CHAPTER 6

Short time existence and uniqueness: preliminary version

This version is in progress: please, take this into account. All corrections and comments are welcome.

In this chapter we prove the existence and uniqueness of a smooth compact mean curvature flow.

6.0.1. Preliminary lemmas. Let us begin with the inclusion principle between smooth compact mean curvature flows. The following lemma compares the mean curvature of two boundaries which are locally tangent, with a local inclusion between the sets.

Lemma 6.0.3. *Let $\partial E_1, \partial E_2 \in \mathcal{C}^\infty$, and assume that there exist $x \in \mathbb{R}^n$ and $\rho > 0$ with the following properties:*

$$x \in \partial E_1 \cap \partial E_2, \qquad E_1 \cap B_\rho(x) \subseteq E_2 \cap B_\rho(x).$$

Then $H^{E_1}(x) \geq H^{E_2}(x)$.

Proof. We can assume that x is the origin of the coordinates. Since the mean curvature is rotationally invariant, we can assume that $\mathrm{n}^{E_1}(x) = \mathrm{n}^{E_2}(x) = -e_n$, $\partial E_1 \cap B_\rho(x) = \mathrm{graph}(f_1)$, $\partial E_2 \cap B_\rho(x) = \mathrm{graph}(f_2)$, where f_1 and f_2 are two smooth functions defined on an open set of $\mathbb{R}^{n-1} = \mathrm{span}\{e_1, \ldots, e_{n-1}\}$ such that $f_1 \geq f_2$ locally around 0. Then $f_1 - f_2$ has a local minimum at 0, so that $\nabla f_1(0) = \nabla f_2(0) = 0$ and $\Delta f_1(0) \geq \Delta f_2(0)$. Then

$$H^{E_1}(x) = \Delta f_1(0) \geq \Delta f_2(0) = H^{E_2}(x).$$

□

We now need a preliminary useful result.

Lemma 6.0.4. *Let $h \geq 1$ and let M be an h-dimensional smooth compact orientable manifold without boundary. Let $u \in \mathcal{C}^1(M \times [a,b])$. Define, for any $t \in [a,b]$,*

$$u_{\min}(t) := \min_{p \in M} u(p,t), \qquad P^u_{\min}(t) := \{m \in M : u(m,t) = u_{\min}(t)\}, \tag{6.1}$$

$$u_{\max}(t) := \max_{p \in M} u(p,t), \qquad P^u_{\max}(t) := \{m \in M : u(m,t) = u_{\max}(t)\}. \tag{6.2}$$

Then for any $t \in [a,b)$

$$\lim_{\tau \to 0^+} \frac{1}{\tau}\Big(u_{\min}(t+\tau) - u_{\min}(t)\Big) = \min\left\{\frac{\partial u}{\partial t}(m,t) : m \in P^u_{\min}(t)\right\}, \tag{6.3}$$

$$\lim_{\tau \to 0^+} \frac{1}{\tau}\Big(u_{\max}(t+\tau) - u_{\max}(t)\Big) = \max\left\{\frac{\partial u}{\partial t}(m,t) : m \in P^u_{\max}(t)\right\}. \tag{6.4}$$

Proof. Let us show (6.3). For any $t \in [a,b)$, $m \in P^u_{\min}(t)$, $\tau > 0$ small enough so that $t+\tau \leq b$, we have

$$\begin{aligned} u_{\min}(t+\tau) \leq u(m,t+\tau) &= u(m,t) + \tau \frac{\partial u}{\partial t}(m,t) + o(\tau) \\ &= u_{\min}(t) + \tau \frac{\partial u}{\partial t}(m,t) + o(\tau). \end{aligned}$$

Since $\tau > 0$, the previous inequality can be rewritten as

$$\frac{1}{\tau}\left(u_{\min}(t+\tau) - u_{\min}(t)\right) \leq \frac{\partial u}{\partial t}(m,t) + o(1).$$

Therefore $\limsup_{\tau\to 0^+} \frac{1}{\tau}\left(u_{\min}(t+\tau) - u_{\min}(t)\right) \leq \frac{\partial u}{\partial t}(m,t)$. Since this inequality is valid for any $m \in P^u_{\min}(t)$ we deduce

$$\limsup_{\tau\to 0^+} \frac{1}{\tau}\left(u_{\min}(t+\tau) - u_{\min}(t)\right) \leq \min\left\{\frac{\partial u}{\partial t}(m,t) : m \in P^u_{\min}(t)\right\}. \tag{6.5}$$

To conclude the proof of the lemma, we need to show that

$$\liminf_{\tau\to 0^+} \frac{1}{\tau}\left(u_{\min}(t+\tau) - u_{\min}(t)\right) \geq \min\left\{\frac{\partial u}{\partial t}(m,t) : m \in P^u_{\min}(t)\right\}. \tag{6.6}$$

Fix $\varepsilon > 0$ and for $t \in [a,b]$ define $P_\varepsilon(t) := \{q \in M : u(q,t) < u_{\min}(t) + \varepsilon\}$. For any $t \in [a,b)$, $q \in P_\varepsilon(t)$ and $\tau > 0$ small enough so that $t+\tau \leq b$, we have

$$\begin{aligned} u(q,t+\tau) &= u(q,t) + \tau \frac{\partial u}{\partial t}(q,t) + o(\tau) \\ &\geq u_{\min}(t) + \tau \frac{\partial u}{\partial t}(q,t) + o(\tau) \\ &\geq u_{\min}(t) + \tau \inf_{q \in P_\varepsilon(t)} \frac{\partial u}{\partial t}(q,t) + o(\tau). \end{aligned} \tag{6.7}$$

On the other hand, if $p \in M \setminus P_\varepsilon(t)$ we have $u(p,t) \geq u_{\min}(t) + \varepsilon$, and therefore

$$\begin{aligned} u(p,t+\tau) &= u(p,t) + \tau \frac{\partial u}{\partial t}(p,t) + o(\tau) \\ &\geq u_{\min}(t) + \varepsilon + \tau \frac{\partial u}{\partial t}(p,t) + o(\tau) \qquad (6.8) \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad (6.9) \\ &\geq u_{\min}(t) + \varepsilon - \tau L + o(\tau), \end{aligned}$$

where $L := \max_{(p,t)\in M\times[a,b]} \left|\frac{\partial u}{\partial t}(p,t)\right|$. For $t \in [a,b)$ and $0 < \tau < \frac{\varepsilon}{2L}$ we have

$$u_{\min}(t) + \varepsilon - \tau L \geq u_{\min}(t) + \tau L \geq u_{\min}(t) + \tau \inf_{q \in P_\varepsilon(t)} \frac{\partial u}{\partial t}(q,t). \tag{6.10}$$

From (6.7), (6.8) and (6.10) we deduce, for $0 < \tau < \frac{\varepsilon}{2L}$ such that $t + \tau \le b$, and for any $p \in M$,

$$u(p, t + \tau) \ge u_{\min}(t) + \tau \inf_{q \in P_\varepsilon(t)} \frac{\partial u}{\partial t}(q, t) + o(\tau). \tag{6.11}$$

From (6.11) it follows, for the same values of t and τ,

$$u_{\min}(t + \tau) \ge u_{\min}(t) + \tau \inf_{q \in P_\varepsilon(t)} \frac{\partial u}{\partial t}(q, t) + o(\tau).$$

Hence

$$\liminf_{\tau \to 0^+} \frac{1}{\tau} \left(u_{\min}(t + \tau) - u_{\min}(t) \right) \ge \inf_{q \in P_\varepsilon(t)} \frac{\partial u}{\partial t}(q, t). \tag{6.12}$$

Since (6.12) holds for any $\varepsilon > 0$, we deduce

$$\liminf_{\tau \to 0^+} \frac{1}{\tau} \left(u_{\min}(t + \tau) - u_{\min}(t) \right) \ge \sup_{\varepsilon > 0} \inf_{q \in P_\varepsilon(t)} \frac{\partial u}{\partial t}(q, t) = \min_{m \in P^u_{\min}(t)} \frac{\partial u}{\partial t}(m, t),$$

where the last equality follows from the continuity of the function $\frac{\partial u}{\partial t}(\cdot, t)$ and the fact that the map $\varepsilon > 0 \to \inf_{q \in P_\varepsilon(t)} \frac{\partial u}{\partial t}(q, t)$ is nonincreasing.

The assertion for $u_{\max}$ follows by setting $v := -u$, so that $u_{\max} = -v_{\min}$, $P^u_{\max}(t) = P^v_{\min}(t)$, and

$$\begin{aligned} \lim_{\tau \to 0^+} \frac{1}{\tau} \Big(u_{\max}(t + \tau) - u_{\max}(t) \Big) &= - \lim_{\tau \to 0^+} \frac{1}{\tau} \Big(v_{\min}(t + \tau) - v_{\min}(t) \Big) \\ = - \min \left\{ \frac{\partial v}{\partial t}(m, t) : m \in P^v_{\min}(t) \right\} &= \max \left\{ \frac{\partial u}{\partial t}(m, t) : m \in P^u_{\max}(t) \right\}, \end{aligned} \tag{6.13}$$

where we used (6.3). □

Remark 6.0.5. Conclusion (6.3) of Lemma 6.0.4 is still valid (with the same proof) if we drop the assumption that M is compact, provided we assume that $\inf_{p \in M} u(p, t) = \min_{p \in M} u(p, t)$, that $P^u_{\min}(t)$ is compact for any $t \in [a, b]$, and that $\sup_{(p,t) \in M \times [a,b]} |\frac{\partial u(p,t)}{\partial t}| < +\infty$. A similar comment applies for conclusion 6.4.

