CHAPTER 2

Distance from a smooth boundary: preliminary version

This version is in progress: please, take this into account. All corrections and comments are welcome.

In this chapter we collect some of the main properties of the distance function from a smooth boundary.

We denote by $\{e_1, \ldots, e_n\}$ the canonical basis of $\mathbb{R}^n$. Coordinates of points z in $\mathbb{R}^n$ are denoted as $(z^1, \ldots, z^n)$, and we write $z = z^k e_k = (z^1, \ldots, z^n)$, where we adopt the convention of summation on repeated indices. $\mathbb{R}^n$ is endowed with the euclidean norm $|\cdot|$, induced by the euclidean scalar product. The tangent space $T_z\mathbb{R}^n$ to $\mathbb{R}^n$ at z is a copy of $\mathbb{R}^n$ which is independent of z. There is an identification between a point $z \in \mathbb{R}^n$ and the position vector of z, which belongs to $T_z\mathbb{R}^n$; we will use this identification in the sequel.

Let $Y = (Y^1, \ldots, Y^m) : \mathbb{R}^n \to \mathbb{R}^m$ be a smooth vector field. The Jacobian $(m \times n)$ matrix representing the differential $dY(z)$ of Y at z is indicated by $JY(z)$. If $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$, the ij-entry $(JY(z))_{ij}$ of $JY(z)$ is $\frac{\partial Y^i}{\partial z^j}(z)$, so that the i-th column of the transposed matrix $JY(z)^T$ is $\nabla Y^i(z)$. We write $\nabla Y(z) = JY(z)^T$. If $n = m$, the determinant of the linear map $dY(z)$ is denoted by $\det(JY(z))$ or by $\det(\nabla Y(z))$.

We set

$$\mathcal{K}(\mathbb{R}^n) := \{K \subset \mathbb{R}^n : K \text{ compact}\}.$$

If $F \subseteq \mathbb{R}^n$ is a nonempty set and $z \in \mathbb{R}^n$, we let

$$\mathrm{dist}(z, F) := \inf_{x \in F} |x - z|.$$

We also let $\mathrm{dist}(z, \emptyset) := +\infty$.

Remark 2.0.1. Note that

- $\mathrm{dist}(z, F) = \mathrm{dist}(z, \overline{F})$ and $\mathrm{dist}(z, \mathbb{R}^n \setminus F) = \mathrm{dist}(z, \mathbb{R}^n \setminus \mathrm{int}(F))$, where $\overline{F}$ and $\mathrm{int}(F)$ denote the topological closure and the topological interior of F respectively;
- $(\mathrm{dist}(z, F))^2 = \inf_{y \in F} |y - z|^2$;
- $\mathrm{dist}(\cdot, F)$ is Lipschitz continuous, and therefore almost everywhere differentiable by Rademacher's theorem. Moreover, if we indicate by $\nabla = (\frac{\partial}{\partial z^1}, \ldots, \frac{\partial}{\partial z^n})$ the gradient vector[1], we have that $|\nabla \mathrm{dist}(\cdot, F)| \leq 1$ almost everywhere.

In what follows we will use the notation $\nabla_i = \frac{\partial}{\partial z^i}$. If $f : \mathbb{R}^n \to \mathbb{R}$ is a function of class $\mathcal{C}^1$, we identify the one-covector $df(z)$ and the vector $\nabla f(z)$ in the usual way, i.e.,

[1] Vector fields defined on $\mathbb{R}^n$ are considered as columns; we omit the symbol of transpositon when we write the vector fields in components.

$\langle df(z), v\rangle = \nabla f(z) \cdot v$ for any $v \in \mathbb{R}^n$, where $\langle \cdot, \cdot \rangle$ denotes the duality between $\mathbb{R}^n$ and its dual space, and $\cdot$ is the euclidean scalar product between vectors. Consequently, the scalar product will be also denoted as $\langle \cdot, \cdot \rangle$.

Given $z \in \mathbb{R}^n$, we set

$$\mathrm{pr}_F(z) := \{x \in F : |z - x| = \mathrm{dist}(z, F)\}. \tag{2.1}$$

REMARK 2.0.2. Let $F \subseteq \mathbb{R}^n$ be a nonempty closed set and $z \in \mathbb{R}^n \setminus F$. It is possible to prove that $\mathrm{dist}(\cdot, F)$ is differentiable at z if and only if $\mathrm{pr}_F(z)$ consists of only one element, namely $\mathrm{pr}_F(z) = \{x\}$, $x \in F$. In this case $\nabla \mathrm{dist}(z, F) = \frac{z-x}{|z-x|}$, so that

$$x = \mathrm{pr}_F(z) = z - \mathrm{dist}(z, F)\nabla \mathrm{dist}(z, F);$$

moreover for any $\lambda \in (0, 1]$ we have that $\mathrm{pr}_F(\lambda z + (1-\lambda)x) = \{x\}$. In particular, $\mathrm{dist}(\cdot, F)$ is differentiable at any point $\lambda z + (1 - \lambda)x$. Furthermore, if $\mathrm{pr}_F(z) = \{x\}$, then pr_F is continuous at x, i.e., for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\mathrm{pr}_F(B_\delta(z)) \subset B_\varepsilon(x)$.

