Representations of reflection groups on the cohomology of varieties

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Outline

Background

Theme

If W is a finite group generated by reflections of a real vector space V, there are some associated varieties X on which W acts. One gets representations of W on the cohomologies $H^i(X(\mathbb{C}), \mathbb{Q})$ and $H^i(X(\mathbb{R}), \mathbb{Q})$. The characters of these representations give a more complete description than the Betti numbers alone. The relationship between $H^i(X(\mathbb{C}), \mathbb{Q})$ and $H^i(X(\mathbb{R}), \mathbb{Q})$ can be subtle.

Main example: $W = S_n$ $(n \geq 2)$, with its irreducible representation

$$V_n = \mathbb{R}\{e_1, e_2, \cdots, e_n\}/\mathbb{R}\{e_1 + e_2 + \cdots + e_n\}.$$

The transpositions $(i\ j)$ act as reflections in the hyperplanes

$$H_{ij} = \{\sum a_k e_k \mid a_i = a_j\}, \text{ for } i, j \in [n], i \neq j,$$

which form the famous braid arrangement.

To encode the characters of representations U_n of the symmetric groups S_n, it is helpful to use the generating function

$$\sum_{n\geq 0} \mathrm{ch}_{S_n}(U_n) \in \mathbb{Q}[\![p_1, p_2, p_3, \cdots]\!] =: A,$$

where ch denotes the Frobenius characteristic

$$\mathrm{ch}_{S_n}(U_n) = \frac{1}{n!} \sum_{w\in S_n} \mathrm{tr}(w, U_n) \prod_i p_i^{\#(i\text{-cycles of } w)}.$$

Note that $\sum_{n\geq 0} \mathrm{ch}_{S_n}(\mathbf{1}_n) = \prod_i (\sum_a \frac{p_i^a}{i^a a!}) = \exp(\sum_i \frac{p_i}{i}) =: \mathrm{Exp}.$

In the power series ring A we define plethystic substitution:

1. $(f+g)[h] = f[h] + g[h]$, $(fg)[h] = f[h]g[h]$,
2. $p_i[f+g] = p_i[f] + p_i[g]$, $p_i[fg] = p_i[f]p_i[g]$,
3. $p_i[p_j] = p_{ij}$, and $p_i[t] = t^i$ if t is an additional indeterminate.

This is an associative operation with identity p_1.

The algebraic torus

Assume W is the Weyl group of a root system. Let $\Lambda \subset V$ be the lattice dual to the root lattice. W acts on the torus $T = \Lambda \otimes_{\mathbb{Z}} \mathbb{G}_m$. If $W = S_n$, take $\Lambda_n = \mathbb{Z}\{e_1, e_2, \cdots, e_n\}/\mathbb{Z}\{e_1 + e_2 + \cdots + e_n\}$ and

$$T_n = \frac{\{(a_1, a_2, \cdots, a_n) \in \mathbb{A}^n \mid a_i \neq 0, \forall i\}}{(a_1, a_2, \cdots, a_n) \sim (\lambda a_1, \lambda a_2, \cdots, \lambda a_n)} \cong (\mathbb{G}_m)^{n-1}.$$

Proposition

The cohomology ring $H^*(T(\mathbb{C}), \mathbb{Q})$ is canonically isomorphic to the exterior algebra $\bigwedge^*(\widehat{V_{\mathbb{Q}}})$. In particular, for any $w \in W$,

$$\sum_i \mathrm{tr}(w, H^i(T(\mathbb{C}), \mathbb{Q}))\,(-t)^i = \det_V(1 - tw).$$

For the symmetric groups, it is easy to deduce that

$$p_1 + \sum_{n \geq 2} \sum_i \mathrm{ch}_{S_n}(H^i(T_n(\mathbb{C}), \mathbb{Q}))\,(-t)^i = \frac{\mathrm{Exp}[(1-t)p_1] - 1}{1 - t}.$$