Example 6.0.6. Let $M \subset \mathbb{R}^2$ be the interval $[-2, 2]$ with the two boundary points identified. Let $v \in \mathcal{C}^\infty(M \times [-1, 1]; (0, +\infty))$ be a function such that the graph of $v(\cdot, t)$ has the form depicted in Figure 1, for $t \in [-1, 0)$, $t = 0$, and $t \in (0, 1]$ respectively. We assume $v(-1, t) \equiv 1$ for any $t \in [-1, 1]$, $v(1, t) > 1$ for any $t \in [-1, 0)$, $v(1, 0) = 1$, and $\frac{\partial v}{\partial t}(1, 0) < 0$. For any $t \in [-1, 1]$ and $x \in \mathrm{graph}(v(\cdot, t))$, let $u(x, t)$ be the distance between x and the first axis, and let $u_{\min}(t) := \min\{u(x, t) : x \in \mathrm{graph}(v(\cdot, t))\}$. Then the function $u_{\min}$ is not differentiable at $t = 0$.

6.1. Inclusion principle: the simplest case

We begin with the following weak form, where we assume that initially one sets is inside the other one, and the boundary of the two sets do not intersect.

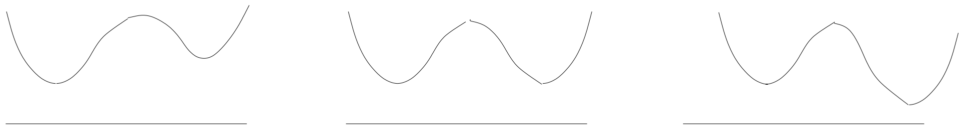

FIGURE 1. ***

THEOREM 6.1.1. *Let $f_1 : [a,b] \to \mathcal{P}(\mathbb{R}^n)$ be a smooth compact mean curvature flow, and let $f_2 : [a,b] \to \mathcal{P}(\mathbb{R}^n)$ be a smooth mean curvature flow. Define the function*

$$\delta(t) := \mathrm{dist}(f_1(t), \mathbb{R}^n \setminus f_2(t)), \qquad t \in [a,b]. \tag{6.14}$$

Assume that

$$\delta(a) > 0. \tag{6.15}$$

Then

$$\text{for any } t \in [a,b) \text{ there exists } \lim_{\tau\to 0^+} \frac{1}{\tau}\Big(\delta(t+\tau) - \delta(t)\Big) \in [0,+\infty). \tag{6.16}$$

Hence δ is nondecreasing in $[a,b]$. In particular

$$f_1(a) \subseteq f_2(a), \quad \partial f_1(a) \cap \partial f_2(a) = \emptyset \quad \Rightarrow \quad f_1(t) \subseteq f_2(t), \qquad t \in [a,b].$$

PROOF. Since f_1 and f_2 are smooth flows, it follows that there exists a $(n-1)$-dimensional smooth compact manifold (resp. a smooth manifold) without boundary $\mathcal{S}_1$ (resp. $\mathcal{S}_2$) and there exist smooth maps $\varphi_1 : \mathcal{S}_1 \times [a,b] \to \mathbb{R}^n$ and $\varphi_2 : \mathcal{S}_2 \times [a,b] \to \mathbb{R}^n$ with the following properties:

- $\varphi_1(\cdot,t)$ is a bijection between $\mathcal{S}_1$ and $\partial f_1(t)$ and $\varphi_2(\cdot,t)$ is a bijection between $\mathcal{S}_2$ and $\partial f_2(t)$ for any $t \in [a,b]$;
- for any $s \in \mathcal{S}_1$ (resp. $\hat{s} \in \mathcal{S}_2$) and $t \in [a,b]$ the differential $d\varphi_1(s,t)$ (resp. $d\varphi_2(\hat{s},t)$) with respect to s (resp. $\hat{s}$) is injective.

Let $M := \mathcal{S}_1 \times \mathcal{S}_2$ and define the function $u : M \to [0,+\infty)$ as

$$u(s,\hat{s},t) := |\varphi_1(s,t) - \varphi_2(\hat{s},t)|. \tag{6.17}$$

Observe that

$$\delta(t) = \min\left\{u(s,\hat{s},t) : (s,\hat{s}) \in M\right\}, \qquad t \in [a,b], \tag{6.18}$$

and that $\delta \in \mathrm{Lip}([a,b])$. Define $\sigma := \inf\{t \in [a,b] : \delta(t) = 0\}$. Thanks to the smoothness of the flows, we have that $\sigma > a$. Hence $\delta(t) > 0$ in $[a,\sigma)$, and therefore the function u is smooth on $M \times [a,\sigma)$. Thus we can apply Lemma 6.0.4 and deduce

$$\begin{aligned} &\lim_{\tau\to 0^+} \frac{\delta(t+\tau) - \delta(t)}{\tau} \\ &= \min\left\{\frac{\partial u}{\partial t}(s,\hat{s},t) : (s,\hat{s}) \in M, u(s,\hat{s},t) = \delta(t)\right\}, \qquad t \in [a,\sigma). \end{aligned} \tag{6.19}$$

We claim that

$$\lim_{\tau\to 0^+} \frac{1}{\tau}(\delta(t+\tau) - \delta(t)) \geq 0 \qquad \forall t \in [a,\sigma). \tag{6.20}$$

Let $t \in [a,\sigma)$, let $(s(t),\hat{s}(t)) \in M$ be such that

$$\frac{\partial u}{\partial t}(s(t),\hat{s}(t),t) = \min\left\{\frac{\partial u}{\partial t}(s,\hat{s},t) : (s,\hat{s}) \in M, u(s,\hat{s},t) = \delta(t)\right\}, \tag{6.21}$$

and set $x_t := \varphi_1(s(t),t) \in \partial f_1(t)$ and $\hat{x}_t := \varphi_2(\hat{s}(t),t) \in \partial f_2(t)$. Note that the relations $u(s(t),\hat{s}(t),t) = |x_t - \hat{x}_t| = \delta(t)$ imply

$$\frac{\hat{x}_t - x_t}{|\hat{x}_t - x_t|} = \mathrm{n}^{f_2(t)}(\hat{x}_t) = \mathrm{n}^{f_1(t)}(x_t),$$

namely $\frac{\hat{x}_t - x_t}{|\hat{x}_t - x_t|}$ coincides with outward unit normal vector to $f_2(t)$ at $\hat{x}_t$, which in turn coincides with outward unit normal vector to $f_1(t)$ at x_t. Denote such a unit vector by ν.

From (6.17) we compute

$$\frac{\partial u}{\partial t}(s(t),\hat{s}(t),t) = \langle \frac{\hat{x}_t - x_t}{|\hat{x}_t - x_t|}, \frac{\partial \varphi_2}{\partial t}(\hat{s}(t),t) - \frac{\partial \varphi_1}{\partial t}(s(t),t)\rangle = \langle \nu, \frac{\partial \varphi_2}{\partial t}(\hat{s}(t),t) - \frac{\partial \varphi_1}{\partial t}(s(t),t)\rangle. \tag{6.22}$$

From Definition 4.0.14 we have

$$\langle \nu, \frac{\partial \varphi_2}{\partial t}(\hat{s}(t),t)\rangle \nu = \mathbf{V}_{f_2}(\hat{s}(t),t), \qquad \langle \nu, \frac{\partial \varphi_1}{\partial t}(s(t),t)\rangle \nu = \mathbf{V}_{f_1}(s(t),t), \tag{6.23}$$

where $\mathbf{V}_{f_i}$ is given in (4.4), with f replaced by f_i, $i = 1,2$. On the other hand f_1 and f_2 are smooth mean curvature flows, so that

$$\langle \nu, \frac{\partial \varphi_2}{\partial t}(\hat{s}(t),t)\rangle = -H^{f_2(t)}(\hat{x}_t), \qquad \langle \nu, \frac{\partial \varphi_1}{\partial t}(s(t),t)\rangle = -H^{f_1(t)}(x_t). \tag{6.24}$$

From (6.19), (6.21), (6.22), and (6.24) we get

$$\lim_{\tau \to 0^+} \frac{1}{\tau}(\delta(t+\tau) - \delta(t)) = -H^{f_2(t)}(\hat{x}_t) + H^{f_1(t)}(x_t). \tag{6.25}$$

Let us now consider the translated set

$$f_1^{\mathrm{tr}}(t) := f_1(t) + \delta(t)\nu.$$

Then

$$f_1^{\mathrm{tr}}(t) \subseteq f_2(t) \quad \text{and} \quad \hat{x}_t \in \partial(f_1^{\mathrm{tr}}(t)) \cap \partial f_2(t). \tag{6.26}$$

Moreover

$$H^{f_1^{\mathrm{tr}}(t)}(\hat{x}_t) = H^{f_1(t)}(x_t). \tag{6.27}$$

By (6.26), using Lemma 6.0.3 we deduce $H^{f_1^{\mathrm{tr}}(t)}(\hat{x}_t) \geq H^{f_2(t)}(\hat{x}_t)$. From (6.27) we then get

$$H^{f_1(t)}(x_t) \geq H^{f_2(t)}(\hat{x}_t). \tag{6.28}$$

The claim then follows from (6.25) and (6.28).

Let us now show that from (6.20) it follows that δ is nondecreasing in $[a,\sigma]$. Assume by contradiction that there exist $a \leq t_1 < t_2 \leq \sigma$ such that $\delta(t_2) < \delta(t_1)$. Let $P : \mathbb{R} \to \mathbb{R}$ be a linear decreasing function such that $P(t_1) = \delta(t_1)$ and $P(t_2) > \delta(t_2)$. Let

$$t^* := \sup\{t \in [t_1,\sigma] : \delta(t) \leq P(t)\}.$$

Then $P(t^*) = \delta(t^*)$, $t^* < \sigma$, and $\frac{\delta(t^*+\tau)-\delta(t^*)}{\tau} < \frac{P(t^*+\tau)-P(t^*)}{\tau}$ for $\tau > 0$ small enough. Therefore

$$\lim_{\tau\to 0^+} \frac{1}{\tau}\left(\delta(t^*+\tau)-\delta(t^*)\right) \leq P'(t^*) < 0,$$

a contradiction.

