Satisfying:

1) Identity. $\kappa_1(Id_{\mathbf{I}^n}, x) = x$ for all $x \in X$.

2) Symmetry. $\kappa_i(L.\sigma, \sigma.x) = \kappa_i(L, x)$ for all $L \in \mathcal{C}_n(j)$, $x \in X^j$ and $\sigma \in \Sigma_n$.

3) Associativity. Let $Y_i = \mathcal{C}_n(k_i) \times X^{k_i}$ then

$$\begin{array}{ccc} \mathcal{C}_n(j) \times Y_1 \times \cdots \times Y_j & \longrightarrow & \mathcal{C}_n(j) \times X^j \\ \downarrow & & \downarrow \\ \mathcal{C}_n(k_1 + \cdots + k_j) \times X^{k_1 + \cdots + k_j} & \longrightarrow & X \end{array}$$

commutes.

The point of the cubes operad

Observation: Let

$$\Omega^n(X,*) = Maps((I^n, \partial I^n), (X,*))$$

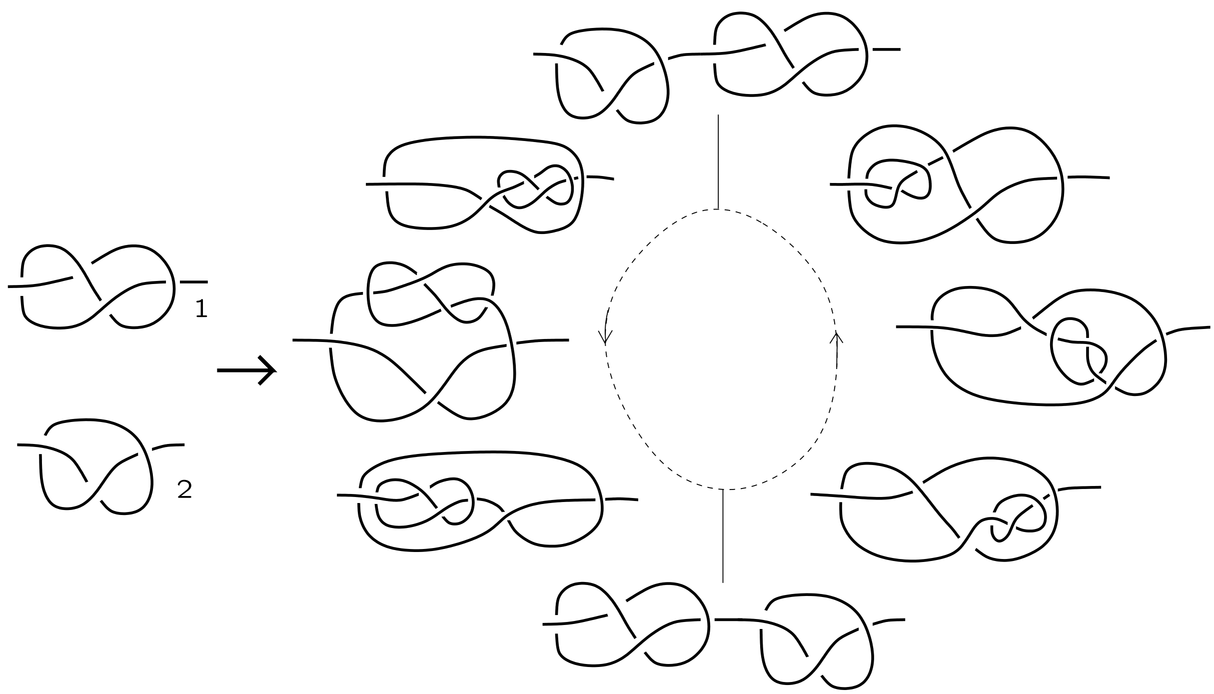
then $\Omega^n(X,*)$ admits an action of $\mathcal{C}_n$.

Observation: If a space X admits an action of $\mathcal{C}_n$, then $\pi_0 X$ is a monoid.

Theorem: (Boardman-Vogt '68, May '74) If X admits an action of $\mathcal{C}_n$ making $\pi_0 X$ into a group, then $X \simeq \Omega^n X'$ for some space X'.

Cubes operad and knot spaces

A $\mathcal{C}_2$-action on $\mathcal{K}_{3,1}$ which extends the connect-sum operation would give a 'pull one knot through the other' family of maps.



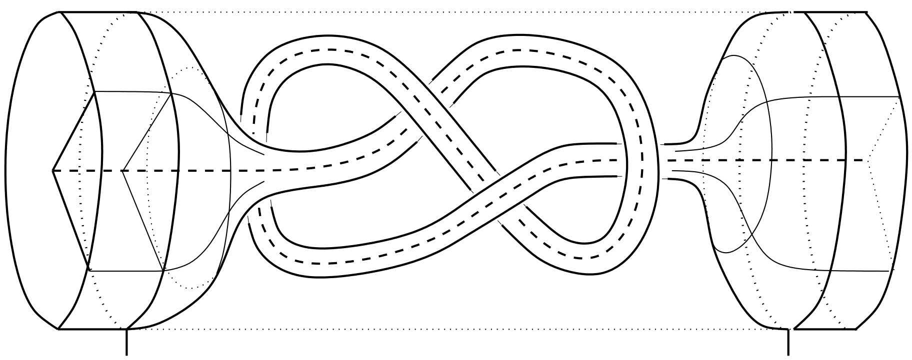
This existance of such an action was conjectured by Victor Turchin.