By contrast, since $\mathbb{R}^\times$ retracts onto $\{1,-1\}$, $T(\mathbb{R})$ is canonically homotopy equivalent to the finite set $\Lambda/2\Lambda$. So for any $w \in W$,

$$\mathrm{tr}(w, H^0(T(\mathbb{R}), \mathbb{Q})) = \#(\text{elements of } \Lambda/2\Lambda \text{ fixed by } w) =: \pi^{(2)}(w).$$

If $W = S_n$, $\pi^{(2)}(w)$ counts the w-stable subsets of $[n]$ which have even size. We deduce that

$$1 + p_1 + \sum_{n \geq 2} \mathrm{ch}_{S_n}(H^0(T_n(\mathbb{R}), \mathbb{Q})) = \mathrm{Exp.Cosh},$$

where Cosh is the sum of the even-degree terms of Exp.
Although the answers for $T(\mathbb{C})$ and $T(\mathbb{R})$ look very different, there is a partial connection between them: if w has odd order, then

$$\pi^{(2)}(w) = 2^{\dim V^w} = \det_V(1 + w),$$

so $\mathrm{tr}(w, H^0(T(\mathbb{R}), \mathbb{Q})) = \sum_i \mathrm{tr}(w, H^i(T(\mathbb{C}), \mathbb{Q}))$ in this case.

The complement of the reflecting hyperplanes

Let M be the affine variety defined by the hyperplane complement. There is a stark topological contrast between $M(\mathbb{C})$ and $M(\mathbb{R})$:

- $M(\mathbb{C})$ is a $K(\pi, 1)$ space for the pure Artin group of W.
- $M(\mathbb{R})$ is a union of contractible cones called chambers. The group W permutes these chambers simply transitively. Hence $H^0(M(\mathbb{R}), \mathbb{Q})$ is the regular representation of W, which has character $\mathrm{tr}(w, H^0(M(\mathbb{R}), \mathbb{Q})) = |W|\delta_{1w}$.

If $W = S_n$, then

$$M_n = \{\sum a_k e_k \in V_n \mid a_i \neq a_j, \forall i \neq j\}$$

is the configuration space of ordered n-tuples in $\mathbb{A}^1$. Note that the image $\mathcal{M}_n$ of M_n in $\mathbb{P}(V_n)$ is the configuration space $\mathcal{M}_{0,n+1}$ of ordered $(n+1)$-tuples in $\mathbb{P}^1$ (since the last point can be set to ∞). So here the action of S_n actually extends to S_{n+1}.

There are two general ways to describe $H^*(M(\mathbb{C}), \mathbb{Q})$:

- as the Orlik–Solomon algebra of the hyperplane arrangement;
- as the Whitney homology of the lattice Π_W of reflection subgroups of W (= the lattice of hyperplane intersections).

If $W = S_n$ there is an inductive approach. The whole vector space $V_n(\mathbb{C})$ can be stratified according to which coordinates are equal: $M_n(\mathbb{C})$ is one stratum, and every other stratum is homeomorphic to $M_m(\mathbb{C})$ for some $m < n$. The alternating sum $\sum(-1)^i H_c^i$ is additive on stratifications, and one can distinguish the H_c^i's using Hodge weights (hyperplane complements are minimally pure).

Theorem (Lehrer)

$$1 + p_1 + \sum_{n \geq 2} \sum_i \mathrm{ch}_{S_n}(H^i(M_n(\mathbb{C}), \mathbb{Q}))\,(-t)^i = \mathrm{Exp}[t^{-1} L[tp_1]],$$

where $L = \sum \frac{\mu(d)}{d} \log(1 + p_d)$ *is the solution of* $\mathrm{Exp}[L] = 1 + p_1$.

Lehrer found similar character formulas for types B and D.