Hence δ is nondecreasing in $[a,\sigma]$, and therefore $\delta(\sigma) \geq \delta(a) > 0$. If $\sigma = b$ the proof is concluded. Assume now that $\sigma \in [a,b)$, and assume by contradiction that δ is not nondecreasing in $[\sigma,b]$. Set $\bar{t} := \inf\{t \in [a,b] : \exists \{t_n\} \subset (t,b), \delta(t_n) < \delta(t), \lim_{n\to+\infty} \delta(t_n) = \delta(t)\}$. Then $\bar{t}$ is a minimum, $\bar{t} \geq a+\sigma$, and $\delta(\bar{t}) \geq \delta(a) > 0$. If $\bar{t} < b$, arguing as before with $\bar{t}$ in place of a, we find $\bar{\sigma} > 0$ such that δ is nondecreasing in $[\bar{t}, \bar{t}+\bar{\sigma}]$, which contradicts the definition of $\bar{t}$. □

Remark 6.1.2. We can state Theorem 7.3 in the following equivalent form. Assume that $f_1(a) \cap f_2(a) = \emptyset$. Define $\delta(t) := \mathrm{dist}(f_1(t), f_2(t))$ for any $t \in [a,b]$. δ is nondecreasing in $[a,b]$. In particular, $f_1(t) \cap f_2(t) = \emptyset$ for any $t \in [a,b]$.

Remark 6.1.3. As we shall see in Section **??**, under the (weaker) assumption $f_1(a) \subseteq f_2(a)$ in place of (6.15), a conclusion even stronger than the one of Theorem 7.3 is valid in $[a,b]$, namely that δ is strictly increasing.

Corollary 6.1.4. *Let $f_1, f_2 \in \mathcal{KF}$ be two smooth compact mean curvature flows in a common time interval $[a,b]$. Assume that $f_1(a) \subseteq \mathrm{int}(f_2(a))$. Then $f_1(t) \subseteq \mathrm{int}(f_2(t))$ for all $t \in [a,b]$.*

Observe that Theorem 7.3 is still valid if we assume that f_2 is a smooth mean curvature flow in $[a,b]$, namely if we drop the compactness assumption on $\partial f(t)$.

We conclude this section with another interesting property, that can be proved by refining the arguments in the proof of Theorem 7.3 is described in the following remark[1].

Remark 6.1.5. Let $\mathcal{S}$ be an $(n-1)$-dimensional smooth compact manifold without boundary and let $\varphi \in \mathcal{C}^\infty(\mathcal{S} \times [a,b], \mathbb{R}^n)$. For any $t \in [a,b]$ set $\Gamma(t) := \varphi(\mathcal{S}, t)$. Assume that

(i) $\varphi(\cdot, a)$ is a bijection between $\mathcal{S}$ and $\Gamma(a)$;
(ii) for any $s \in \mathcal{S}$ and any $t \in [a,b]$ the differential $d\varphi(s,t)$ with respect to s is injective;
(iii) the orthogonal projection of $\frac{\partial \varphi}{\partial t}(s,t)$ on $N_{\varphi(s,t)}(\Gamma(t))$ equals the mean curvature of $\Gamma(t)$ at $\varphi(s,t)$ for any $s \in S$ and any $t \in [a,b]$.

Then $\varphi(\cdot, t)$ is a bijection between $\mathcal{S}$ and $\Gamma(t)$ for any $t \in [a,b]$.

6.2. The approach of Evans-Spruck

Denote by Sym_n the set of all real symmetric $(n\times n)$-matrices, and for $X \in \mathrm{Sym}_n$ let $\{\lambda_1(X), \ldots, \lambda_n(X)\}$ be the set of the eigenvalues of X. Set

$$D := \{(\mathrm{u}, X) \in \mathbb{R} \times \mathrm{Sym}_n : 1 - \mathrm{u}\lambda_i(X)) \neq 0,\ i = 1, \ldots, n\},$$

[1]The conclusion of Remark 6.1.5 is not valid in general for mean curvature flow with a forcing term.

which we consider as a subspace of $\mathbb{R}\times\mathbb{R}^{n^2}$, with the norm induced by the euclidean norm. Let $F : D \to \mathbb{R}$ be defined as

$$F(\mathrm{u}, X) := \sum_{i=1}^{n} \frac{\lambda_i(X)}{1-\mathrm{u}\lambda_i(X)}, \qquad (\mathrm{u}, X) \in D. \tag{6.29}$$

In this chapter we prove the following theorem, due to Evans and Spruck.

THEOREM 6.2.1. *Let $E \subset \mathbb{R}^n$ be a bounded open set with boundary of class $\mathcal{C}^{2+\alpha}$, for some $\alpha \in (0, 1)$, and set $u_0(\cdot) := d(\cdot, E)$. Then there exist $\rho_0 > 0$ and $t_0 > 0$ such that, setting $\mathrm{U} := (\partial E)^+_{\rho_0}$, the problem*

$$\begin{cases} u \in \mathcal{C}^{2+\alpha,1+\alpha/2}(\overline{\mathrm{U}} \times [0, t_0]), & \\ u_t = F(u, \nabla^2 u) & \text{in } \mathrm{U} \times (0, t_0), \\ |\nabla u|^2 = 1 & \text{on } \partial\mathrm{U} \times [0, t_0], \\ u(\cdot, 0) = u_0(\cdot) & \text{in } \mathrm{U} \end{cases} \tag{6.30}$$

has a unique solution.

The definitions of parabolic Hölder spaces and corresponding norms are given in Section 6.3. Observe that, from (2.5), it follows that the compatibility condition $|\nabla u(\cdot, 0)|^2 = 1$ is satisfied.

6.2.1. Some properties of the function F. In order to prove Theorem 6.2.1 we need some preparation. If $(\mathrm{u}, X) \in D$ the two matrices $(\mathrm{Id} - \mathrm{u}X)^{-1} \in \mathrm{Sym}_n$ and $X(\mathrm{Id} - \mathrm{u}X)^{-1} \in \mathrm{Sym}_n$ commute with X; if $X = \mathrm{diag}(\lambda_1(X), \ldots, \lambda_n(X))$ is diagonal in suitable bases of $\mathbb{R}^n$, then $(\mathrm{Id} - \mathrm{u}X)^{-1}$ is diagonal in the same bases, and $(\mathrm{Id} - \mathrm{u}X)^{-1} = \mathrm{diag}((1 - \lambda_1(X))^{-1}, \ldots, (1 - \lambda_n(X))^{-1})$. Since the trace of a matrix is independent of the choice of the basis, we have

$$F(\mathrm{u}, X) = \mathrm{tr}(X(\mathrm{Id} - \mathrm{u}X)^{-1}), \qquad (\mathrm{u}, X) \in D. \tag{6.31}$$

Set

$$\widehat{D} := \{(\mathrm{u}, X) \in \mathbb{R} \times M_n : \mathrm{Id} - \mathrm{u}X \text{ is invertible}\},$$

where M_n is the set of all $(n \times n)$ real matrices. Observe that $\widehat{D}$ is an open subset of $\mathbb{R} \times \mathbb{R}^{n^2}$. The function defined as

$$\mathrm{tr}(X(\mathrm{Id} - \mathrm{u}X)^{-1}), \qquad (\mathrm{u}, X) \in \widehat{D}, \tag{6.32}$$

coincides with F on D, and will be still denoted by the same symbol. From now on we will denote the function F with the symbol F. From (6.31) it follows that F is analytic on $\widehat{D}$, and being

$$(\mathrm{Id} - \mathrm{u}X)^{-1} = \sum_{k\geq 0} \mathrm{u}^k X^k, \qquad (\mathrm{u}, X) \in \widehat{D}, \tag{6.33}$$

we have

$$F(\mathrm{u}, X) = \mathrm{tr}\left(\sum_{k\geq 0} \mathrm{u}^k X^{k+1}\right), \qquad (\mathrm{u}, X) \in \widehat{D}. \tag{6.34}$$

If $\xi, \eta \in \mathbb{R}^n$ we indicate by $\xi \otimes \eta$ the matrix whose ij-entry is given by $\xi_i \eta_j$. Let us denote by $F_{X_{ij}}$ the derivative of F with respect to the ij-th component of X, i.e., $F_{X_{ij}}(\mathrm{u}, X) = \frac{dF}{dX_{ij}}(\mathrm{u}, X) := \lim_{h \to 0} \frac{1}{h}(F(\mathrm{u}, X + h e_i \otimes e_j) - F(\mathrm{u}, X))$ where $e_1, \dots, e_n$ is the canonical basis of $\mathbb{R}^n$. We denote by $F_X(\mathrm{u}, X)$ the matrix whose ij-entry is $F_{X_{ij}}(\mathrm{u}, X)$.

Lemma 6.2.2. *For any $(u, X) \in D$ and any $M \in M_n$ we have*

$$\mathrm{tr}(M F_X(\mathrm{u}, X)) = \mathrm{tr}(M(\mathrm{Id} - \mathrm{u}X)^{-2}). \tag{6.35}$$

Proof. We first observe that

$$F_{X_{ij}}(\mathrm{u}, X) = \mathrm{tr}\left(\frac{d}{dX_{ij}}\left(X \sum_{k \geq 0} \mathrm{u}^k X^k\right)\right),$$

where $\frac{d}{dX_{ij}}\left(X \sum_{k\geq 0} \mathrm{u}^k X^k\right) := \lim_{h \to 0} \frac{(X + h e_i \otimes e_j) \sum_{k \geq 0} \mathrm{u}^k (X + h e_i \otimes e_j)^k - X \sum_{k \geq 0} \mathrm{u}^k X^k}{h}$. Then

$$F_{X_{ij}}(\mathrm{u}, X) = \mathrm{tr}\left(e_i \otimes e_j \left(\sum_{k \geq 0} \mathrm{u}^k X^k + \mathrm{u}X \sum_{k \geq 0} (k+1) \mathrm{u}^k X^k\right)\right).$$

Since $\left(\sum_{k\geq 0} \mathrm{u}^k X^k\right)\left(\sum_{m \geq 0} \mathrm{u}^m X^m\right) = \sum_{k \geq 0}(k+1)\mathrm{u}^k X^k$ we deduce

$$F_{X_{ij}}(\mathrm{u}, X) = \mathrm{tr}\left(e_i \otimes e_j \left(\sum_{k \geq 0} \mathrm{u}^k X^k + \mathrm{u}X(\sum_{k \geq 0} u^k X^k)^2\right)\right).$$

Being $(\sum_{k \geq 0} u^k X^k)^2(\mathrm{Id} - uX) = \sum_{k \geq 0} u^k X^k$, we obtain

$$F_{X_{ij}}(\mathrm{u}, X) = \mathrm{tr}\left(e_i \otimes e_j \left((\sum_{k \geq 0} \mathrm{u}^k X^k)^2 (\mathrm{Id} - \mathrm{u}X + \mathrm{u}X)\right)\right) = \mathrm{tr}\left(e_i \otimes e_j (\mathrm{Id} - \mathrm{u}X)^{-2}\right).$$

Then the assertion follows. □

Corollary 6.2.3. *Let* $(\mathrm{u}, X) \in D$. *Then*

$$F_{X_{ij}}(\mathrm{u}, X)\xi_i \xi_j \geq C(\mathrm{u}, X)|\xi|^2, \qquad \xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n, \tag{6.36}$$

where $C(\mathrm{u}, X) := \min\{(1 - \mathrm{u}\lambda_i(X))^{-2} : i = 1, \dots, n\} > 0$.