A knot space on which cubes act

$$\mathsf{EC}(k, M) =$$

$$\{f \in Emb(\mathbf{R}^k \times M, \mathbf{R}^k \times M), supp(f) \subset \mathbf{I}^k \times M\}$$

$f \in \mathsf{EC}(1, D^2)$



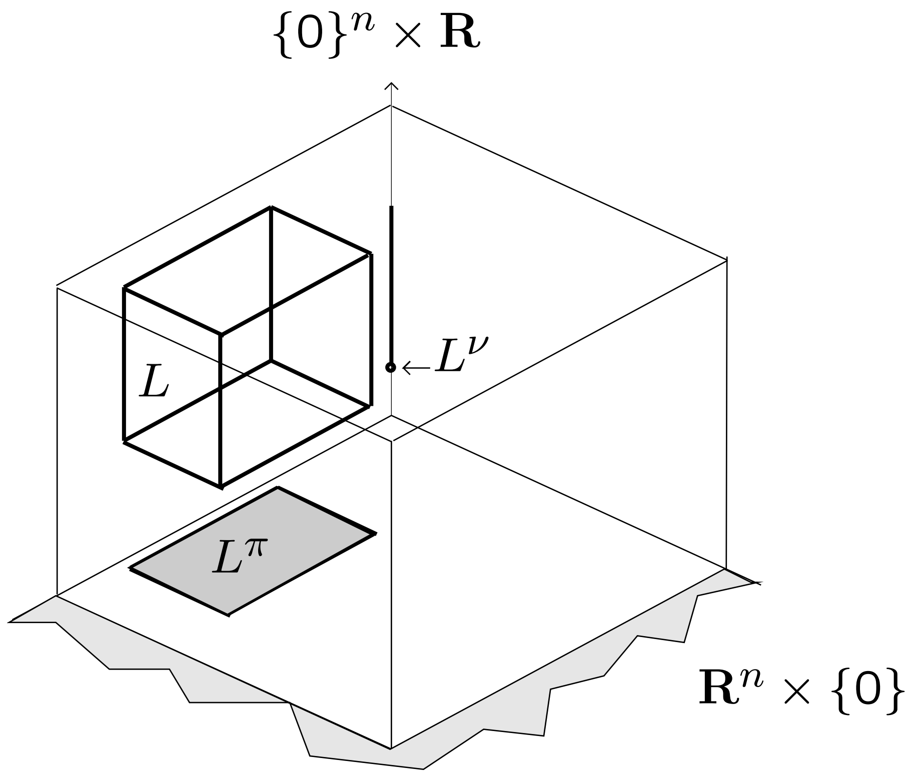
Theorem: (B '07) $\mathsf{EC}(k, M)$ admits an action of the operad of little $(k+1)$-cubes.

The relation between $\mathsf{EC}(1, D^2)$ and $\mathcal{K}_{3,1}$ is the trivial fibration

$$\Omega SO_2 \to \mathsf{EC}(1, D^2) \to \mathcal{K}_{3,1}$$

Construction of the action

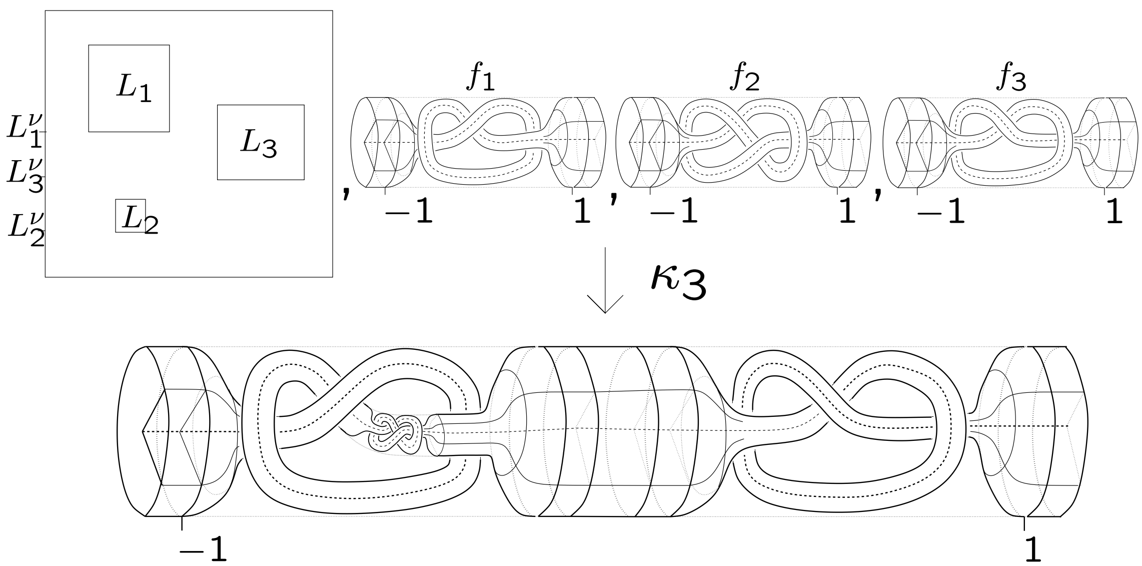
Given a little $(n+1)$-cube L we define a little n-cube L^π and a real number L^ν