For any $\rho > 0$ we write

$$F^+_\rho := \{z \in \mathbb{R}^n : \mathrm{dist}(z, F) < \rho\}, \qquad F^-_\rho := \{z \in \mathbb{R}^n : \mathrm{dist}(z, \mathbb{R}^n \setminus F) > \rho\}. \tag{2.2}$$

Since $\overline{\mathbb{R}^n \setminus F} = \mathbb{R}^n \setminus \mathrm{int}(F)$, we have

$$F^+_\rho = \left(\overline{F}\right)^+_\rho, \qquad F^-_\rho = (\mathrm{int}(F))^-_\rho. \tag{2.3}$$

2.1. First order properties of the distance function

Given a set $E \subseteq \mathbb{R}^n$, we let

$$d(z, E) := \mathrm{dist}(z, E) - \mathrm{dist}(z, \mathbb{R}^n \setminus E), \qquad z \in \mathbb{R}^n \tag{2.4}$$

be the oriented distance function from the topological boundary ∂E of E, negative inside E. Note that $d(\cdot, E) = -d(\cdot, \mathbb{R}^n \setminus E)$.

Notation: when there is no ambiguity in the choice of the set E, for simplicity we use the notation

$$d(\cdot) = d(\cdot, E).$$

Moreover, if we want to remark that a quantity depends on ∂E rather than on E itself, we will use the notation

$$\Sigma = \partial E.$$

EXAMPLE 2.1.1. Let $\rho > 0$ and $d(\cdot) := d(\cdot, B_\rho(z_0))$ be the oriented distance from the boundary of the open ball $B_\rho(z_0)$ centered at $z_0 \in \mathbb{R}^n$ with radius ρ. Then $d(z) = |z-z_0|-\rho$, and for $z \neq z_0$ we have $\nabla d(z) = \frac{z-z_0}{|z-z_0|}$.

DEFINITION 2.1.2. *Let $E \subset \mathbb{R}^n$ be a set. We write $\partial E \in \mathcal{C}^\infty$ if there exists an open set* U *containing ∂E such that $d(\cdot, E) \in \mathcal{C}^\infty(\mathrm{U})$.*

When $\partial E \in \mathcal{C}^\infty \cap \mathcal{K}(\mathbb{R}^n)$ we can take U of the form $\mathrm{U} = (\partial E)^+_\rho$ for some $\rho > 0$.

REMARK 2.1.3. It is possible to show that $\partial E \in \mathcal{C}^\infty \cap \mathcal{K}(\mathbb{R}^n)$ if and only if ∂E is an $(n-1)$-dimensional compact manifold of class $\mathcal{C}^\infty$.

We indicate by $\mathrm{n}^E : \partial E \to \mathbb{R}^n$ the unit normal vector field to ∂E pointing toward $\mathbb{R}^n \setminus E$; when no confusion is possible, we write $\mathrm{n}^E = e_k \mathrm{n}^k = \mathrm{n} = (\mathrm{n}^1, \ldots, \mathrm{n}^n)$, considered as a column vector.

THEOREM 2.1.4. *Let* $\partial E \in \mathcal{C}^\infty \cap \mathcal{K}(\mathbb{R}^n)$ *and let* U *be a tubular neighbourhood such that* $d \in \mathcal{C}^\infty(\mathrm{U})$. *Then*

(i) d *satisfies the eikonal equation in* U*:*

$$|\nabla d(z)|^2 = 1, \qquad z \in \mathrm{U}; \tag{2.5}$$

(ii) $\mathrm{pr}_\Sigma(z)$ *is a singleton for any* $z \in \mathrm{U}$, *and*

$$\mathrm{pr}_\Sigma(z) = \{z - d(z)\nabla d(z)\}.$$

Moreover

$$\nabla d(z) = \nabla d(\mathrm{pr}_\Sigma(z)). \tag{2.6}$$

In view of (2.5) we have

$$\nabla d(x) = \mathrm{n}^E(x), \qquad x \in \partial E.$$

Note that the ij-component of $\nabla \mathrm{pr}_\Sigma(z)$ reads as

$$\mathrm{Id}_{ij} - \nabla_i d \nabla_j - d\nabla^2_{ij} d \qquad \text{in } \mathrm{U}, \tag{2.7}$$

where Id_{ij} is the ij-component of the identity map Id in $\mathbb{R}^n$, and $\nabla^2 = (\nabla^2_{ij})$ is the Hessian matrix[2] in $\mathbb{R}^n$, where we use the notation $\nabla^2_{ik} = \frac{\partial^2}{\partial z^i \partial z^k}$.

In particular, if $x \in \Sigma$, $\nabla \mathrm{pr}_\Sigma(x)$ coincides with the orthogonal projection $P_{T_x\Sigma} : \mathbb{R}^n \to T_x\Sigma$ on the tangent space $T_x\Sigma \subset \mathbb{R}^n$ to Σ at x,

$$\nabla \mathrm{pr}_\Sigma(x) = P_{T_x\Sigma}.$$

We denote by $N_x\Sigma$ the normal line to ∂E at x, and by $P_{N_x\Sigma} = \mathrm{Id} - P_{T_x\Sigma}$ the orthogonal projection on $N_x\Sigma$.

2.1.1. Extensions. Let $u \in \mathcal{C}^\infty(\partial E)$, and let $u^e \in \mathcal{C}^\infty(\mathrm{U})$ be a smooth extension of u on U. The vector field $\nabla u^e - \langle \nabla u^e, \nabla d\rangle \nabla d$ restricted to ∂E is independent of the particular extension of u; it is called the tangential gradient of u on ∂E and denoted by $\nabla^\Sigma u$. Clearly $\nabla^\Sigma u(x) = P_{T_x\Sigma}(\nabla u^e(x))$ for $x \in \partial E$. The tangential gradient is sometimes denoted as $\delta = \nabla - \langle \nabla, \mathrm{n}\rangle \mathrm{n} = (\delta_1, \ldots, \delta_n)$, $\delta_i = \nabla^\Sigma_i$.