Proof. Let $\{v_1, \dots, v_n\}$ be an orthonormal basis of $\mathbb{R}^n$ so that v_i is an eigenvalue of X with $\lambda_i(X)$ as eigenvector; note that in the same basis $(\mathrm{Id} - uX)^{-2} = \mathrm{diag}((1 - \lambda_1(X))^{-2}, \dots, (1 - \lambda_n(X))^{-2})$. Let us apply (6.35) with $M = \xi \otimes \xi$. We have

$$\begin{aligned} F_{X_{ij}}(\mathrm{u}, X)\xi_i \xi_j =& \mathrm{tr}\left(\xi \otimes \xi (\mathrm{Id} - \mathrm{u}X)^{-2}\right) = \sum_{i=1}^n \frac{\langle \xi, v_i \rangle^2}{(1 - \mathrm{u}\lambda_i(X))^2} \\ \geq& C(u, X) \sum_{i=1}^n \langle \xi, v_i \rangle^2 = C(u, X)|\xi|^2. \end{aligned}$$

□

If we denote by F_{u} the partial derivative of F with respect to u, with similar computations made as in Lemma 6.2.2 we have that $F_{\mathrm{u}}(\mathrm{u}, X) = \mathrm{tr}(X^2(\mathrm{Id} - \mathrm{u}X)^{-2})$ for any $(u, X) \in D$.

6.2.2. Existence. We begin by proving the existence statement of Theorem 6.2.1. Since the boundary of E is of class $\mathcal{C}^{2+\alpha}(\mathbb{R}^n)$, for $\rho > 0$ small enough we have $u_0 \in \mathcal{C}^{2+\alpha}(\overline{(\partial E)^+_\rho})$ and

$$|u_0(z)\lambda_i(\nabla^2 u_0(z))| \leq \frac{1}{2}, \qquad z \in \overline{(\partial E)^+_\rho},\ i = 1, \ldots, n.$$

In particular, $(u_0(z), \nabla^2 u_0(z)) \in D$ for any $z \in \overline{(\partial E)^+_\rho}$.

We will reduce the problem to a linear one. Given $t_0 > 0$ and $w \in \mathcal{C}^{2+\alpha, 1+\alpha/2}(\overline{\mathrm{U}} \times [0, t_0])$ to be selected later, we look for solutions u of (6.30) of the form

$$u(z, t) = u_0(z) + tF(u_0(z), \nabla^2 u_0(z)) + w(z, t), \qquad (z, t) \in \mathrm{U} \times (0, t_0). \tag{6.37}$$

Inserting (6.37) into the first equation in (6.30) and adding and subtracting the quantity $F_{\mathrm{u}}(u_0, \nabla^2 u_0)w + F_{X_{ij}}(u_0, \nabla^2 u_0)\nabla_{ij} w$, we get

$$w_t - \mathcal{A}(z, w, \nabla^2 w) = \mathfrak{f}(z, t, w, \nabla^2 w), \qquad (z, t) \in \mathrm{U} \times (0, t_0), \tag{6.38}$$

where

$$\mathcal{A}(z, \mathrm{u}, X) := F_{X_{ij}}(u_0(z), \nabla^2 u_0(z))X_{ij} + F_{\mathrm{u}}(u_0(z), \nabla^2 u_0(z))\mathrm{u}$$

is linear with respect to X and u and where, setting for simplicity

$$\ell(z) := F(u_0(z), \nabla^2 u_0(z)),$$

the function $\mathfrak{f}$ is defined as

$$\begin{aligned} \mathfrak{f}(z, t, \mathrm{u}, X) :=& F\Big(u_0(z) + t\ell(z) + \mathrm{u}, \nabla^2 u_0(z) + t\nabla^2 \ell(z) + X\Big) \\ & - \ell(z) - F_{\mathrm{u}}\left(u_0(z), \nabla^2 u_0(z)\right)\mathrm{u} - F_{X_{ij}}(u_0(z), \nabla^2 u_0(z))\, X_{ij}. \end{aligned} \tag{6.39}$$

Inserting (6.37) into the second equation in (6.30), setting as usual

$$d(\cdot) := d(\cdot, E),$$

and observing that, thanks to the eikonal equation (2.5),

$$1 = |\nabla d + t\nabla \ell + \nabla w|^2 = 1 + |t\nabla \ell + \nabla w|^2 + 2\langle \nabla d, \nabla w\rangle + 2t\langle \nabla d, \nabla \ell\rangle,$$

we get $\langle \nabla d, \nabla w\rangle = -\frac{1}{2}|t\nabla \ell + \nabla w|^2 - t\langle \nabla d, \nabla \ell\rangle$. Hence

$$\frac{\partial w}{\partial \nu} = \beta(z, t, \nabla w) \qquad \text{on } \partial\mathrm{U} \times [0, t_0], \tag{6.40}$$

where ν is the outer unit normal to ∂U, so that $\nu = \nabla d(\cdot)$ on $\{d(\cdot) > 0\} \cap \partial\mathrm{U}$ and $\nu = -\nabla d$ on $\{d(\cdot) < 0\} \cap \partial\mathrm{U}$ and $\beta \in \mathcal{C}^{2+\alpha, 1+\alpha/2}(\mathrm{U} \times (0, t_0) \times \mathbb{R}^n)$ is defined as

$$\beta(z, t, q) := \begin{cases} -\frac{1}{2}|t\nabla \ell(z) + q|^2 - t\frac{\partial \ell(z)}{\partial \nu} & \text{on } (\{d(\cdot) > 0\} \cap \partial\mathrm{U}) \times (0, t_0) \times \mathbb{R}^n, \\ \\ \frac{1}{2}|t\nabla \ell(z) + q|^2 - t\frac{\partial \ell(z)}{\partial \nu} & \text{on } (\{d(\cdot) < 0\} \cap \partial\mathrm{U}) \times (0, t_0) \times \mathbb{R}^n. \end{cases} \tag{6.41}$$

Finally, inserting (6.37) into the last equation of (6.30) we get

$$w(\cdot, 0) = 0 \qquad \text{on } \mathrm{U}. \tag{6.42}$$

Collecting together the equations (6.38), (6.40) and (6.42) for w we have:

$$\begin{cases} w_t - \mathcal{A}(z, w, \nabla^2 w) = \mathfrak{f}(z, t, w, \nabla^2 w) & \text{in } \mathrm{U} \times (0, t_0), \\ \frac{\partial w}{\partial \nu} = b(z, t, \nabla w) & \text{on } \partial\mathrm{U} \times [0, t_0], \\ w(\cdot, 0) = 0 & \text{in } \mathrm{U}. \end{cases} \tag{6.43}$$

In Proposition 6.2.4 it is shown that problem (6.43) has a unique solution $w \in \mathcal{C}^{2+\alpha,1+\alpha/2}(\overline{\mathrm{U}} \times [0, t_0])$: note that $\mathfrak{f}$ depends nonlinearly on second derivatives of w and β depends nonlinearly on first derivatives of w. We will make use of Theorem 6.3.1, that we will apply with the choice

$$a_{ij}(z, t) = a_{ij}(z) := F_{X_{ij}}(u_0(z), \nabla^2 u_0(z)), \qquad i, j = 1, \dots, n, \tag{6.44}$$
$$b_i \equiv 0, \qquad i = 1, \dots, n$$

and

$$c(z, t) = c(z) := F_{\mathrm{u}}(u_0(z), \nabla^2 u_0(z)),$$

so that $a_{ij} \in \mathcal{C}^{\alpha,\alpha/2}(\overline{\mathrm{U}})$ and $c \in \mathcal{C}^{\alpha,\alpha/2}(\overline{\mathrm{U}})$, and with the choice $\beta_i(z, t) = \beta_i(z) = \nu_i(z)$, $\gamma \equiv 0$, so that $\beta_i \in \mathcal{C}^{1+\alpha,(1+\alpha)/2}(\partial\mathrm{U} \times [0, t_0])$. Recall that (6.36) implies that (6.70) is satisfied, since the smoothness and compactness of ∂E imply that there exists a constant $C > 0$ such that $F(u_0(x), \nabla^2 u_0(x))\xi_i\xi_j \geq C|\xi|^2$ for any $x \in \partial E$ and $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$.