Definition of the cubes action on $\mathsf{EC}(n, M)$

$$\mathcal{C}_{n+1}(j) \times \mathsf{EC}(n,M)^j \xrightarrow{\kappa_j} \mathsf{EC}(n,M)$$

$$(L_1, \cdots, L_j, f_1, \cdots, f_j) \longrightarrow L^{\pi}_{\sigma(j)} . f_{\sigma(j)} \circ \cdots \circ L^{\pi}_{\sigma(1)} . f_{\sigma(1)}$$



$$L^{\pi}_{\sigma(i)} . f := (L^{\pi}_{\sigma(i)} \times Id_M) \circ f \circ (L^{\pi}_{\sigma(i)} \times Id_M)^{-1}$$

$\sigma : \{1, \cdots, j\} \to \{1, \cdots, j\}$ is any permutation such that $L^{\nu}_{\sigma(j)} \geq L^{\nu}_{\sigma(j-1)} \geq \cdots \geq L^{\nu}_{\sigma(1)}$

Some related results

Theorem: (Cerf, Morlet '69)

$$\mathsf{EC}(n,*) \simeq \Omega^{n+1}(PL_n/O_n)$$

Theorem: (Salvatore '06) $\mathcal{C}_2$ acts on $\mathcal{K}_{n,1}$ and $\mathsf{EC}(1, D^{n-1})$ provided $n \geq 4$.

Theorem: (Sakai '08) Provided n is even, the Browder operation on $H_*\mathsf{EC}(1, D^{n-1})$ is non-trivial. In particular, the bracket $[e, v] \in H_{3n-8}\mathsf{EC}(1, D^{n-1})$ is non-zero where

- $v \in H_{2n-6}\mathsf{EC}(1, D^{n-1})$
- $e \in H_{n-3}\Omega SO_{n-1}$ adjoint to $\Sigma S^{n-3} \to SO_{n-1}$ the clutching map for TS^{n-1}.

Overlapping cubes operad

Definition: A collection of j *overlapping* n-*cubes* is an equivalence class of pairs (L, σ) where

$$L = (L_1, \cdots, L_j)$$

is j little n cubes.

$$(L, \sigma) \sim (L', \sigma') \quad \text{iff} \quad L = L' \quad \text{and}$$

$$\sigma^{-1}(i) < \sigma^{-1}(k) \Longleftrightarrow \sigma'^{-1}(i) < \sigma'^{-1}(k)$$

$$\text{whenever } L_i^\circ \cap L_k^\circ \neq \emptyset.$$

$$\mathcal{C}'_n(j) = \{(L, \sigma)\}$$

Overlapping cubes and $\mathsf{EC}(k, M)$

Proposition: The map $\mathcal{C}_{n+1} \to \mathcal{C}'_n$ is an equivalence of operads.

$$(L_1, \cdots, L_j) \longmapsto (L_1^\pi, \cdots, L_j^\pi, \sigma)$$

where $L^\nu_{\sigma(j)} \geq L^\nu_{\sigma(j-1)} \geq \cdots \geq L^\nu_{\sigma(1)}$

Proposition: $\mathcal{C}'_k$ acts on $\mathsf{EC}(k, M)$ moreover the action of $\mathcal{C}_{k+1}$ on $\mathsf{EC}(k, M)$ factors through this action.

Splicing diagrams for $\mathrm{EC}(n, M)$

Definition: An *k-splicing diagram* is an equivalence class of pair (L, σ) where $\sigma \in \Sigma_k$, $L_0 \in \mathrm{EC}(n, M)$ and $L_i : I^n \times M \to I^n \times M$ an embedding.

$$(L, \sigma) \sim (L', \sigma') \iff L = L' \text{ and if}$$

$$L_i((I^n)^\circ \times M) \cap L_j((I^n)^\circ \times M) \neq \emptyset$$

then

$$\sigma^{-1}(i) < \sigma^{-1}(j) \iff \sigma'^{-1}(i) < \sigma'^{-1}(j)$$

For $0 \leq i < j \leq k$ there is also the *continuity constraint* on splicing diagrams:

$$\overline{L_{\sigma_i}(I^n \times M)) \setminus L_{\sigma_j}(I^n \times M)} \cap L_{\sigma_j}((I^n)^\circ \times \partial M) = \emptyset$$

Where we interpret $\sigma(0) = 0$ for the purposes of the continuity constraint.