DEFINITION 2.1.5. *We define* $\overline{u} : \mathrm{U} \to \mathbb{R}$ *as* $\overline{u}(z) := u(\mathrm{pr}_\Sigma(z))$ *for any* $z \in \mathrm{U}$.

Then $\overline{u} \in \mathcal{C}^\infty(\mathrm{U})$ is an extension of u on U, and

$$\nabla \overline{u} = \nabla^\Sigma u \qquad \text{on } \partial E. \tag{2.8}$$

[2] Sometimes, to simplify notation, we will write $\nabla^2 d = B$.

REMARK 2.1.6. For any $k = 1, \ldots, n$ let $\pi_k(z) := \langle z, e_k \rangle = z^k$, so that

$$\mathrm{Id} = (\pi_1, \ldots, \pi_n).$$

Then $\overline{\pi_j}(z) = z^j - d(z)\nabla_j d(z)$,

$$\langle e_i, \nabla^\Sigma \pi_j \rangle = P_{T_x ij}, \qquad i, j = 1, \ldots, n.$$

Let $X = (X^1, \ldots, X^n) : \partial E \to \mathbb{R}^n$ be a smooth vector field. The scalar quantity $\langle e_k, \nabla^\Sigma X^k \rangle$ is called the tangential divergence of X and denoted by $\mathrm{div}_\Sigma X$. If $X^e = (X^{1e}, \ldots, X^{ne}) : \mathrm{U} \to \mathbb{R}$ is any smooth extension of X on U we have

$$\mathrm{div}_\Sigma X = \mathrm{div} X^e - \langle \nabla X^e \mathrm{n}, \mathrm{n} \rangle = \mathrm{div} X^e - \frac{\partial X^{ie}}{\partial z^j} \mathrm{n}^i \mathrm{n}^j \qquad \text{on } \Sigma, \tag{2.9}$$

where div is the divergence in $\mathbb{R}^n$. Note that $\mathrm{div}_\Sigma X(x) = \mathrm{tr}(P_{T_x\Sigma} \nabla X^e(x))$ for $x \in \partial E$.

Also $P_{T_x\Sigma} \nabla X^e P_{T_x\Sigma}$ is independent of the particular extension of X, and is denoted by $\nabla^\Sigma X$, and we have $\mathrm{div}_\Sigma X(x) = \mathrm{tr}(\nabla^\Sigma X(x))$.

DEFINITION 2.1.7. *We define* $\overline{X} : \mathrm{U} \to \mathbb{R}^n$ *as* $\overline{X}(z) := X(\mathrm{pr}_\Sigma(z))$ *for any* $z \in \mathrm{U}$.

Then $\overline{X} \in \mathcal{C}^\infty(\mathrm{U}; \mathbb{R}^n)$ is an extension of X, and

$$\mathrm{div}\overline{X} = \mathrm{div}_\Sigma X \qquad \text{on } \partial E.$$

Note that

$$\overline{\mathrm{n}}^E(z) = \nabla d(z), \qquad z \in \mathrm{U}.$$

$\overline{\mathrm{n}}^E$ is the natural extension of the vector field n^E on the whole of U, in the sense that $\overline{\mathrm{n}}^E \in C^\infty(\mathrm{U})$, $\overline{\mathrm{n}}^E = \mathrm{n}^E$ on ∂E, and $\overline{\mathrm{n}}^E$ keeps the constraint $|\overline{\mathrm{n}}^E(z)| = 1$ for any $z \in \mathrm{U}$. When no confusion is possible, we will write $\overline{\mathrm{n}}^E = \overline{\mathrm{n}} = e_k \overline{\mathrm{n}}^k = (\overline{\mathrm{n}}^1, \ldots, \overline{\mathrm{n}}^n)$.

Observe that $\overline{\delta_k u}$ does not coincide, in general, with $\nabla_k \overline{u}$ in U, since

$$\overline{\delta_k u}(z) = \delta_k u(\mathrm{pr}(z)) = \nabla_k \overline{u}(\mathrm{pr}(z)), \qquad z \in \mathrm{U},$$

while from $\overline{u}(z) = \overline{u}(\mathrm{pr}(z))$ it follows

$$\begin{aligned} \nabla_k \overline{u}(z) =& \nabla_j \overline{u}(\mathrm{pr}(z))(\mathrm{Id}_{jk} - \nabla_k d(z) \nabla_j d(z) - d(z) \nabla_{jk} d(z)) \\ =& \nabla_j \overline{u}(\mathrm{pr}(z))(\mathrm{Id}_{jk} - d(z) \nabla_{jk} d(z)), \qquad z \in \mathrm{U}, \end{aligned}$$

so that $\nabla_k \overline{u}(\mathrm{pr}(z)) = \nabla_j \overline{u}(z)(\mathrm{Id}_{jk} - d(z)\nabla^2_{jk} d(z))^{-1}$. **** check ****

Given $u \in \mathcal{C}^\infty(\partial E)$, we denote by $\Delta_\Sigma u$ the tangential laplacian of u on ∂E, defined as

$$\Delta_\Sigma u := \mathrm{div}_\Sigma (\nabla^\Sigma u).$$

REMARK 2.1.8. Note that

$$\Delta_\Sigma u = \Delta \overline{u} \qquad \text{on } \partial E, \tag{2.10}$$

where $\Delta = \mathrm{div}\nabla$ is the Laplacian in $\mathbb{R}^n$. Indeed by definition

$$\Delta_\Sigma u = \mathrm{div}_\Sigma \nabla^\Sigma u = \langle e_k, \nabla^\Sigma \overline{\delta_k u} \rangle \qquad \text{on } \partial E.$$

Since

$$\overline{\delta_k u} = \nabla_k \overline{u} \qquad \text{on } \partial E,$$

we have

$$\Delta_\Sigma u = \langle e_k, \nabla^\Sigma \nabla_k \overline{u} \rangle = \langle e_k, \nabla \nabla_k \overline{u} \rangle - \langle e_k, \mathrm{n} \rangle \langle \nabla \nabla_k \overline{u}, \mathrm{n} \rangle \qquad \text{on } \partial E,$$

and (2.10) follows since $\langle \nabla \nabla_k \overline{u}, \mathrm{n} \rangle = 0$[3].

