Curves Homothetically Shrinking by Curvature

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ABSTRACT. Note for students with a proof of the fact that the circle is the only smooth embedded curve in the plane homothetically shrinking by curvature.
This result is due to Abresch and Langer [**1**] and, independently, to Epstein and Weinstein [**3**].

Let $\gamma \subset \mathbb{R}^2$ be a smooth connected embedded curve, fixing a reference point on the curve γ we have an arclength parameter s which gives a unit tangent vector field and a unit normal vector field ν, which is the counterclockwise rotation of $\pi/2$ in $\mathbb{R}^2$ of the vector τ. Then, the curvature is given by $k = \langle \partial_s \tau \,|\, \nu \rangle$.

If γ is homothetically shrinking around the origin of $\mathbb{R}^2$ when moving by curvature, it is easy to see that the normal projection of the position vector γ at every point must be proportional to the curvature *vector* $k\nu = \overline{k} = \partial^2_{ss}\gamma$ which gives the velocity of the evolving curve. That is, $\lambda \overline{k} + \langle \gamma \,|\, \nu \rangle = 0$ for a nonnegative constant λ. Dilating the curve by a factor $1/\sqrt{\lambda}$ we can assume that $\lambda = 1$. Multiplying then this equation for ν we get the characterizing equation $k + \langle \gamma \,|\, \nu \rangle = 0$.

THEOREM 1. *The only smooth complete embedded curves in $\mathbb{R}^2$ satisfying $k + \langle \gamma \,|\, \nu \rangle = 0$ are the lines through the origin and the unit circle.*

PROOF. The relation $k = -\langle \gamma \,|\, \nu \rangle$ implies $k_s = k\langle \gamma \,|\, \tau \rangle$. Suppose that at some point $k = 0$, it follows that also $k_s = 0$ at the same point, hence, by the uniqueness theorem applied to such first order ODE for the curvature k, we can conclude that k is identically zero and we are dealing with a line L, which then, as $\langle x \,|\, \nu \rangle = 0$ for every $x \in L$, it must contain the origin of $\mathbb{R}^2$.

From now on we will suppose that k is everywhere nonzero and, possibly reversing the orientation of the curve, we can assume that $k > 0$ at every point, that is the curve is strictly convex.

Computing the derivative of $|\gamma|^2$,

$$\partial_s |\gamma|^2 = 2\langle \gamma \,|\, \tau \rangle = 2k_s/k = 2\partial_s \log k$$

we get $k = Ce^{|\gamma|^2/2}$ for some constant $C > 0$.

We consider the new coordinate $\theta = \arccos \langle e_1 \,|\, \nu \rangle$, which is global since the curve is convex.
Differentiating with respect to the arclength parameter we have $\partial_s \theta = k$ and

$$k_\theta = k_s / k = \langle \gamma \,|\, \tau \rangle \qquad k_{\theta\theta} = \frac{\partial_s k_\theta}{k} = \frac{1 + k\langle \gamma \,|\, \nu \rangle}{k} = \frac{1}{k} - k\,. \tag{1}$$

Multiplying both sides of the last equation by $2k_\theta$ we get $\partial_\theta [k_\theta^2 + k^2 - \log k^2] = 0$, that is, the quantity $k_\theta^2 + k^2 - \log k^2$ is equal to some constant E along all the curve. Notice that such quantity E cannot be less than 1, moreover, if $E = 1$ we have k must be constant and equal to one along the curve which consequently must be the unit circle centered in the origin of $\mathbb{R}^2$.
Considering the case $E > 1$, from the strict convexity of the function $x - \log x$, it follows that k must be uniformly bounded from above and below and it can never be equal to zero. Hence, remembering that $k = Ce^{|\gamma|^2/2}$, the image of the curve is contained in a compact set of $\mathbb{R}^2$ and, by the embeddedness hypothesis, the curve must be closed and simple.

We now look at the critical points of the curvature k. Since $k_{\theta\theta} = \frac{1}{k} - k$, it holds that $k_{\theta\theta} \neq 0$ when $k_\theta = 0$, otherwise, the second order ODE for k would imply $k_\theta = 0$ identically. This way we would have $k = 1$ identically and we would be in the case of the unit circle centered in the origin of $\mathbb{R}^2$ as before. Since $k_{\theta\theta} \neq 0$ when $k_\theta = 0$, the critical points of the curvature are not degenerate hence, by the compactness of the curve, they are isolated and finite.
Moreover, by looking at the equation for the curvature (1) we can see easily that $k_{\min} < 1$ and $k_{\max} > 1$.