The splicing diagram operad of $\mathsf{EC}(n, M)$

Definition: $\mathcal{SD}_n^M(k) = \{(L, \sigma) : k\text{-splicing diagram}\}$.

We want $\mathcal{SD}_n^M = \sqcup_k \mathcal{SD}_n^M(k)$ to be an operad, and

$$\mathcal{SD}_n^M(k) \times \mathsf{EC}(n, M)^k \to \mathsf{EC}(n, M)$$

defined by

$$(L_0, L_1, \cdots, L_k, \sigma), (f_1, \cdots, f_k) \longmapsto$$

$$(L_{\sigma(k)} \circ f_{\sigma(k)} \circ L_{\sigma(k)}^{-1}) \circ \cdots \circ (L_{\sigma(1)} \circ f_{\sigma(1)} \circ L_{\sigma(1)}^{-1}) \circ L_0$$

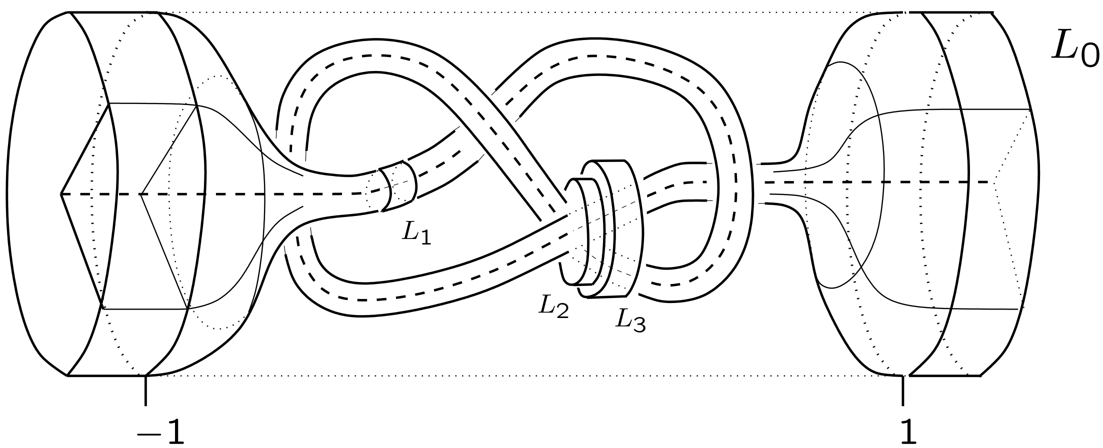
$$\equiv \left(\bigcirc_{i=1}^k L_{\sigma(i)} f_{\sigma(i)} L_{\sigma(i)}^{-1} \right) L_0$$

to be an action of $\mathcal{SD}_n^M$ on $\mathsf{EC}(n, M)$ where $L_{\sigma(i)} \circ f_{\sigma(i)} \circ L_{\sigma(i)}^{-1}$ is defined to be the identity outside of the image of $L_{\sigma(i)}$.

Splicing in pictures

a splicing diagram

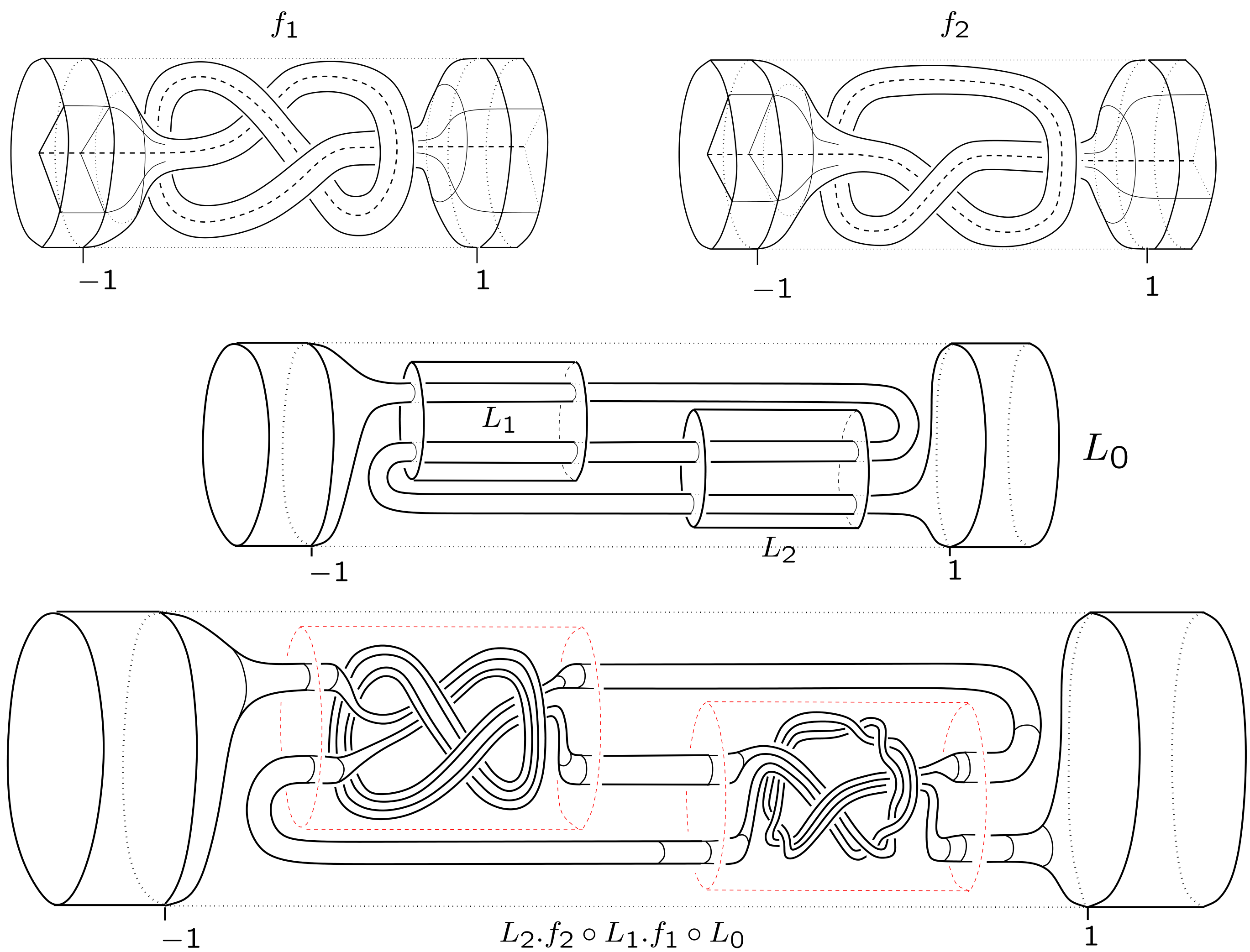
$(L,\sigma) \in \mathcal{SD}_1^{D^2}(3)$