2.2. Second order properties of the distance function

Let $\partial E \in \mathcal{C}^\infty(\mathbb{R}^n)$, and U be as in Definition 2.1.2. Differentiating the equality (2.5) with respect to z^i it follows

$$\nabla^2_{ik} d \nabla_k d = 0 \qquad \text{in U}, \quad i \in \{1, \dots, n\}, \tag{2.11}$$

i.e.,

$$\nabla d(z) \in \ker(\nabla^2 d(z)), \qquad z \in \mathrm{U}. \tag{2.12}$$

Hence $\nabla d(z)$ is a unit zero eigenvector of $\nabla^2 d(z)$; therefore, given $z \in U$, it is possible to choose an orthonormal basis of $\mathbb{R}^n$ which diagonalizes $\nabla^2 d(z)$ for which the last vector is $\nabla d(z)$[4].

From (2.12) it follows that

$$\mathrm{div}_\Sigma(\nabla d) = \mathrm{div}(\nabla d) = \Delta d \qquad \text{on } \partial E. \tag{2.13}$$

Note that if we apply (2.10) to $u = \pi_k$, for a given $k \in \{1, \dots, n\}$, where π_k is defined in Remark 2.1.6, we find, using (2.11),

$$\Delta_\Sigma \pi_j = \Delta \overline{\pi_j} = -\Delta d \nabla_j d \qquad \text{on } \partial E. \tag{2.14}$$

2.2.1. Mean curvature. Let us recall the definition of second fundamental form[5]. and of mean curvature

Definition 2.2.1. *Let $\partial E \in \mathcal{C}^\infty$, $i, j, k \in \{1, \dots, n\}$, and $x \in \partial E$. The ijk-th component of the second fundamental form of ∂E at x is defined as*

$$\nabla_{ij} d(z) \nabla_k d(z).$$

The mean curvature vector of ∂E at x is defined as as $\Delta d(x) \nabla d(x)$, and the mean curvature of ∂E at x as $\Delta d(x)$.

Note that $\Delta d \nabla d$ is unchanged if we substitute $\mathbb{R}^n \setminus E$ to E in (2.4). Moreover Δd is positive for a smooth uniformly convex set E, so that in this case $\Delta d \nabla d$ points toward $\mathbb{R}^n \setminus E$.

In what follows we set

$$|\nabla^2 d|^2 := \mathrm{tr}(\nabla^2 d \nabla^2 d),$$

where tr is the trace operator in $\mathbb{R}^n$.

---

[3]Note that, with the notation of Note **??**, we have $\Delta_\Sigma = \delta_h \delta_h$.

[4]Note that, with the notation of Note **??**, we have $\nabla_{ij} d = \delta_i \mathrm{n}_j$ on Σ. In particular, the $(n \times n)$-matrix $\delta_i \mathrm{n}_j$ is symmetric.

[5]What is usually called second fundamental form is the restriction of the concept given in Definition 2.2.1 to the tangent space to ∂E.

The eigenvalues of $\nabla^2 d(x)$ are denoted by $\kappa_1^E(x), \dots, \kappa_n^E(x)$; if we take $\nabla d(x)$ as the last eigenvector, from (2.12) we have that $\kappa_n^E(x) = 0$, and

$$|\nabla^2 d|^2 = \sum_{i=1}^{n-1} (\kappa_i)^2.$$

REMARK 2.2.2. The mean curvature can be expressed also using the squared distance function as follows. Let $\eta := d^2/2$. Then $\eta \in C^\infty(\mathrm{U})$, $\eta = 0$ on ∂E, and

$$\nabla \eta = d \nabla d = 0 \qquad \text{on } \partial E. \tag{2.15}$$

Moreover, if $i, j \in \{1, \dots, n\}$ we have $\nabla^2_{ij} \eta = \nabla_i d \nabla_j d + d \nabla^2_{ij} d$ in U, so that

$$\nabla^2_{ij} \eta = \nabla_i d \nabla_j d \qquad \text{on } \partial E. \tag{2.16}$$

Hence $\nabla^2 \eta(x) = P_{N_x \Sigma}$. Finally, if $i, j, k \in \{1, \dots, n\}$ we have

$$\nabla^3_{ijk} \eta = \nabla_i d \nabla^2_{jk} d + \nabla_j d \nabla^2_{ik} d + \nabla_k d \nabla^2_{ij} d + d \nabla^3_{ijk} d \tag{2.17}$$

on U, where we use the notation $\nabla^3_{ijk} = \frac{\partial^3}{\partial z^i \partial z^j \partial z^k}$.

Therefore, recalling (2.12), we find

$$\Delta \nabla \eta = \Delta d \nabla d \qquad \text{on } \partial E. \tag{2.18}$$

The mean curvature can be expressed by differentiating the projection as follows.