Suppose now that $k(0) = k_{\max}$ and $k(\overline{\theta})$ for $\overline{\theta} > 0$ are a pair of consecutive critical values for k then, the curvature is strictly decreasing in the interval $[0, \overline{\theta}]$ and, again by the second order ODE, the curvature function (hence also the curve) function is symmetric with respect to $\theta = 0$ and $\theta = \overline{\theta}$. This clearly implies that $k(\overline{\theta})$ must be the minimum $k_{\min}$ as every critical point is not degenerate.

By the four vertex theorem **[5, 6]**, there are at least four critical points of k and consequently the curve is composed by at least four pieces like the one above, hence, by the assumption that the curve is embedded, the curvature $k(\theta)$ must be a periodic function with period $T > 0$ not larger than π (since 2π is an obvious multiple of the period) and $\overline{\theta} = T/2$. Precisely, by the previous symmetry argument, the period must be π/n for some even $n \in \mathbb{N}$.

By a straightforward computation, starting by differentiating the equation $k_{\theta\theta} = \frac{1}{k} - k$, one gets $(k^2)_{\theta\theta\theta} + 4(k^2)_\theta = 4k_\theta/k$, then we compute

$$\begin{aligned}4\int_0^{T/2} \sin 2\theta\,\frac{k_\theta}{k}\,d\theta &= \int_0^{T/2} \sin 2\theta\,[(k^2)_{\theta\theta\theta} + 4(k^2)_\theta]\,d\theta\\ &= \sin 2\theta (k^2)_{\theta\theta}\big|_0^{T/2} - 2\int_0^{T/2}\cos 2\theta (k^2)_{\theta\theta}\,d\theta + 4\int_0^{T/2}\sin 2\theta (k^2)_\theta\,d\theta\\ &= 2\sin T[k(T/2)k_{\theta\theta}(T/2) + k_\theta^2(T/2)] - 2\cos 2\theta (k^2)_\theta\big|_0^{T/2}\\ &\quad - 4\int_0^{T/2}\sin 2\theta (k^2)_\theta\,d\theta + 4\int_0^{T/2}\sin 2\theta (k^2)_\theta\,d\theta\\ &= 2\sin T[k(T/2)k_{\theta\theta}(T/2) + k_\theta^2(T/2)]\\ &\quad - 4\cos T k(T/2)k_\theta(T/2) + 4k(0)k_\theta(0)\,.\end{aligned}$$

Now, since $k_\theta(0) = k_\theta(T/2) = 0$, $k(0) = k_{\max}$ and $k(T/2) = k_{\min}$, using the equation for the curvature $k_{\theta\theta} = 1/k - k$, we get

$$4\int_0^{T/2}\sin 2\theta\,\frac{k_\theta}{k}\,d\theta = 2\sin T(1 - k_{\min}^2)\,,$$

and this last term is nonnegative as $k_{\min} < 1$ and $0 < T \leq \pi$.
Looking at the left hand integral we see instead that the factor $\sin 2\theta$ is always nonnegative as $T \leq \pi$ and k_θ is always nonpositive in the interval $[0, T/2]$, as we assumed that we were moving from the maximum $k_{\max}$ at $\theta = 0$ to the minimum $k_{\min}$ at $\theta = T/2$, without crossing any critical point of k. This gives a contradiction and concludes the proof. □

REMARK 2. The original proof of Abresch and Langer (or the one by Epstein and Weinstein) is different, actually this result is a consequence of their general classification theorem.
To my knowledge, this "shortcut" in the embedded case is due to Chou and Zhu [**2**, Proposition 2.3].

We discuss a while the analysis in the case of an immersed closed curve.
The initial part of the proof in theorem still holds, that is, that is the curve is a line or $k \neq 0$ everywhere, it is bounded, and the ODEs (1) hold. The quantity $k_\theta^2 + k^2 - \log k^2$ is equal to some constant E which must be larger than one (otherwise we are dealing with a circle). Again the curve is symmetric with respect to the critical points of the curvature, which are all nondegenerate, isolated and finite. Hence, the curvature function is oscillating between its maximum and its minimum with some period $T > 0$ which is an integer fraction (at least by a factor 4) of an integer multiple (at least 2) of 2π.
Notice that there are two parameters here around, the rotation number of the closed curve and the number of critical points of the curvature.

Suppose that $k_{\min} < k_{\max}$ are these two consecutive critical values of k, It follows that they are two distinct positive zeroes of the function $E + \log k^2 - k^2$ when $E > 1$ with $0 < k_{\min} < 1 < k_{\max}$.
We have that the change $\Delta\theta$ in the angle θ along the piece of curve delimited by two consecutive points where the curvature assumes the values $k_{\min}$ and $k_{\max}$, must be the semiperiod $T/2$. Then, the analysis reduce to understand what are the admissible T. Such quantity $\Delta\theta$ is given by the integral

$$I(E) = \int_{k_{\min}}^{k_{\max}} \frac{dk}{\sqrt{E - k^2 + \log k^2}}\,.$$

Abresch and Langer (and also Epstein and Weinstein) by studying the behavior of this integral were able to classify *all* the immersed closed curves in $\mathbb{R}^2$ satisfying the structural equation $k + \langle \gamma \,|\, \nu \rangle = 0$.
They are a family of curves indexed by two parameters called *Abresch–Langer curves*, see [**1**] for the description, proofs and details.