We also need the following expression of $\mathfrak{f}$, obtained by Taylor expanding $\mathfrak{f}$ in (6.39) to second order (with integral remainder):

$$\begin{aligned} \mathfrak{f}(z, t, \mathrm{u}, X) &= F_{\mathrm{u}}(u_0(z), \nabla^2 u_0(z))\ t\ell(z) + F_{X_{ij}}(u_0(z), \nabla^2 u_0(z))\ t\nabla_{ji}\ell(z) \\ &+ \int_0^1 (1-\sigma) F_{X_{ij}X_{kl}}(u_0 + \sigma t\ell + \sigma\mathrm{u}, \nabla^2 u_0 + \sigma t \nabla^2 \ell + \sigma X)\ u_0\sigma\ \ (t\nabla_{ij}\ell + X_{ij})\,(t\nabla_{kl}\ell + X_{kl}) \\ &+ 2\int_0^1 (1-\sigma) F_{X_{ij}\mathrm{u}}(u_0 + \sigma t\ell + \sigma\mathrm{u}, \nabla^2 u_0 + \sigma t \nabla^2 \ell + \sigma X)\ u_0\sigma\ (t\nabla_{ij}\ell + X_{ij})(t\ell + \mathrm{u}) \\ &+ \int_0^1 (1-\sigma) F_{\mathrm{uu}}(u_0 + \sigma t\ell + \sigma\mathrm{u}, \nabla^2 u_0 + \sigma t \nabla^2 \ell + \sigma X)\ d\sigma\ (t\ell + \mathrm{u})^2, \end{aligned} \tag{6.45}$$

where u_0 and ℓ are evaluated at z.

Proposition 6.2.4. *There exists $t_0 > 0$ such that problem* (6.43) *has a unique solution $w \in \mathcal{C}^{2+\alpha,1+\alpha/2}(\overline{\mathrm{U}} \times [0, t_0])$.*

Proof. The proof is based on Theorem 6.3.1 and on a fixed point argument. Define

$$Y := \left\{ u \in \mathcal{C}^{2+\alpha,1+\alpha/2}(\overline{\mathrm{U}} \times [0, t_0]) : u(\cdot, 0) = 0 \right\}.$$

Y turns out to be a Banach space. We define the map $\Gamma : Y \to Y$ as follows: given $u \in Y$, then

$$\Gamma(u) := w,$$

where w is the solution of (6.73) given by Theorem 6.3.1, with the choices

$$f(z,t) := \mathfrak{f}(z,t,u(z,t),\nabla^2 u(z,t)), \qquad \mathrm{g}(z,t) := \beta(z,t,\nabla u(z,t)), \qquad w_0 \equiv 0; \tag{6.46}$$

note that, as $u \in \mathcal{C}^{2+\alpha,1+\alpha/2}(\overline{\mathrm{U}} \times [0,t_0])$, it follows that

$$f \in \mathcal{C}^{\alpha,\alpha/2}(\overline{\mathrm{U}} \times [0,t_0]), \qquad \mathrm{g} \in \mathcal{C}^{1+\alpha,(1+\alpha)/2}(\partial U \times [0,t_0]), \tag{6.47}$$

and therefore the assumptions of Theorem 6.3.1 are satisfied.

Given $R > 0$ set

$$Y_{t_0,R} := \left\{ u \in Y : \|u\|_{\mathcal{C}^{2+\alpha,1+\alpha/2}(\overline{\mathrm{U}}\times[0,t_0])} \le R \right\}.$$

Since $Y_{t_0,R}$ is closed in Y, also $Y_{t_0,R}$ is a Banach space. We will prove the following two properties:

(i) there exist $t_0 > 0$ and $R > 0$ such that $\Gamma : Y_{t_0,R} \to Y_{t_0,R}$;

(ii) there exist $t_0 > 0$ and $R > 0$ such that

$$\|\Gamma(u) - \Gamma(v)\|_{\mathcal{C}^{2+\alpha,1+\alpha/2}(\overline{\mathrm{U}}\times[0,t_0])} \le \frac{1}{2}\|u - v\|_{\mathcal{C}^{2+\alpha,1+\alpha/2}(\overline{\mathrm{U}}\times[0,t_0])}, \qquad u,v \in Y_{t_0,R}.$$

Let us prove (i). Let $u \in Y_{t_0,r_0}$, so that

$$\|u\|_{\mathcal{C}^{\alpha,\alpha/2}(\overline{\mathrm{U}}\times[0,t_0])} \le r_0. \tag{6.48}$$

Observe that

$$\|t\|_{\mathcal{C}^{\alpha,\alpha/2}(\overline{\mathrm{U}}\times[0,t_0])} = \|t\|_{\mathcal{C}^{\alpha/2}([0,t_0])} = t_0^{1-\alpha/2}. \tag{6.49}$$

and recall that $\|uv\|_{\mathcal{C}^{\alpha,\alpha/2}(\overline{\mathrm{U}}\times[0,t_0])} \le C\|u\|_{\mathcal{C}^{\alpha,\alpha/2}(\overline{\mathrm{U}}\times[0,t_0])}\|v\|_{\mathcal{C}^{\alpha,\alpha/2}(\overline{\mathrm{U}}\times[0,t_0])}$. [2] Then, from (6.49), (6.45), (6.48) it follows that there exists $C_1 > 0$ such that

$$\|\mathfrak{f}(z,t,u(z,t),\nabla^2 u(z,t))\|_{\mathcal{C}^{\alpha,\alpha/2}(\overline{\mathrm{U}}\times[0,t_0])} \le C_1(r_0^2 + t_0^{1-\alpha/2}). \tag{6.50}$$

Similarly,

$$\|t\|_{\mathcal{C}^{1+\alpha,(1+\alpha)/2}(\partial\mathrm{U}\times[0,t_0])} = t_0^{(1-\alpha)/2}. \tag{6.51}$$

Hence, using (6.41), (6.51), (6.48),

$$\|\beta(z,t,u(z,t))\|_{\mathcal{C}^{1+\alpha,(1+\alpha)/2}(\partial\mathrm{U}\times[0,t_0])} \le C_2(r_0^2 + t_0^{(1-\alpha)/2}) \tag{6.52}$$

---

[2] We have $|u(z,t)v(z,t) - u(y,s)v(y,s)| \le |u(x,t)-u(y,s)||v(x,t)| + |v(x,t)-v(y,s)||u(y,s)|$ $\|u\|_{\mathcal{C}^{\alpha,\alpha/2}} = 2\|u\|_\infty + [u]_{\mathcal{C}^{\alpha,0}} + [u]_{\mathcal{C}^{0,\alpha/2}}$. Le seminorme holderiane del prodotto si maggiorano, aggiungendo e togliendo, con

$$[uv]_{\mathcal{C}^{\alpha,0}} \le \|u\|_\infty [v]_{\mathcal{C}^{\alpha,0}} + [u]_{\mathcal{C}^{\alpha,0}}\|v\|_\infty,$$
$$[uv]_{\mathcal{C}^{0,\alpha/2}} \le \|u\|_\infty [v]_{\mathcal{C}^{0,\alpha/2}} + [u]_{\mathcal{C}^{0,\alpha/2}}\|v\|_\infty$$

e quindi

$$\begin{aligned}\|uv\|_{\mathcal{C}^{\alpha,\alpha/2}} &= 2\|uv\|_\infty + [uv]_{\mathcal{C}^{\alpha,0}} + [uv]_{\mathcal{C}^{0,\alpha/2}} \le \\ &\le 2\|u\|_\infty\|v\|_\infty + \|u\|_\infty([v]_{\mathcal{C}^{\alpha,0}} + [v]_{\mathcal{C}^{0,\alpha/2}}) + \|v\|_\infty([u]_{\mathcal{C}^{\alpha,0}} + [u]_{\mathcal{C}^{0,\alpha/2}}) \\ &\le \|u\|_{\mathcal{C}^{\alpha,\alpha/2}}\|v\|_{\mathcal{C}^{\alpha,\alpha/2}}.\end{aligned}$$

for some $C_2 > 0$. From (6.50), (6.52), (6.46), $1 - \alpha/2 > (1-\alpha)/2$, the definition of w and (6.74) we have

$$\|w\|_{\mathcal{C}^{\alpha,\alpha/2}(\overline{\mathrm{U}}\times[0,t_0])} \le C_3(r_0^2 + t_0^{(1-\alpha)/2}),$$

where $C_3 := C(C_1 + C_2)$. Taking $r_0^2 \le 1/C_3$ we have $C_3 r_0^2 \le r_0/2$, so that

$$\|w\|_{\mathcal{C}^{\alpha,\alpha/2}(\overline{\mathrm{U}}\times[0,t_0])} \le \frac{r_0}{2} + C_3 t_0^{(1-\alpha)/2}.$$

Taking $t_0 \le \frac{r_0}{(2C_3)^{1/(1-\alpha/2)}}$ we get $\frac{r_0}{2} + C_3 t_0^{(1-\alpha)/2} \le r_0$, so that

$$\|w\|_{\mathcal{C}^{\alpha,\alpha/2}(\overline{\mathrm{U}}\times[0,t_0])} \le r_0,$$

and assertion (i) follows.