REMARK 2.2.3. Differentiating the rl-component of (2.7) with respect to z^s, we obtain

$$\nabla^2_{sr} \mathrm{pr}_l = -\nabla_l d \nabla^2_{rs} d - \nabla_r d \nabla^2_{sl} d - \nabla_s d \nabla^2_{rl} d - d \nabla^3_{slr} d \qquad \text{on } U$$

In particular, using (2.11),

$$\Delta \mathrm{pr}_l = -\Delta d \nabla_l d \qquad \text{on } \partial E.$$

REMARK 2.2.4. Let $X \in \mathcal{C}^\infty(\partial E; \mathbb{R}^n)$; split X as $X = X_\Sigma + X_\perp$, where $X_\Sigma := X - \langle X, \mathrm{n} \rangle \mathrm{n}$ is the orthogonal projection of X on the tangent space to Σ. Then, writing $X_\perp = \xi \mathrm{n}$, where $\xi := \langle X, \mathrm{n} \rangle$, we have

$$\mathrm{div}_\Sigma X_\perp = \mathrm{div}_\Sigma(\xi \mathrm{n}) = \langle \nabla^\Sigma \xi, \mathrm{n} \rangle + \xi \mathrm{div}_\Sigma \mathrm{n} = \xi \mathrm{div}_\Sigma \mathrm{n} = \xi \Delta d, \tag{2.19}$$

so that

$$\mathrm{div}_\Sigma X = \mathrm{div}_\Sigma (X_\Sigma + X_\perp) = \mathrm{div}_\Sigma X_\Sigma + \Delta d \langle X, n \rangle,$$

The mean curvature can be expressed looking at ∂E as a level set of any smooth function with nonvanishing gradient as follows.

REMARK 2.2.5. Let $\partial E \in \mathcal{C}^\infty \cap \mathcal{K}(\mathbb{R}^n)$. If $u : \mathbb{R}^n \to \mathbb{R}$ is a smooth function with $E = \{u < 0\}$, $\partial E = \{u = 0\}$, and $\nabla u \neq 0$ on ∂E, then $n^E = \frac{\nabla u}{|\nabla u|}$, and ijk-component of the second fundamental form is

$$(P_{T_x \Sigma} \nabla^2 u P_{T_x \Sigma})_{ij} \frac{\nabla_k u}{|\nabla u|},$$

where we recall that $P_{T_x\Sigma ij} = \mathrm{Id}_{ij} - \frac{\nabla_i u(x)}{|\nabla u(x)|}\frac{\nabla_j u(x)}{|\nabla u(x)|}$. Then the mean curvature vector equals

$$\mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right)\frac{\nabla u}{|\nabla u|}, \tag{2.20}$$

and the mean curvature equals

$$\mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right) = \mathrm{tr}(P_{T_x\Sigma}\nabla^2 u). \tag{2.21}$$

If $\{u = 0\}$ is, in a neighbourhood O of a point, the graph of a smooth function v defined on an open set $\Omega \subset \mathbb{R}^{n-1}$, i.e., $\{u = 0\} \cap O = \{(s, z_n) \in \Omega \times \mathbb{R} : z_n = v(s)\}$, and $\{u < 0\} \cap O = \{(s, z_n) \in \Omega \times \mathbb{R} : v(s) < z_n\}$, then the mean curvature vector equals

$$\mathrm{div}\left(\frac{\nabla v}{\sqrt{1 + |\nabla v|^2}}\right)\frac{(\nabla v, -1)}{\sqrt{1 + |\nabla v|^2}}, \tag{2.22}$$

where in (2.22) the symbols div and ∇ are the gradient and the divergence with respect to s, respectively. Note that if $s \in \Omega$ is such that $\nabla v(s) = 0$ then the mean curvature of the graph of v at $(s, v(s))$ equals $\Delta v(s)$.

Remark 2.2.6. If $E \in \mathcal{C}^\infty(\mathbb{R}^n)$ note that

$$\Delta d\nabla d = \mathrm{div}_\Sigma \mathrm{n}\ \mathrm{n} \qquad \text{on } \partial E \tag{2.23}$$

and

$$\Delta d = \mathrm{div}_\Sigma \mathrm{n} \qquad \text{on } \partial E. \tag{2.24}$$

2.2.1.1. *Mean curvature using parametrizations.* Let $\mathcal{S}$ be a smooth $(n-1)$-dimensional orientable manifold without boundary, let $\varphi : \mathcal{S} \to \mathbb{R}^n$ be a smooth bijection between $\mathcal{S}$ and

$$\partial E = \varphi(\mathcal{S}),$$

and such that for any $s \in \mathcal{S}$ the differential $d\varphi(s)$ is injective.

Let $x = \varphi(s) \in \partial E$ and let $s^1, \ldots, s^{n-1}$ be local coordinates on $\mathcal{S}$. We set

$$\nu(s) := \mathrm{n}(x).$$

Let us define the map $\mathcal{B}_x = (\mathcal{B}_x^1, \ldots, \mathcal{B}_x^n) : T_x\Sigma \times T_x\Sigma \to N_x\Sigma \subset \mathbb{R}^n$ as follows: if $i, j \in \{1, \ldots, n-1\}$, $k \in \{1, \ldots, n\}$, and $\tau_i(s) := \frac{\partial\varphi}{\partial s^i}(s)$,

$$\mathcal{B}_x^k(\tau_i(s), \tau_j(s)) := \langle \nu(s), \frac{\partial \tau_j(s)}{\partial s^i}\rangle\ \nu^k(s) = \langle \nu(s), \frac{\partial^2\varphi(s)}{\partial s^i \partial s^j}\rangle\ \nu^k(s).$$