We conclude stating and partially proving the main properties of the integral $I(E)$. It should be noticed that, by the discussion about the semiperiod T, the last statement implies Theorem 1.

PROPOSITION 3. *The function* $I : (1, +\infty) \to \mathbb{R}$ *satisfies*

(1) $\lim_{E\to 1^+} I(E) = \pi/\sqrt{2}$,
(2) $\lim_{E\to +\infty} I(E) = \pi/2$,
(3) $I(E)$ *is monotone nonincreasing.*

As a consequence $I(E) > \pi/2$.

PROOF. Notice that the study of the quantity $I(E)$ is equivalent to the study of the semi–period for the one dimensional Hamiltonian system with Hamiltonian function given by $H(k_\theta, k) = k_\theta^2 + k^2 - \log k^2$.

(1) As the global minimum 1 of the strictly convex potential $V(k) = k^2 - \log k^2$ is assumed at $k = 1$, the limiting value for the period of the Hamiltonian system when $E \to 1^+$ is equal to the period of the corresponding linearized system (see [**4**, Chap. 12]). The linearized Hamiltonian is $H_L(\hat{k}_\theta, \hat{k}) = \hat{k}_\theta^2 + \hat{k}^2$, which gives $\hat{k}_{\theta\theta} = -2\hat{k}$. The solution to this last ODE is clearly $\sqrt{2}\pi$–periodic, hence its semi–period is equal to $\pi/\sqrt{2}$.

(2) As $0 < k_{\min} < 1 < k_{\max}$ for $E > 1$, we can write

$$I(E) = \int_{k_{\min}}^{1} \frac{dk}{\sqrt{E - k^2 + \log k^2}} + \int_{1}^{k_{\max}} \frac{dk}{\sqrt{E - k^2 + \log k^2}} = I_-(E) + I_+(E)\,.$$

We want to prove that $\lim_{E\to+\infty} I_-(E) = 0$ and $\lim_{E\to+\infty} I_+(E) = \pi/2$. This claim is motivated by the structure of the potential, that is, as E increases more and more, the motion of the system can be seen as an "instantaneous reflection" against a wall situated at $k_{\min} \simeq 0$.

Introducing the variable $w = k/k_{\min}$ the first integral becomes

$$I_-(E) = k_{\min} \int_1^{1/k_{\min}} \frac{dw}{\sqrt{k_{\min}^2(1-w^2) + \log w^2}} \,.$$

Notice that, given a real number $0 < \alpha < 1$, it is always possible to find $\tilde{k}(\alpha)$ such that $|k_{\min}(1-w^2)| \leq \alpha |\log w^2|$ with $w \in [1, 1/k_{\min}]$ and $k_{\min} \leq \tilde{k}$. Fixing such an α, we have

$$\begin{aligned} 0 \leq I_-(E) \; &\leq \frac{k_{\min}}{\sqrt{1-\alpha}} \int_1^{1/k_{\min}} \frac{dw}{\sqrt{2\log w}} \\ &\leq \frac{k_{\min}}{\sqrt{1-\alpha}} \Big(\int_1^n \frac{dw}{\sqrt{2\log w}} + \int_n^{1/\sqrt{k_{\min}}} \frac{dw}{\sqrt{2\log w}} + \int_{1/\sqrt{k_{\min}}}^{1/k_{\min}} \frac{dw}{\sqrt{2\log w}} \Big) \\ &\leq k_{\min}(C_1 + C_2/\sqrt{k_{\min}} + o_{k_{\min}}(1)/k_{\min}) \,, \end{aligned}$$

hence, the claim on $I_-(E)$ follows.
Regarding $I_+(E)$, we proceed in a similar way changing again integration variable to $w = k/k_{\max}$. This way we obtain

$$\lim_{E\to+\infty} I_+(E) = \lim_{E\to+\infty} \int_{1/k_{\max}}^1 \frac{dw}{\sqrt{1 - w^2 + \frac{2\log w}{k_{\max}^2}}} = \lim_{E\to+\infty} \int_0^1 \chi_{[1/k_{\max},1]} \frac{dw}{\sqrt{1-w^2+\frac{2\log w}{k_{\max}^2}}} = \pi/2 \,,$$

where in the last equality we applied the dominated convergence theorem.

(3) See the original paper of Abresch and Langer **[1]** or the general result by Zevin and Pinsky **[7]**. □

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