To prove (ii), set $B^u(z,t) := \mathfrak{f}(z,t,u(z,t),\nabla^2 u(z,t))$, $\mathrm{g}^u(z,t) := \beta(z,t,\nabla u(z,t))$, $B^v(z,t) := \mathfrak{f}(z,t,v(z,t),\nabla^2 v(z,t))$, $\mathrm{g}^v(z,t) := \beta(z,t,\nabla v(z,t))$. From (6.74) and the linearity of the equation in (6.73) we have

$$\|\Gamma(u) - \Gamma(v)\|_{\mathcal{C}^{2+\alpha,1+\alpha/2}(\overline{\mathrm{U}}\times[0,t_0])} \le C(\|B^u - B^v\|_{\mathcal{C}^{\alpha,\alpha/2}(\overline{\mathrm{U}}\times[0,t_0])} + \|\mathrm{g}^u - \mathrm{g}^v\|_{\mathcal{C}^{1+\alpha,(1+\alpha)/2}(\partial\mathrm{U}\times[0,t_0])}).$$

From properties (i) and (ii) and the fixed point theorem it follows that there exist $t_0 > 0$ and $R > 0$ such that Γ has a unique fixed point in $Y_{t_0,R}$. This concludes the proof of Proposition 6.2.4. □

6.2.3. Uniqueness. Let us now show uniqueness of solutions to (6.30). Let $u, v \in \mathcal{C}^{2+\alpha,1+\alpha/2}(\overline{\mathrm{U}} \times [0,t_0])$ be two solutions of (6.30), and set $\omega := u - v$. Then ω satisfies

$$\begin{cases} \omega_t = a_{ij}\nabla_{ij}\omega + c\omega & \text{in } \mathrm{U}\times(0,t_0), \\ b_i(z,t)\nabla_i\omega(z,t) = 0 & \text{on } \partial\mathrm{U}\times[0,t_0], \\ \omega(\cdot,0) = 0 & \text{on } \mathrm{U} \end{cases} \tag{6.53}$$

where

$$a_{ij}(z,t) := \int_0^1 F_{X_{ij}}(\sigma u(z,t) + (1-\sigma)v(z,t), \sigma\nabla^2 u(z,t) + (1-\sigma)\nabla^2 v(z,t))\, d\sigma$$

$$c(z,t) := \int_0^1 F_{\mathrm{u}}(\sigma u(z,t) + (1-\sigma)v(z,t), \sigma\nabla^2 u(z,t) + (1-\sigma)\nabla^2 v(z,t))\, d\sigma$$

and, setting $g(p) := |p|^2 - 1$,

$$b_i(z,t) := \int_0^1 \nabla_i g(\sigma\nabla u(z,t) + (1-\sigma)\nabla v(z,t))\, d\sigma = \frac{1}{2}\nabla_i u(z,t) + \frac{1}{2}\nabla_i v(z,t).$$

From (6.36) we have that a_{ij} satisfy (6.70), and b_i satisfies (6.71). Moreover $a_{ij} \in \mathcal{C}^{\alpha,\alpha/2}(\overline{\mathrm{U}}\times [0,t_0])$, and $c \in \mathcal{C}^{\alpha,\alpha/2}(\overline{\mathrm{U}} \times [0,t_0])$. Then ω solves a uniformly parabolic linear problem, so that by the classical maximum principle it follows that $\omega \equiv 0$. □

The solution u given by Theorem 6.2.1 can be continued on a larger time interval, taking $u(\cdot, t_0 + \delta)$ as initial datum. Repeating the argument, in this way one can find $T > 0$ and a solution $u : \mathrm{U} \times [0,T) \to \mathbb{R}$ such that $u \in \mathcal{C}^{2+\alpha,1+\alpha/2}(\mathrm{U} \times [0,\tau])$, for any

$\tau \in (0, T)$, and such that, if $T < +\infty$, then there does not exist any solution of (6.30) belonging to $\mathcal{C}^{2+\alpha,1+\alpha/2}(\mathrm{U} \times [0, T])$.

Looking at the linear evolution equation[3] satisfied by $\frac{u(z+he_k,t)-u(z,t)}{h}$ and passing to the limit as $h \to 0$ it is the possible to show the following result.

Proposition 6.2.5. *Assume that the boundary of E is of class $\mathcal{C}^{3+\alpha}$. Let u be the solution given by Theorem 6.2.1. Then $\nabla u \in \mathcal{C}^{2+\alpha,1+\alpha/2}(\Omega \times [0, t_0])$.*

****check (krylov?) la regolarita' C^∞ all'interno se ∂E e' $\mathcal{C}^\infty$: vedere la u come soluzione di una eq. lineare del tipo

$$u_t = tr(A(x,t)D^2(u))$$

dove la matrice $A(x)$ e' l'inversa di $(Id - uD^2(u))$, ma pensata come matrice di coefficienti in funzione della sola x e del tempo. A questo punto usare la massima regolarita' della u, per dedurne la massima regolarita' sulla $A(x)$, e da questa, usando l'equazione pensata come eq. lineare, dedurre ulteriore regolarita' della u, quindi ulteriore regolarita' della $A(x)$, e cosi' via. servirebbe la regolarita' fino al tempo zero.***

Theorem 6.2.6. *Let $E \in \mathcal{C}^\infty_b(\mathbb{R}^n)$ and let $t_0 > 0$ be as in Theorem 6.2.1. Then there exists a unique smooth compact mean curvature flow $f : [0, t_0] \to \mathcal{P}(\mathbb{R}^n)$ starting from E at time 0.*

Proof. Let U and $u \in C^{***}(\mathrm{U} \times [0, t_0])$ be given by Theorem 6.2.1. We first show that

$$|\nabla u|^2 = 1 \qquad \text{in } \mathrm{U} \times [0, t_0]. \tag{6.54}$$

We set $v := |\nabla u|^2 - 1$; by Proposition 6.2.5 we have $v \in \mathcal{C}^{2+\alpha,1+\alpha/2}(\mathrm{U} \times [0, t_0])$. By (6.30) we have

$$v = 0 \qquad \text{on } \partial\mathrm{U} \times [0, t_0] \tag{6.55}$$

and by the properties of $d(\cdot, E)$ also

$$v = 0 \qquad \text{on } \mathrm{U} \times \{t = 0\}. \tag{6.56}$$

In addition $\nabla_{ij} v = 2\nabla_k u \nabla_{ijk} u + 2\nabla_{ik} u \nabla_{kj} u$. Differentiating the equation in (6.30) with respect to z_k,

$$\begin{aligned} v_t = 2\nabla_k u \nabla_k u_t &= 2\nabla_k u \big[F_{X_{ij}}(u, \nabla^2 u)\nabla_{ijk} u + F_{\mathrm{u}}(u, \nabla^2 u)\nabla_k u\big] \\ &= F_{X_{ij}}(u, \nabla^2 u)\nabla_{ij} v - 2F_{X_{ij}}(u, \nabla^2 u)\nabla_{ik} u \nabla_{kj} u + 2F_{\mathrm{u}}(u, \nabla^2 u)|\nabla u|^2. \end{aligned}$$

Observe now that

$$F_{\mathrm{u}}(u, \nabla^2 u) = F_{X_{ij}}(u, \nabla^2 u)\nabla_{ik} u \nabla_{kj} u. \tag{6.57}$$

Indeed, by (6.35) applied with $M = \nabla^2 u \nabla^2 u$ we have

$$F_{X_{ij}}(u, \nabla^2 u)\nabla_{ik} u \nabla_{kj} u = \mathrm{tr}\left(\nabla^2 u \nabla^2 u (\mathrm{Id} - u\nabla^2 u)^{-2}\right) = \sum_{i=1}^n \frac{(\lambda_i(\nabla^2 u))^2}{(1 - u\lambda_i(\nabla^2 u))^2}. \tag{6.58}$$

[3] which turns out to be uniformly parabolic

On the other hand it is immediate to check that $F_{\mathrm{u}}(u, \nabla^2 u)$ coincides with the right hand side of (6.58).

From (6.57) we then have

$$v_t = F_{X_{ij}}(u, \nabla^2 u)\nabla_{ij} v + 2F_{\mathrm{u}}(u, \nabla^2 u)v. \tag{6.59}$$

Equation (6.59) is a linear partial differential equation in the unknown w, which is uniformly parabolic thanks to Lemma 6.2.3. Hence, from (6.55), (6.56), it follows that $v \equiv 0$ in $\mathrm{U} \times [0, t_0]$.

In particular, for any $t \in [0, t_0]$ the boundary of the set $E(t) := \{u(\cdot, t) \leq 0\}$ is a hypersurface of class $\mathcal{C}^{3+\alpha}$ without boundary in U[4]. ***in realta' $\mathcal{C}^\infty$ *** Using (6.54) it is possible to prove that

$$u(z, t) = \mathrm{dist}(z, E(t)) - \mathrm{dist}(z, \mathbb{R}^n \setminus E(t)), \qquad (z, t) \in \mathrm{U} \times [0, t_0].$$

Then, recalling also Proposition **??**, it follows that $t \in [0, t_0] \to \{u(\cdot, t) \leq 0\}$ is the smooth mean curvature flow starting from E. □

6.2.4. Improvements of the inclusion principle. Let $E_1, E_2 \in \mathcal{C}^\infty$. We say that ∂E_1 and ∂E_2 are close if there exists an open set $A \subset \mathbb{R}^n$ such that $\partial E_1 \subset A$, $\partial E_2 \subset A$, and the oriented distance functions from ∂E_1 and from ∂E_2 belong to $\mathcal{C}^\infty(A)$.

THEOREM 6.2.7. *Let $f_1, f_2 : [a, b] \to \mathcal{P}(\mathbb{R}^n)$ be two smooth compact mean curvature flows. Assume that*

$$f_1(a) \subseteq f_2(a), \qquad \partial f_1(a) \text{ and } \partial f_2(a) \text{ are close.}$$

Then

$$f_1(t) \subseteq f_2(t), \qquad t \in [a, b]. \tag{6.60}$$

If moreover $\partial f_i(a)$ are connected for $i = 1, 2$ and $f_1(a) \neq f_2(a)$, then

$$\partial f_1(t) \cap \partial f_2(t) = \emptyset, \qquad t \in (a, b]. \tag{6.61}$$

PROOF. We can suppose that each $\partial f_i(a)$ is connected, since the argument can be repeated separately for each connected component. Without loss of generality, assume that $f_1(a) \neq f_2(a)$. Let $d_i(\cdot, t)$ be the oriented distance function from $\partial f_i(t)$. By assumption there exists an open set $A \subset \mathbb{R}^n$ such that $d_i(\cdot, 0) \in \mathcal{C}^\infty(A)$. Recalling Theorem (6.2.6) we have that there exists $\tau > 0$ such that d_1 and d_2 are two solutions of equation (6.30) in $A \times [a, a + \tau]$. Define $w := d_1 - d_2$. Since $f_1(a) \subseteq f_2(a)$, it follows that

$$w(\cdot, a) \geq 0. \tag{6.62}$$

Then, from the maximum principle applied to the uniformly parabolic equation (6.53), it follows that $w(z, t) \geq 0$ for any $(z, t) \in Q$, so that (6.103) holds for any $t \in [a, a + \tau]$. From the strong maximum principle applied to (6.53), observing that $w(\cdot, a) \neq 0$, we deduce that $f_1(t) \subset f_2(t)$ and $\partial f_1(t) \cap \partial f_2(t) = \emptyset$ for any $t \in (a, a + \tau]$. Then (6.103) and (6.104) follow from Theorem 7.3. □

[4]Hence the second fundamental form of $\partial E(t)$ is of class $\mathcal{C}^{1+\alpha}$.