$\sigma^{-1}(2) < \sigma^{-1}(3)$ the only restriction $\sigma \in \Sigma_3$

Splicing in pictures

the splicing process



The definition of the operad $\mathcal{SD}_n^M$

$$\mathcal{SD}(k)\times\mathcal{SD}(j_1)\times\cdots\times\mathcal{SD}(j_k)\to\mathcal{SD}(j_1+\cdots+j_k)$$

$$(J_0,J_1,\cdots,J_k,\alpha),(L_1,\sigma_1),\cdots,(L_k,\sigma_k)\longmapsto J.L$$

$J.L$ is defined to have 0-th entry

$$\left(\bigcirc_{i=1}^{k}(J_{\alpha(i)}L_{\alpha(i)0}J_{\alpha(i)}^{-1})\right)J_0$$

(a,b)-th coordinate entry is given by

$$\left(\bigcirc_{i=\alpha^{-1}(a)+1}^{k}(J_{\alpha(i)}L_{\alpha(i)0}J_{\alpha(i)}^{-1})\right)J_aL_{a,b}$$

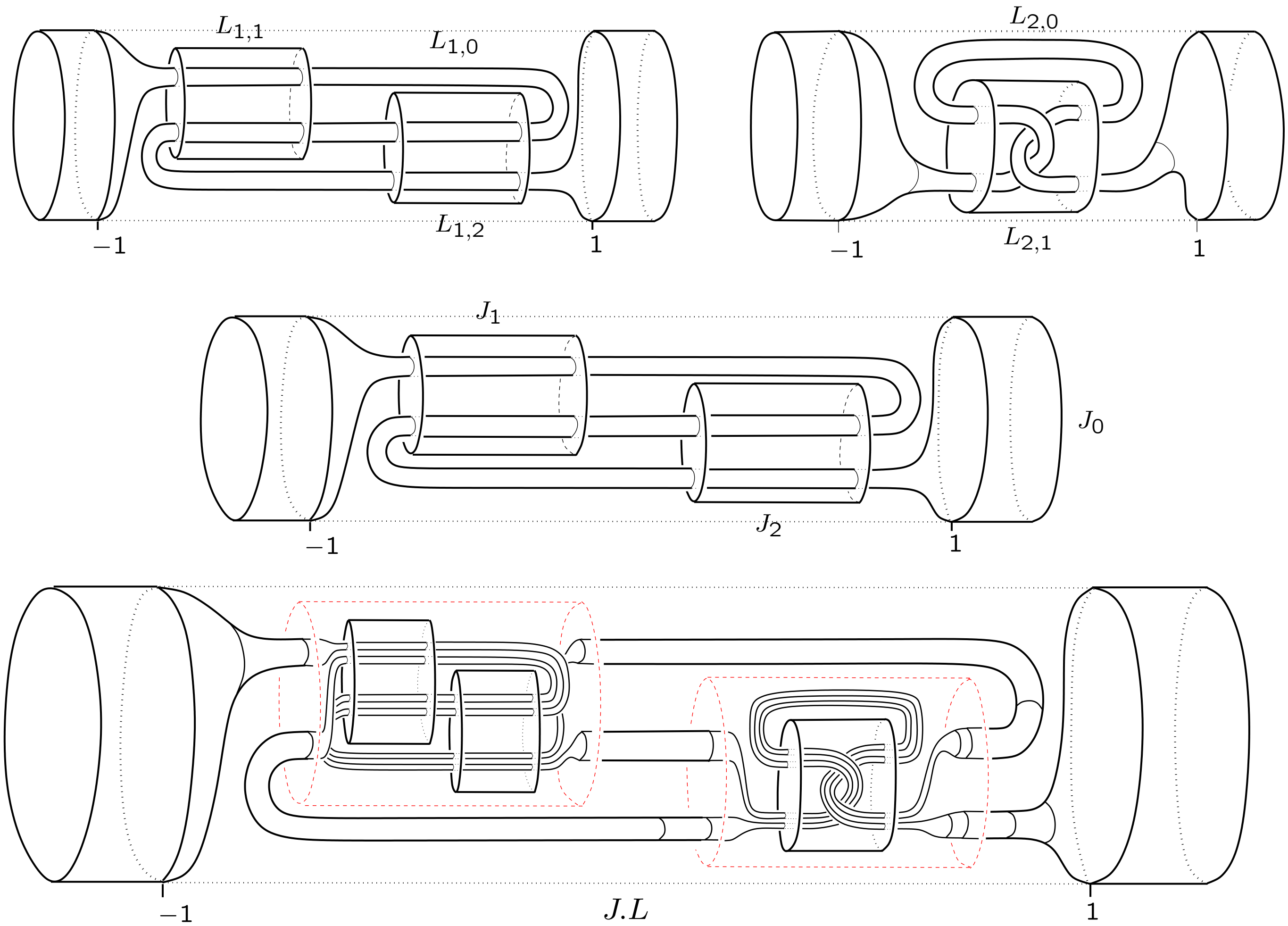
where we lexicographically identify

$$\{(a,b):a\in\{1,\cdots,k\},b\in\{1,\cdots,j_a\}\}$$

$$\text{with }\ \{1,2,\cdots,j_1+\cdots+j_k\}$$

Theorem: $\mathcal{SD}_n^M\equiv\mathcal{SD}$ is a $\Sigma^*\wr Diff(I^n\times M)$-operad, where $\Sigma^*\wr Diff(I^n\times M)=\sqcup_j Diff(I^n\times M)\times\left(\Sigma_j\ltimes Diff(I^n\times M)^j\right)$.