Then $\mathcal{B}$ is a symmetric bilinear form and, for $x = \varphi(s)$ one defines

$$\mathbf{H}(s) := g^{ij}(s)\mathcal{B}_x(\tau_i(s), \tau_j(s)), \qquad \mathrm{H}(s) := g^{ij}(s)\langle \nu(s), \frac{\partial^2\varphi(s)}{\partial s^i \partial s^j}\rangle, \tag{2.25}$$

where $g^{ij}(s)$ is the ij-component of the inverse matrix of $g_{ij}(s) := \langle \frac{\partial\varphi(s)}{\partial s^i}, \frac{\partial\varphi(s)}{\partial s^j}\rangle$. It turns out that

$$-\mathbf{H}(s) = \Delta d(x)\nabla d(x), \qquad \mathrm{H}(s) = \Delta d(x), \qquad x = \varphi(s).$$

If we define $\hat{\mathcal{B}}_x = (\hat{\mathcal{B}}_x^1, \dots, \hat{\mathcal{B}}_x^n) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ as $\hat{\mathcal{B}}_x(v, w) = \mathcal{B}_x(P_{T_x}(v), P_{T_x}(w)) \in N_x\Sigma$ for every pair $v, w \in \mathbb{R}^n$, where $P_{T_x\Sigma} := \mathrm{Id} - P_{N_x\Sigma}$, then $\mathcal{B}_{ij}^k = \langle \hat{\mathcal{B}}(e_i, e_j), e_k \rangle = \nabla^2_{ij} d \nabla_k d$. Note that

$$|\mathcal{B}|^2 = |\nabla^2 d|^2. \tag{2.26}$$

Finally, another expression of the mean curvature vector at $x = \varphi(s)$ is given by

$$\mathbf{H}(s) = \Delta_g \varphi(s) = (\Delta_g \varphi_1(s), \dots, \Delta_g \varphi_n(s)),$$

where $\Delta_g \varphi_k := g^{ij} \left(\frac{\partial^2 \varphi_k}{\partial s^i \partial s^j} - \Gamma_{ij}^h \frac{\varphi_k}{\partial s^h} \right) = \frac{1}{\sqrt{G}} \frac{\partial}{\partial s^i} \left(\sqrt{G} g^{ij} \frac{\partial \varphi_k}{\partial s^j} \right)$ for any $k \in \{1, \dots, n\}$, and $\Gamma_{ij}^k := \frac{1}{2} g^{kh} \left(\frac{\partial g_{jh}}{\partial s^i} + \frac{\partial g_{ih}}{\partial s^j} - \frac{\partial g_{ij}}{\partial s^k} \right)$ and $G := \det(g_{ij})$.

Notation: if $x = \varphi(s)$, we sometimes will use the notation

$$\mathrm{H}(s) = H(x) = H^E(x).$$

EXAMPLE 2.2.7. Let $n = 1$, $E \subset \mathbb{R}$ be a finite union of intervals. Then d is linear around the boundary points of the intervals, and hence $\Delta d = 0$ on ∂E.

EXAMPLE 2.2.8. Let $n = 2$ and $\partial E = \gamma(\mathbb{S}^1)$, where $\mathbb{S}^1$ is the unit circle and $\gamma : \mathbb{S}^1 \to \mathbb{R}^2$ is a smooth embedding of $\mathbb{S}^1$ in $\mathbb{R}^2$. Then the (mean) curvature vector of ∂E at $x = \gamma(s)$ is given $\frac{1}{|\gamma'|^2} \left(\gamma'' - \langle \gamma'', \frac{\gamma'}{|\gamma'|^2} \rangle \gamma' \right)$.

EXAMPLE 2.2.9. In Example 2.1.1 we have $\Delta d(z) = \frac{n-1}{|z - z_0|}$ and, for $z \neq z_0$, $\nabla^2_{ij} d(z) = \frac{1}{|z - z_0|} \left(\mathrm{Id}_{ij} - \frac{(z^i - z_0^i)}{|z - z_0|} \frac{(z^j - z_0^j)}{|z - z_0|} \right)$.

EXAMPLE 2.2.10. Let $n = 2$ and $\partial E \in \mathcal{C}^\infty \cap \mathcal{K}(\mathbb{R}^2)$. Then $\nabla^2 d = \Delta d (\nabla d^\perp)_i (\nabla d^\perp)_j$, where $\nabla d^\perp$ is the $\pi/2$-counterclockwise rotation of ∇d.

EXAMPLE 2.2.11. Let $v \in C^\infty(\mathbb{R}, (0, +\infty))$, and let $E := \{(z_1, z_2, z_3) \in \mathbb{R}^3 : (v(z_1))^2 \leq z_2^2 + z_3^2\}$, which is a solid of revolution, having as boundary the rotation of the graph of v around the z_1-axis, $\partial E = \{u(z) = 0\}$, $u(z) := \frac{1}{2}((v(z_1))^2 - z_2^2 - z_3^2)$. Direct computations give, for points of $\mathbb{R}^3$,

$$\mathrm{div} \left(\frac{\nabla u}{|\nabla u|} \right) = \left(v^2 (v')^2 + z_2^2 + z_3^2 \right)^{-3/2} \Big\{ ((v')^2 + v v'' - 2)(v^2 (v')^2 + z_2^2 + z_3^2) \tag{2.27}$$

$$- \left[v^2 (v')^2 ((v')^2 + v v'') - z_2^2 - z_3^2 \right] \Big\} \tag{2.28}$$

so that, if $(v(z_1))^2 = z_2^2 + z_3^2$,

$$H^E(z_1, z_2, z_3) = \left(v^2 (v')^2 + v^2 \right)^{-3/2} v^2 \left[v v'' - ((v')^2 + 1) \right] \tag{2.29}$$

$$= \frac{1}{(1 + (v')^2)^{1/2}} \left(\frac{v''}{1 + (v')^2} - \frac{1}{v} \right) \tag{2.30}$$

$$= \left(\frac{v'}{(1 + (v')^2)^{1/2}} \right)' - \frac{1}{v (1 + (v')^2)^{1/2}} \tag{2.31}$$

where the right hand side is evaluated at z_1.