The De Concini–Procesi model of the arrangement

This projective variety $\overline{\mathcal{M}}$ is defined as the closure of the image of

$$M \to \prod_{W' \in \Pi_W^{\mathrm{irr},\mathrm{rk}\geq 2}} \mathbb{P}(V/V^{W'}).$$

If $W = S_n$, $\overline{\mathcal{M}}_n$ is the moduli space $\overline{\mathcal{M}}_{0,n+1}$ of stable genus 0 curves with $n+1$ marked points ($\Pi_{S_n}^{\mathrm{irr},\mathrm{rk}\geq 2} \leftrightarrow \{K \subseteq [n], |K| \geq 3\}$). De Concini and Procesi gave a presentation of $H^*(\overline{\mathcal{M}}(\mathbb{C}), \mathbb{Q})$ (generalizing the $\overline{\mathcal{M}}_n$ case due to Keel), where the generators in H^2 are the classes of the divisors labelled by $W' \in \Pi_W^{\mathrm{irr},\mathrm{rk}\geq 2}$.
It follows that $H^{2i}(\overline{\mathcal{M}}(\mathbb{C}), \mathbb{F}_2) \cong H^i(\overline{\mathcal{M}}(\mathbb{R}), \mathbb{F}_2)$, and one can conclude that if $w \in W$ has odd order,

$$\sum_i (-1)^i \operatorname{tr}(w, H^{2i}(\overline{\mathcal{M}}(\mathbb{C}), \mathbb{Q})) = \sum_i (-1)^i \operatorname{tr}(w, H^i(\overline{\mathcal{M}}(\mathbb{R}), \mathbb{Q})).$$

But overall, $H^*(\overline{\mathcal{M}}(\mathbb{C}), \mathbb{Q})$ and $H^*(\overline{\mathcal{M}}(\mathbb{R}), \mathbb{Q})$ are very different.

$\overline{\mathcal{M}}_n$ has a stratification where the strata are indexed by rooted trees with n leaves (these indicate the intersection pattern of the components of the curve); each stratum is isomorphic to a product $\mathcal{M}_{n_1} \times \mathcal{M}_{n_2} \times \cdots \times \mathcal{M}_{n_k}$ where $n_1 + n_2 + \cdots + n_k = n$.

Theorem (Ginzburg–Kapranov 1994)

The following elements of A are inverses for plethystic substitution:

$$p_1 + \sum_{n \geq 2} \sum_i \mathrm{ch}_{S_n}(H^{2i}(\overline{\mathcal{M}}_n(\mathbb{C}), \mathbb{Q}))\, t^i \text{ and}$$

$$p_1 - \sum_{n \geq 2} \sum_i \mathrm{ch}_{S_n}(H^{i}(\mathcal{M}_n(\mathbb{C}), \mathbb{Q}))\, (-1)^i t^{n-2-i}.$$

Since Lehrer's result determines the latter, this can be viewed as a very complicated recursion for $\mathrm{ch}_{S_n}(H^*(\overline{\mathcal{M}}_n(\mathbb{C}), \mathbb{Q}))$. There are some results about other W: bases for $H^*(\overline{\mathcal{M}}(\mathbb{C}), \mathbb{Q})$ in types B and D (Yuzvinsky), character formula in type B (myself).

Surprisingly, $\overline{\mathcal{M}}_n(\mathbb{R})$ is analogous to $M_n(\mathbb{C})$.

- As shown by Davis–Januszkiewicz–Scott, it is a $K(\pi, 1)$ space for the pure cactus group.
- Etingof–Henriques–Kamnitzer–Rains gave a presentation for $H^*(\overline{\mathcal{M}}_n(\mathbb{R}), \mathbb{Q})$ which is like the Orlik–Solomon algebra, but with generators labelled by $K \subseteq [n]$, $|K| = 3$.

There is also an analogue of the Whitney homology description:

Theorem (Rains 2006)

Let $\Pi_W^{(2)}$ be the subposet of Π_W consisting of W' whose irreducible components all have even rank. There is a natural isomorphism

$$H^*(\overline{\mathcal{M}}(\mathbb{R}), \mathbb{Q}) \cong \bigoplus_{W' \in \Pi_W^{(2)}} \widetilde{H}_*^{\Pi_W^{(2)}}((\{1\}, W'), \mathbb{Q}) \otimes \mathrm{or}(V/V^{W'}).$$

Rains deduced a formula for $\mathrm{ch}_{S_n}(H^*(\overline{\mathcal{M}}_n(\mathbb{R}), \mathbb{Q}))$, similar to Lehrer's for $M_n(\mathbb{C})$. He and I generalized this to types B and D.