6.3. Definitions of parabolic Hölder spaces and linear theory

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set of class $\mathcal{C}^2$ and $T > 0$. We recall the definition of the following parabolic Hölder spaces: for $\alpha > 0$

$$\mathcal{C}^{0,\alpha}(\overline{\Omega} \times [0,T]) := \Big\{ u \in \mathcal{C}(\overline{\Omega} \times [0,T]) : u(z,\cdot) \in \mathcal{C}^{\alpha}([0,T]) \ \forall z \in \Omega, \tag{6.63}$$

$$\|u\|_{\mathcal{C}^{0,\alpha}(\overline{\Omega}\times[0,T])} := \sup_{z \in \overline{\Omega}} \|u(z,\cdot)\|_{\mathcal{C}^{\alpha}([0,T])} < +\infty \Big\} \tag{6.64}$$

$$\mathcal{C}^{\alpha,0}(\overline{\Omega} \times [0,T]) := \Big\{ u \in \mathcal{C}(\overline{\Omega} \times [0,T]) : u(\cdot,t) \in \mathcal{C}^{\alpha}(\overline{\Omega}) \ \forall t \in [0,T], \tag{6.65}$$

$$\|u\|_{\mathcal{C}^{\alpha,0}(\overline{\Omega}\times[0,T])} := \sup_{t \in [0,T]} \|u(\cdot,t)\|_{\mathcal{C}^{\alpha}(\overline{\Omega})} < +\infty \Big\} \tag{6.66}$$

where, if $0 < \theta < 1$ and O is a bounded open subset of $\mathbb{R}^m$, $m \geq 1$,

$$\mathcal{C}^{\theta}(\overline{O}) := \left\{ u \in \mathcal{C}(\overline{O}) : [u]_{\mathcal{C}^{\theta}} := \sup_{x,y \in \overline{O}, x \neq y} \frac{|u(x) - u(y)|}{|x-y|^{\theta}} < +\infty \right\}$$

$$\|u\|_{\mathcal{C}^{\theta}(\overline{O})} := \|u\|_{\infty} + [u]_{\mathcal{C}^{\theta}},$$

and for $k \in \mathbb{N}$, $k \geq 1$,

$$\mathcal{C}^{k+\theta}(\overline{O}) := \big\{ u \in \mathcal{C}^{k}(\overline{O}) : \nabla_{i_1 \dots i_k} u \in \mathcal{C}^{\theta}(\overline{O}), i_1, \dots, i_k \in \{1, \dots, n\} \big\}$$

with

$$\|u\|_{\mathcal{C}^{k+\theta}(\overline{O})} := \|u\|_{\mathcal{C}^{k}(\overline{O})} + \sum_{i_1,\dots,i_k \in \{1,\dots,n\}} [\nabla_{i_1 \dots i_k} u]_{\mathcal{C}^{\theta}(\overline{O})}.$$

For $0 < \alpha < 2$

$$\mathcal{C}^{\alpha,\alpha/2}(\overline{\Omega} \times [0,T]) := \mathcal{C}^{0,\alpha/2}(\overline{\Omega} \times [0,T]) \cap \mathcal{C}^{\alpha,0}(\overline{\Omega} \times [a,b]), \tag{6.67}$$

endowed with the norm [5],[6]

$$\|u\|_{\mathcal{C}^{\alpha,\alpha/2}(\overline{\Omega}\times[0,T])} := \|u\|_{\mathcal{C}^{0,\alpha/2}(\overline{\Omega}\times[0,T])} + \|u\|_{\mathcal{C}^{\alpha,0}(\overline{\Omega}\times[0,T])} \tag{6.68}$$

In addition

$$\begin{aligned} \mathcal{C}^{2,1}(\overline{\Omega} \times [0,T]) := \big\{ & u \in \mathcal{C}(\overline{\Omega} \times [0,T]) : \\ & \exists u_t, \nabla_{ij} u \in \mathcal{C}(\overline{\Omega} \times [0,T]), \ i,j \in \{1, \dots, n\} \big\}, \end{aligned} \tag{6.69}$$

endowed with the norm

$$\|u\|_{\mathcal{C}^{2,1}(\overline{\Omega}\times[0,T])} := \|u\|_{L^{\infty}(\overline{\Omega}\times[0,T])} + \sum_{i=1}^{n} \|\nabla_i u\|_{\infty} + \|u_t\|_{L^{\infty}(\overline{\Omega}\times[0,T])} + \sum_{i,j=1}^{n} \|\nabla_{ij} u\|_{L^{\infty}(\overline{\Omega}\times[0,T])}$$

---

[5] Since $0 < \alpha < 2$, definition (6.67) includes the definition of the space $C^{1+\beta,(1+\beta)/2}(\overline{\Omega} \times [0,T])$ for $0 < \beta < 1$: from (6.67) it follows $\|u\|_{\mathcal{C}^{1+\beta,(1+\beta)/2}} = 2\|u\|_{L^{\infty}(\overline{\Omega}\times[0,T])} + \sup_{x \in \overline{\Omega}} [u(x,\cdot)]_{\mathcal{C}^{(1+\beta)/2}([0,T])} + \sum_{i=1}^{n} \|\nabla_i u\|_{L^{\infty}(\overline{\Omega}\times[0,T];\mathbb{R}^n)} + \sum_{i=1}^{n} \sup_{t \in [0,T]} [\nabla_i u(\cdot,t)]_{\mathcal{C}^{\beta}}$. Note that a term of the form $[\nabla_i u]_{\mathcal{C}^{(1+\beta)/2}}$ does not explicitely appear, but it is recovered using the inequality $\|\varphi\|_{C^1} \leq C \|\varphi\|_{C^{2+\beta}}^{1/(2+\beta)} \|\varphi\|_{\infty}^{1+\beta}$ applied to $\varphi(x) = u(x,t) + u(x,s) - 2u(x,(s+t)/2)$, where C is a constant independent of x.

[6] Another slightly different norm **** is defined as $\|u\|_{\infty} + \sup_{x,y \in \overline{\Omega},\, x \neq y,\, t,s \in [0,t_0], t \neq s} \frac{|u(x,t) - u(y,t)|}{|x-y|^{\alpha} + |t-s|^{\alpha/2}}$.

and for $0 < \alpha < 2$

$$\mathcal{C}^{2+\alpha,1+\alpha/2}(\overline{\Omega} \times [0,T]) := \big\{u \in \mathcal{C}^{2,1}(\overline{\Omega} \times [0,T]) : \\ u_t, \nabla_{ij} u \in \mathcal{C}^{\alpha,\alpha/2}(\overline{\Omega} \times [0,T]),\ i,j \in \{1,\dots,n\}\big\},$$

endowed with the norm

$$\|u\|_{\mathcal{C}^{2+\alpha,1+\alpha/2}(\overline{\Omega}\times[0,T])} = \|u\|_{L^\infty(\overline{\Omega}\times[0,T])} + \sum_{i=1}^{n} \|\nabla_i u\|_{L^\infty(\overline{\Omega}\times[0,T])} \\ + \|u_t\|_{\mathcal{C}^{\alpha,\alpha/2}(\overline{\Omega}\times[0,T])} + \sum_{i,j=1}^{n} \|\nabla_{ij} u\|_{\mathcal{C}^{\alpha,\alpha/2}(\overline{\Omega}\times[0,T])}$$

$$\|u\|_{\mathcal{C}^{1+\alpha,(1+\alpha)/2}(\overline{\Omega}\times[a,b])} := \|u\|_\infty + \sum_{i=1}^{n} \|\nabla_i u\|_\infty + \|u_t\|_{\mathcal{C}^{\alpha,\alpha/2}(\overline{\Omega}\times[a,b])} + \sum_{i,j=1}^{n} \|\nabla_{ij} u\|_{\mathcal{C}^{\alpha,\alpha/2}(\overline{\Omega}\times[a,b])}$$

Finally, we set

$$\|f\|_{C^{(1+\alpha)/2,1+\alpha}([a,b]\times\partial\Omega)} := \inf\{\|v\|_{C^{(1+\alpha)/2,1+\alpha}([a,b]\times\overline{\Omega})} : v = f \text{ su } [a,b]\times\partial\Omega\}.$$

6.3.1. Remarks on the linear theory. It is possible to prove the following theorem on second order linear parabolic partial differential equations.