2.3. Expansion of the Hessian of the oriented distance function

Differentiating (2.11) with respect to z_j we get

$$\nabla^3_{ijk} d \nabla_k d = -\nabla^2_{jk} d \nabla^2_{ik} d \qquad \text{in U}, \quad i, j \in \{1, \ldots, n\}. \tag{2.32}$$

In particular, multiplying by $\nabla^2_{ij} d$ we get

$$\nabla_k d \nabla^2_{ij} d \nabla^3_{ijk} d = -\nabla^2_{ij} d \nabla^2_{jk} d \nabla^2_{ik} d \qquad \text{in U}. \tag{2.33}$$

The following result describes the expansion of the eigenvalues of $\nabla^2 d$ on the whole of U.

THEOREM 2.3.1. *Let $\Sigma = \partial E$, U and d be as in Theorem 2.1.4. Let $z \in$ U and let $x := \mathrm{pr}_\Sigma(z)$ be the orthogonal projection of z on Σ. Fix an orthonormal basis $\{v_1, \ldots, v_n\}$ of $\mathbb{R}^n$ in which $\nabla^2 d(x)$ is diagonal, such that $v_n = \nabla d(x)$ and*

$$\nabla^2 d(x) v_i = \kappa^E_i(x) v_i, \qquad i = 1, \ldots, n, \tag{2.34}$$

where we recall that $\kappa^E_n(x) = 0$. Then $v_n \in \mathrm{Ker}(\nabla^2 d(z))$, the basis $\{v_1, \ldots, v_n\}$ diagonalizes $\nabla^2 d(z)$, and if we denote by $\mu_i(z)$ the eigenvalue corresponding to v_i for $i = 1, \ldots, n$, then

$$\mu_i(z) = \frac{\kappa^E_i(x)}{1 + d(z)\kappa^E_i(x)}. \tag{2.35}$$

PROOF. Define

$$B(\lambda) := \nabla^2 d\,(x + \lambda \nabla d(x))$$

for $\lambda \in \mathbb{R}$, $|\lambda|$ small enough in such a way that $x + \lambda \nabla d(x) \in$ U. Fix $i, j \in \{1, \ldots, n\}$, and consider the ij-th entry $B_{ij}(\lambda)$ of $B(\lambda)$. Then, using (2.6) and (2.32) we get

$$B'_{ij}(\lambda) = \nabla^3_{ijk} d(x + \lambda \nabla d(x)) \nabla_k d(x) = \nabla^3_{ijk} d(x + \lambda \nabla d(x)) \nabla_k d(x + \lambda \nabla d(x)) = -(B^2(\lambda))_{ij},$$

hence

$$B'(\lambda) = -B^2(\lambda). \tag{2.36}$$

Observe that

$$B(0) = \kappa^E_l(x) v_l \otimes v_l, \tag{2.37}$$

where, given two vectors $a = (a_1, \ldots, a_n)$, $b = (b_1, \ldots, b_n) \in \mathbb{R}^n$, the symbol $a \otimes b$ denotes the matrix whose ij-th entry is given by $a_i b_j$. The solution of the system (2.36) with initial condition (2.37) is $B(\lambda) = \frac{\kappa^E_l(x)}{1 + \lambda \kappa^E_l(x)} v_l \otimes v_l$. $\square$

Note that if $z \in U$ is such that $d(z) = \lambda$, then the principal curvatures of $\{d = \lambda\}$ at z are given by $\mu_i(z)$, $i = 1, \ldots, n$.

REMARK 2.3.2. In the statement of Theorem 2.1.4 the neighbourhood U is small enough in such a way that, in particular, $1 + d(z)\kappa^E_i(\mathrm{pr}_\Sigma(z)) > 0$ for any $z \in$ U.

REMARK 2.3.3. From (2.35) we have the following assertions.

(i) For any $i = 1, \dots, n$

$$\kappa_i^E(x) = \frac{\mu_i(z)}{1 - d(z)\mu_i(z)}; \tag{2.38}$$

Hence

$$\nabla^2 d(x) = \nabla^2 d(z) G(z), \qquad G(z) := (\mathrm{Id} - d(z)\nabla^2 d(z))^{-1}. \tag{2.39}$$

In particular

$$\overline{H}^{\Sigma}(z) = \mathrm{tr}\left(\nabla^2 d(z)(\mathrm{Id} - d(z)\nabla^2 d(z))^{-1}\right). \tag{2.40}$$

(ii) For any $i = 1, \dots, n$ we have $\frac{\mu_i(z)}{1-d(z)\mu_i(z)} \leq \mu_i(z)$ in $\mathrm{U} \cap E$, hence

$$\sum_{i=1}^{n} \frac{\mu_i(z)}{1 - d(z)\mu_i(z)} \leq \sum_{i=1}^{n} \mu_i(z) \qquad \text{in } \mathrm{U} \cap \{d \leq 0\}.$$

Similarly $\frac{\mu_i(z)}{1-d(z)\mu_i(z)} \geq \mu_i(z)$ in $\mathrm{U} \cap (\mathbb{R}^n \setminus E)$, hence

$$\sum_{i=1}^{n} \frac{\mu_i(z)}{1 - d(z)\mu_i(z)} \geq \sum_{i=1}^{n} \mu_i(z) \qquad \text{in } \mathrm{U} \cap \{d \geq 0\}.$$