Structure maps for $\mathcal{SD}_1^{D^2}$

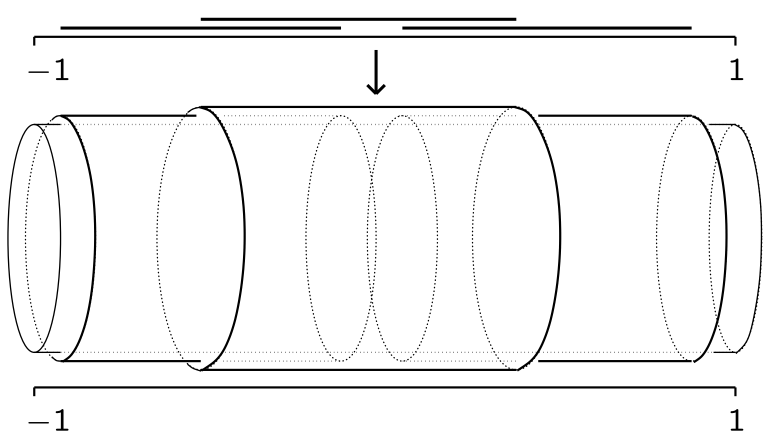


Overlapping cubes in splicing operad

Proposition: The map $\mathcal{C}'_n \to \mathcal{SD}_n^M$ given by

$$(L_1, \cdots, L_k, \sigma) \longmapsto (Id_{\mathbf{R}^n \times M}, L_1 \times Id_M, \cdots, L_k \times Id_M, \sigma)$$

is an inclusion of operads. Moreover, the action of $\mathcal{SD}_n^M$ on EC(n, M) extends the action of $\mathcal{C}'_n$ on EC(n, M).



$n = 1$, $M = D^2$, $k = 3$ example

Freeness of $\mathcal{K}_{3,1}$ over $\mathcal{SP}$

Let $\mathcal{SP} \subset \mathcal{SD}_1^{D^2}$ be the suboperad where the elements $(L_0, L_1, \cdots, L_k, \sigma)$ satisfy:

(a) $L_0 \in \hat{\mathcal{K}}_{3,1}$ always.

(b) $(L_0, L_1, \cdots, L_k)$ is a non-split link.

(c) $\mathcal{SP}(0) = \{Id_{\mathbf{R}\times D^2}\}$.

Let $\mathcal{TH} \subset \hat{\mathcal{K}}_{3,1}$ be the subspace of non-trivial torus knots and hyperbolic knots.