The toric variety of the arrangement

Associated to the lattice $\Lambda \subset V$, and the fan defined by the hyperplane arrangement, there is a toric variety $\overline{T}$, which is nonsingular and projective. (It has an alternative definition as a Hessenberg variety, a certain closed subvariety of the flag variety.) Choose a chamber, and let I denote the set of hyperplanes which bound it. Then the T-orbits on $\overline{T}$ are labelled by the cones in the fan, which are in bijection with $\coprod_{J \subseteq I} W/W_J$.

Using the Stanley–Reisner presentation of $H^*_T(\overline{T}(\mathbb{C}), \mathbb{Q})$, one gets:

Theorem (Procesi)

$$\sum_i \mathrm{tr}(w, H^{2i}(\overline{T}(\mathbb{C}), \mathbb{Q}))\, t^i = \det{}_V(1 - tw) \sum_{J \subseteq I} 1_{W_J}^W(w) (\frac{t}{1-t})^{|I \setminus J|}.$$

For the symmetric groups, Stanley deduced that

$$1 + p_1 + \sum_{n \geq 2} \sum_i \mathrm{ch}_{S_n}(H^{2i}(\overline{T_n}(\mathbb{C}), \mathbb{Q}))\, t^i = \frac{1-t}{\mathrm{Exp}[(t-1)p_1] - t}.$$

When $W = S_n$, $\overline{T_n}$ can be described as a De Concini–Procesi model of the arrangement of coordinate hyperplanes; namely, it is the closure of the image of

$$T_n \to \prod_{K \subseteq [n], |K| \geq 2} \mathbb{P}(\mathbb{A}^K).$$

Identifying this arrangement with the hyperplanes $H_{i,n+1}$ in V_{n+1} gives a birational map $\overline{\mathcal{M}}_{n+1} \to \overline{T_n}$. Hence $H^*(\overline{T_n}(\mathbb{C}), \mathbb{Q})$ maps (injectively?) to the subring of $H^*(\overline{\mathcal{M}}_{n+1}(\mathbb{C}), \mathbb{Q})$ generated by the divisors labelled by the subsets $K \cup \{n+1\} \subseteq [n+1]$.
Rains' theorem applies to general building sets; here, the relevant poset is that of even-size subsets of $[n]$. One deduces:

$$1 + p_1 + \sum_{n \geq 2} \sum_i \mathrm{ch}_{S_n}(H^i(\overline{T_n}(\mathbb{R}), \mathbb{Q}))\,(-t)^i = \mathrm{Exp.}(\mathrm{Cosh}^{\varepsilon}[t^{1/2} p_1])^{-1},$$

where ε indicates multiplying by the sign character. In particular, $\dim H^i(\overline{T_n}(\mathbb{R}), \mathbb{Q}) = \binom{n}{2i} A_{2i}$, where A_{2i} is the Euler secant number.
Problem: find a presentation for $H^*(\overline{T_n}(\mathbb{R}), \mathbb{Q})$.

Lehrer and I have a project to describe $H^*(\overline{T}(\mathbb{R}),\mathbb{Q})$ in general; preferably as a representation of W, but at present even the Betti numbers are unknown. Using the fact that $\sum(-1)^i H^i_c$ is additive on the stratification into T-orbits, we found the 'Euler character':

Theorem (H.–Lehrer 2009)

$$\sum_i (-1)^i \mathrm{tr}(w, H^i(\overline{T}(\mathbb{R}),\mathbb{Q})) = \sum_{J\subseteq I} (-1)^{|J|}\,\mathrm{Ind}_{W_J}^W(\varepsilon.\pi^{(2)}_{W_J})(w).$$

Over $\mathbb{R}$, we have no Hodge weights to distinguish individual H^i's. There is an alternative proof. It is known that the real toric variety $\overline{T}(\mathbb{R})$ can be constructed by gluing together $2^{\dim V}$ copies of the polytope dual to the fan; in the resulting cell chain complex, C_j has character $\sum_{J\subseteq I,|J|=j} \mathrm{Ind}_{W_J}^W(\varepsilon.\pi^{(2)}_{W_J})$.

Problem: compute the homology of this complex.