(iii) As a consequence of (2.35) and the expansion $(1 + \lambda\kappa)^{-1} = 1 - \lambda\kappa + \mathcal{O}(\lambda^2)$, we deduce

$$\Delta d(z) = \Delta d(x) - d(z) \sum_{i=1}^{n-1} (\kappa_i^E(x))^2 + \mathcal{O}(d(z)^2) \tag{2.41}$$

Notes

Remark 2.0.2 is proved in [**34**], see also [**65**, Theorem 4.8, item (4)]. General properties of the distance function from a smooth compact boundary can be found for instance in [**72**], [**5**], [**103**], [**51**]. The if part in the statement of Remark 2.1.3 is proved for instance in [**5**, Theorem 2 statement (i)][6]. The converse statement follows from [**5**, Theorem 9].

THEOREM 2.3.4. *Let $E \in \mathcal{C}^\infty \cap \mathcal{K}(\mathbb{R}^n)$ and let $u : \mathbb{R}^n \to \mathbb{R}$ be a Lipschitz function, so that $\{u < 0\} = \mathrm{int}(E)$, and $\{u = 0\} = \partial E$. Assume that $|\nabla u|^2 = 1$ in $\mathbb{R}^n \setminus E$. Then $u(z) = \mathrm{dist}(z, \{u = 0\})$ for any $z \in \mathbb{R}^n \setminus E$.*

Theorem 2.1.4 is proved in [**5**]. See also [**72**, Appendix B].

Theorem 2.3.1 is proved in [**5**].

Let $k \geq 2$ be an integer; similarly to Definition 2.1.2, we say that a closed set $E \subseteq \mathbb{R}^n$ with compact boundary belongs to $\mathcal{C}^k_{\mathrm{cb}}(\mathbb{R}^n)$ if there exists an open set U containing ∂E such that $d(\cdot, E) \in \mathcal{C}^k(\mathrm{U})$.

THEOREM 2.3.5. *$E \in \mathcal{C}^k_{\mathrm{cb}}(\mathbb{R}^n)$ if and only if E has boundary of class $\mathcal{C}^k$.*

PROOF. See [**49**, Section 5.4], [**50**, Theorems 5.1, 5.2], [**103**, Section 11, Proposition 13.8]. ☐

[6] In statement (i) the author considers the case E bounded. In statement (ii) he proves a far more general result, valid in any codimension, which contains in particular the case ∂E bounded.

Formula (2.10) is proven for instance in [**50**, Chap. 4, Section 1.3], [**98**, Proposition 2.68]

The extension of the distance function approach to manifolds with arbitrary codimension is through the square distance function, as observed in [**47**]. We refer the reader to the papers [**10**], [**9**], [**58**], [**21**].

The tangential gradient δ on Σ is used in [**?**], [**86**], [**72**], [**87**]. We recall (see for instance [**86**]) that, given $h, k \in \{1, \dots, n\}$, the following commutation rule holds:

$$\delta_h \delta_k - \delta_k \delta_h = (\nu_h \delta_k \nu_j - \nu_k \delta_h \nu_j)\delta_j. \tag{2.42}$$

Indeed, let $u \in \mathcal{C}^\infty(\Sigma)$, and let $\overline{u} \in \mathcal{C}^\infty(\mathrm{U})$ be its extension as in Definition 2.1.5. Then, setting $\overline{\ell} := \langle \nabla \overline{u}, \nabla d \rangle$, we have

$$\delta_h \delta_k \overline{u} = \delta_h(\nabla_k \overline{u} - \overline{\ell} \nabla_k d) = \nabla_h(\nabla_k \overline{u} - \overline{\ell}\nabla_k d) - \langle \nabla(\nabla_k \overline{u} - \overline{\ell}\nabla_k d), \nabla d\rangle \nabla_h d \qquad \text{on } \Sigma. \tag{2.43}$$

On the other hand in U we have

$$\begin{aligned} &\nabla^2_{hk}\overline{u} - \nabla_k d\ \nabla_h \overline{\ell} - \overline{\ell}\nabla^2_{hk} d - \nabla_h d \langle \nabla\nabla_k \overline{u} - \nabla_k d \nabla \overline{\ell} - \overline{\ell}\nabla\nabla_k d, \nabla d\rangle \\ =&\nabla^2_{hk}\overline{u} - \nabla_k d\ \nabla_h \overline{\ell} - \overline{\ell}\nabla^2_{hk} d, \end{aligned} \tag{2.44}$$

where we used (2.11), the orthogonality between $\nabla\overline{\ell}$ and ∇d and the orthogonality between $\nabla\nabla_k\overline{u}$ and ∇d in U. Observing that $\nabla_h \overline{\ell} = \langle \nabla\overline{u}, \nabla\nabla_h d\rangle$ in U, Then from (2.43) and (2.44) we deduce

$$\delta_h\delta_k\overline{u} - \delta_k\delta_h\overline{u} = \nabla_h d\langle \nabla\overline{u}, \nabla\nabla_k d\rangle - \nabla_k d \langle \nabla \overline{u}, \nabla\nabla_h d\rangle,$$

which is (2.42).

***bellettini novaga j. convex anal. (citare de giorgi)

*** da sistemare: orientabilita' di $\mathcal{S}$ forse e' conseguenza della richiesta di avere un embedding

quando scrivo $\nabla d = \mathrm{n}$, gli indici di ∇d sono in basso, gli indici di n sono in alto

check la affermazione sulla restrizione dell'hessiano al tangente