Theorem: (B) The action of the operad $\mathcal{SP}$ on $\hat{\mathcal{K}}_{3,1}$ induces a homotopy-equivalence

$$\sqcup_{n=0}^{\infty} \left(\mathcal{SP}(n) \times_{\Sigma_n \wr O_2} \mathcal{TH}^n \right) \to \hat{\mathcal{K}}_{3,1}$$

Splicing and geometrization

Definition: Given a 3-manifold M let $c(M)$ denote the sum of the number of components of M split along its canonical (geometric) decomposition.

Theorem: (Schubert, Burde, Murasugi, JSJ, Thurston) Given $f \in \mathcal{K}_{3,1}$ its complement has complexity one if and only if it is a non-trivial torus or hyperbolic knot. $L \in \mathcal{SP}$ has a complement with complexity one if and only if it is hyperbolic or Seifert. Moreover, every link $L \in \mathcal{SP}$ is an iterated-splice of complexity-one links.

Proposition: The splicing map satisfies

$$\mathcal{SP}(k) \times \prod_{i=1}^{k} \mathcal{SP}(j_i) \to \mathcal{SP}(\sum_{i=1}^{k} j_i)$$

$$c(J.(L_1, \cdots, L_k)) = c(J) + \sum_{i=1}^{k} c(L_i)$$

except in the cases:

(a) Either J or one of the L_i's is a Hopf link.

(b) One of the L_i's is the unknot.

(c) For some i, L_i is not prime, and J contains two parallel components (an untwisted annulus whose boundary is two components of J), of which J_i is one.

An exception we call a *redundant splice*.

Definition: An operad $\mathcal{O}$ is *homotopically generated* by a subspace $\mathcal{X} \subset \mathcal{O}$ if the inclusion $\overline{\mathcal{X}} \to \mathcal{O}$ is a homotopy-equivalence where $\overline{\mathcal{X}}$ is the operadic closure of $\mathcal{X}$ in $\mathcal{O}$.

Theorem: $\mathcal{SP}$ is homotopically generated by the union of the three subspaces

1. $\mathcal{KCL}_k \subset \mathcal{SP}(k)$ of $(k+1)$-component keychain links, $k = 2, 3, \cdots$.

2. $\mathcal{SFL} \subset \mathcal{SP}(1)$ of 2-component Seifert links (Hopf link excluded).

3. $\mathcal{HGL}_k \subset \mathcal{SP}(k)$ of $(k+1)$-component hyperbolic links, $k = 1, 2, 3, \cdots$.

$$\mathcal{SP}(k) \times_{\Sigma_k \wr O_2} \prod_{i=1}^{k} \mathcal{SP}(j_i) \to \mathcal{SP}(\sum_{i=1}^{k} j_i)$$

is a homotopy-equivalence between non-redundant splice components.

Proposition: There are homotopy-equivalences:

- $\mathcal{C}_1'(k) \times O_2^k \to \mathcal{KCL}_k$
- $\sqcup_{p,q} S^1 \times S^1 \to \mathcal{SFL}$.
- $\sqcup (S^1 \times S^1) \times (S^1)^k \to \mathcal{HGL}_k$

The above homotopy-equivalences can be made to be equivariant with respect to the action of $\Sigma^* \wr O_2$ on $\mathcal{SP}$. Moreover, the maps

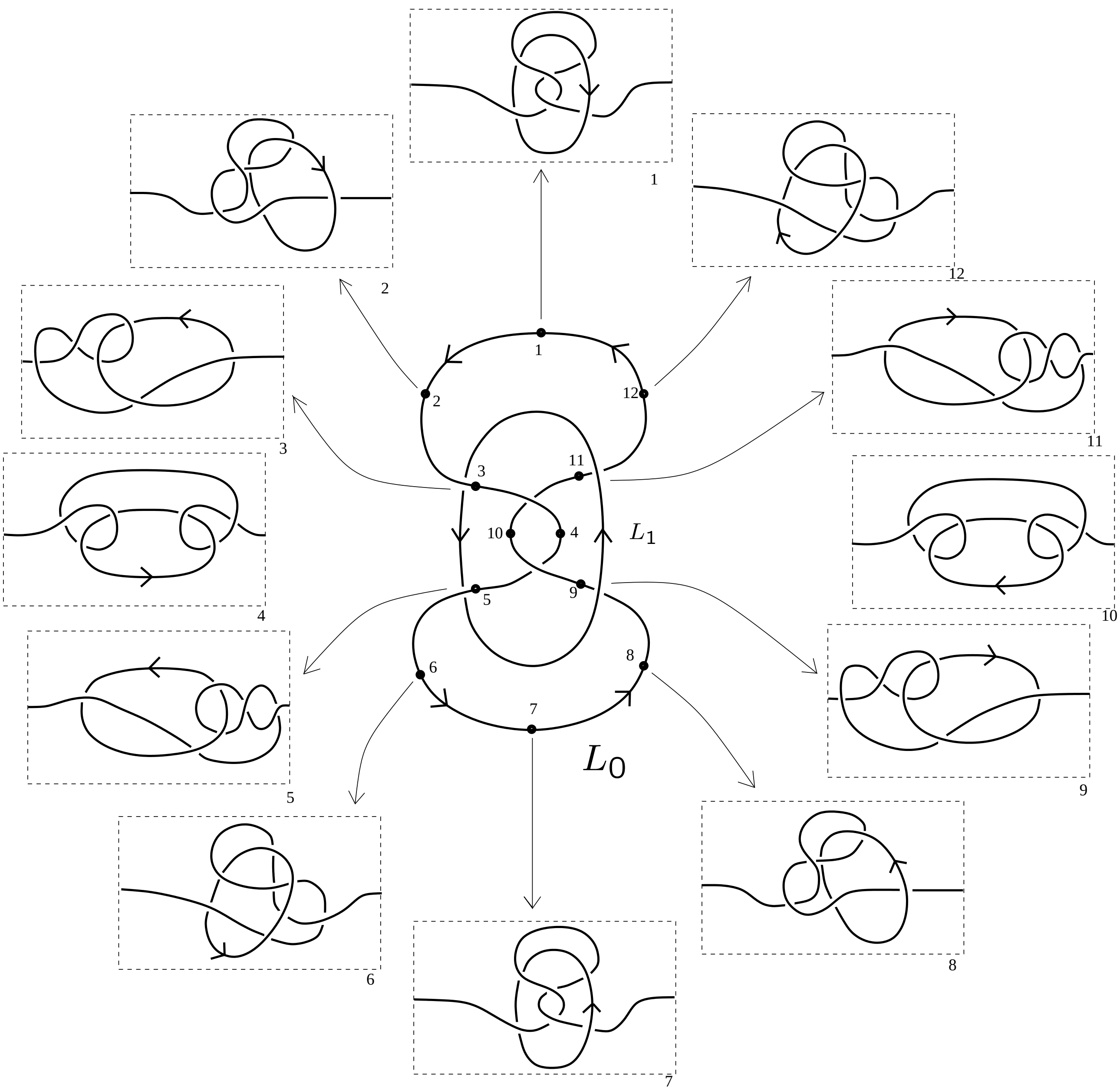
$$\mathcal{SP}(k) \times_{\Sigma_k \wr O_2} \prod_{i=1}^{k} \mathcal{SP}(j_i) \to \mathcal{SP}(\sum_{i=1}^{k} j_i)$$

$$\sqcup_{n=0}^{\infty} \left(\mathcal{SP}(n) \times_{\Sigma_n \wr O_2} \mathcal{TH}^n \right) \to \widehat{\mathcal{K}}_{3,1}$$

are similarly equivariant.

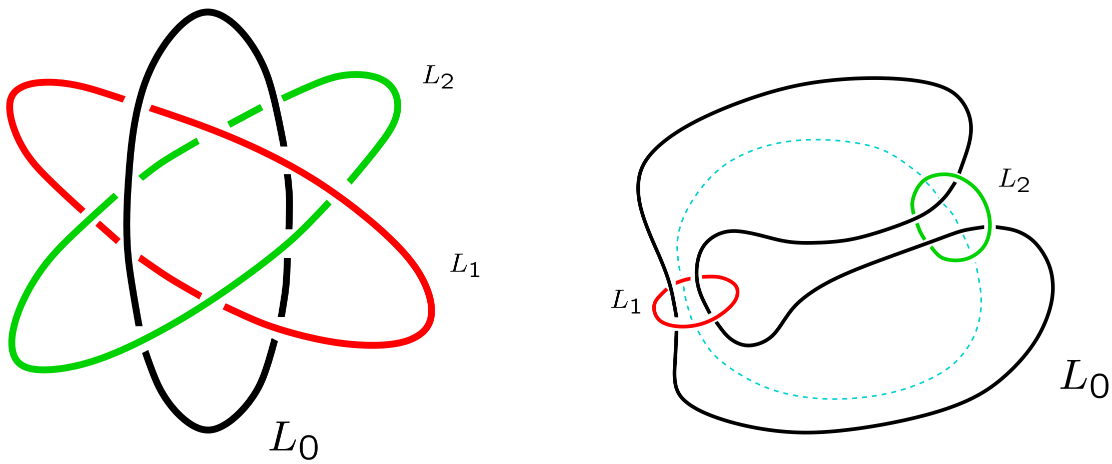
Corollary: $\mathcal{SP}$ contains a suboperad such that each path-component is finite-dimensional.

Whitehead link example



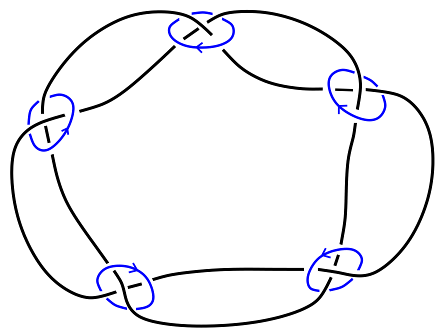
Symmetric links and positions I

Borromean rings, two maximal symmetry positions



Symmetric links and positions II

Sakuma's example



'pure translation' symmetry example