THEOREM 6.3.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set which is uniformly $C^{2+\alpha}$, $0 < \alpha < 1$, and let $T > 0$. Let $a_{ij}, b_i, c, f \in \mathcal{C}^{\alpha,\alpha/2}(\overline{\Omega} \times [0,T])$, and β_i, γ, $g \in C^{1+\alpha,(1+\alpha)/2}(\partial\Omega \times [0,T])$, $w_0 \in \mathcal{C}^{2+\alpha}(\overline{\Omega})$. Assume that there exists a constant $C > 0$ such that*

$$a_{ij}(z,t)\xi_i\xi_j \geq C|\xi|^2, \qquad t \in [0,T],\ z \in \overline{\Omega},\ \xi = (\xi_1,\dots,\xi_n) \in \mathbb{R}^n, \tag{6.70}$$

and the nontangentiality condition

$$\left|\sum_{i=1}^{n} \beta_i(z,t)\nu_i(z)\right| \geq \nu_0,\ \ 0 \leq t \leq T,\ z \in \partial\Omega, \tag{6.71}$$

Set

$$\mathcal{A}(z,t)\varphi = a_{ij}(z,t)\nabla_{ij}\varphi + b_i(z,t)\nabla_i\varphi + c(z,t)\varphi,$$

$$\mathcal{B}(z,t)\varphi = \beta_i(z,t)D_i\varphi + \gamma(z,t)\varphi.$$

Moreover, assume that the following compatibility condition holds:

$$\mathcal{B}(0,z)u_0(z) = \mathrm{g}(0,z),\ \ z \in \partial\Omega. \tag{6.72}$$

Then the problem

$$\begin{cases} \mathrm{w}_t = \mathcal{A}(z,t)w + f & \text{in } \Omega \times (0,T) \\ \mathcal{B}(z,t)u(z,t) = \mathrm{g}(z,t), & \text{on } \partial\Omega \times (0,T], \\ \mathrm{w} = w_0 & \text{on } \overline{\Omega} \times \{t = 0\} \end{cases} \tag{6.73}$$

has a unique solution $\mathrm{w} \in \mathcal{C}^{2+\alpha,1+\alpha/2}(\overline{\Omega} \times [0,T])$, *and there exists a constant* $C > 0$ *such that*

$$\|\mathrm{w}\|_{\mathcal{C}^{2+\alpha,1+\alpha/2}(\overline{\Omega}\times[0,T])} \leq C\left(\|w_0\|_{\mathcal{C}^{2+\alpha}(\overline{\Omega})} + \|f\|_{\mathcal{C}^{\alpha,\alpha/2}(\overline{\Omega}\times[0,T])} + \|\mathrm{g}\|_{\mathcal{C}^{1+\alpha,(1+\alpha)/2}(\partial\Omega\times[0,T])}\right). \tag{6.74}$$

The constant C *depends on* Ω, *on the* $C^{\alpha,\alpha/2}$*-norm of* a_{ij}, b_i, c, *on the* $C^{1+\alpha,(1+\alpha)/2}$*-norm of* β_i, γ, *o n the constants* ν, ν_0, *on the space dimension* n, *and on* T, *and it is increasing with respect to* T.

In view of the role played by Theorem 6.3.1 in the proof of Theorem 6.2.1, we give here some ideas on how to prove that $w \in \mathcal{C}^{2+\alpha,1+\alpha/2}(\overline{\Omega} \times [0,T])$, following the approach described in [**83**, Theorem 5.1.22]. In our application (see (6.43)) we have $w_0 = 0$, $b_i = 0$, and the coefficients a_{ij} and c are independent of t; therefore the statement of Theorem 6.3.1 covers a case which is more general than the one required to prove Proposition 6.2.4. The proof of Theorem 6.3.1 can be reduced to the case in which the coefficients are independent of time (see the proof of [**83**, Theorem 5.1.21]). Therefore, let us assume that

$$a_{ij}, b_i \text{ and } c \text{ do not depend on } t:$$

this is the case considered in [**83**, Theorems 5.1.19, 5.1.20]. Observe that $f \in \mathcal{C}^{\alpha,\alpha/2}(\overline{\Omega} \times [0,T])$ implies that

$$\text{the map } t \to f(t,\cdot) \text{ belongs to } \mathcal{C}^{\alpha/2}([0,T];X) \cap B([0,T];\mathcal{C}^{\alpha}(\overline{\Omega})), \tag{6.75}$$

where the Banach space X is defined as

$$X := \mathcal{C}(\overline{\Omega}),$$

and $B([0,T];\mathcal{C}^{\alpha}(\overline{\Omega}))$ denotes the space of bounded functions from $[0,T]$ into $\mathcal{C}^{\alpha/2}(\overline{\Omega})$.

The strategy of the proof now is the following.

Case 1. Assume $\mathrm{g} = 0$.

We recall that the interpolation space $D_A(\alpha/2,\infty)$ as defined in [**83**, Section 2.2.1] satisfies

$$D_A(\alpha/2,\infty) = \mathcal{C}^{\alpha}(\overline{\Omega}), \tag{6.76}$$

see [**83**, Theorem 3.1.30].

The equation (6.73) is viewed as an ordinary differential equation in X

$$\begin{cases} w'(t) = Aw(t) + f(t), & t \in [0,T], \\ \mathrm{w}(0) = w_0, \end{cases} \tag{6.77}$$

where the domain $D(A)$ of A is, thanks to the assumption $\mathrm{g} = 0$, the *linear* space given by

$$D(A) = \{\varphi \in \cap_{p\geq 1} W^{2,p}(\Omega) : \varphi, A\varphi \in X, \frac{\partial\varphi}{\partial\nu} = 0 \text{ on } \partial\Omega\}.$$

We recall (see [**83**, pag. 50]) that, using also (6.76), the interpolation space $D_A(\alpha/2 + 1,\infty)$ has the following expression:

$$D_A(\alpha/2+1,\infty) = \{\varphi \in D(A) : A\varphi \in \mathcal{C}^{\alpha}(\overline{\Omega})\} = \{\varphi \in \mathcal{C}^{2+\alpha}(\overline{\Omega}) : \frac{\partial \varphi}{\partial \nu} = 0 \text{ on } \partial\Omega\}, \quad (6.78)$$

where the last equality is a consequence of the Schauder estimates [**81**], [**67**].

From (6.75) and (6.76) we have in particular $f \in \mathcal{C}([0,T];X) \cap B([0,T];D_A(\alpha/2,\infty))$. From (6.78) we also have $w_0 \in D_A(\alpha/2+1,\infty)$. Hence we can apply [**83**, Corollary 4.3.9], and we obtain that (6.77) has a unique strict solution v which has the expression

$$v(t) = e^{tA}w_0 + (e^{tA} \star f)(t) = e^{tA}w_0 + \int_0^t e^{(t-s)A} f(s)\, ds,$$

with

$$u', Au \in \mathcal{C}([0,T];X) \cap B([0,T];\mathcal{C}^{\alpha/2}(\overline{\Omega})), \quad (6.79)$$

and

$$Au \in \mathcal{C}^{\alpha/2}([0,T];X) \quad (6.80)$$

Inclusions (6.79) imply the more delicate conclusion, namely that

$$v \in \mathcal{C}^{2+\alpha,1}(\overline{\Omega} \times [0,T]), \quad (6.81)$$

where for $0 < \alpha < 1$

$$\mathcal{C}^{2+\alpha,1}(\overline{\Omega} \times [0,T]) := \{u \in \mathcal{C}^{2,1}(\overline{\Omega} \times [0,T]) : u_t, \nabla_{ij}u \in \mathcal{C}^{\alpha,0}(\overline{\Omega} \times [0,T]) \ \forall i,j\}$$

and $\|u\|_{\mathcal{C}^{2+\alpha,1}(\overline{\Omega}\times[0,T])} := \|u\|_\infty + \sum_{i=1}^n \|\nabla_i u\|_\infty + \|u_t\|_{\mathcal{C}^{\alpha,0}} + \sum_{i=1}^n \|\nabla_{ij}u\|_{\mathcal{C}^{\alpha,0}}$. Formula (6.75) and (6.80) imply that $u = Au + f \in \mathcal{C}^{\alpha/2}([0,T];X)$. Hence, from (6.81) we get $v \in \mathcal{C}^{2+\alpha,1+\alpha/2}(\overline{\Omega} \times [0,T])$.

Case 2. Assume that $\mathrm{g} \neq 0$. We want to solve the problem

$$\begin{cases} u'(t) = Au(t) + f(t), & t \in [0,T], \\ \frac{\partial u}{\partial \nu} = \mathrm{g} \quad \text{on } \partial\Omega \times [0,T], \\ \mathrm{u}(0) = u_0. \end{cases} \quad (6.82)$$

We use the extension operator $\mathcal{N}$ with respect to the variable x, as defined in [**83**, Theorem 0.3.2]. It follows that we can construct a function $B := \mathcal{N}\mathrm{g} \in \mathcal{C}^{2+\alpha,1/2+\alpha/2}(\overline{\Omega} \times [0,T])$ such that $B = \mathrm{g}$ on $\partial\Omega \times [0,T]$. Note that $1/2 + \alpha/2 < 1$, so that B is not differentiable in time. Let us define

$$v := u - B$$

Formally, it follows that v satisfies

$$\begin{cases} v_t = \mathcal{A}v + f(t) + \mathcal{A}B - B_t, & t \in [0,T], \\ \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\Omega \times [0,T], \\ \mathrm{v}(0) = u_0 - B(0,x), \quad x \in \partial\Omega. \end{cases} \quad (6.83)$$

Again at the formal level, the function v has one of the expressions on [**83**, pag. 200], which gives

$$u = u_1 + u_2, \quad (6.84)$$

where

$$
\begin{aligned}
u_1 &:= -A\int_0^t e^{(t-s)A}[B(s,\cdot)-B(0,\cdot)]ds + B(0,\cdot),\\
u_2 &:= e^{tA}(u_0 - B(0,\cdot)) + \int_0^t e^{(t-s)A}[f(s,\cdot)+\mathcal{A}B(s,\cdot)]ds
\end{aligned}
\tag{6.85}
$$

The point is now to show that u in (6.84) has the required regularity (in particular that $u_1, u_2 \in \mathcal{C}^{2+\alpha,1}(\overline{\Omega}\times[0,T])$), and it is the solution of (6.82). The most delicate part is to prove that u_1 has the required regularity (see [**83**, pagg. 201-203]) and that $\frac{\partial u_1}{\partial\nu} = \mathrm{g}$ on $\partial\Omega\times[0,T]$: one shows that $B(s,\cdot)-B(0,\cdot)\in\mathcal{C}^{(1+\alpha)/2}([0,T];\mathcal{C}^1(\Omega))\subset\mathcal{C}^{(1+\alpha)/2}([0,T];D_A(1/2,\infty)$ and then applies [**83**, Theorem 4.3.16] with $\theta=\frac{1+\alpha}{2}$, $\beta=\frac{1}{2}$. Concerning u_2, one shows that $f(s,\cdot)+\mathcal{A}B(s,\cdot)\in\mathcal{C}([0,T];X)\cap B([0,T];\mathcal{C}^\alpha(\overline{\Omega}))$ so that $f(s,\cdot)+\mathcal{A}B(s,\cdot)\in B([0,T];D_A(\alpha/2,\infty))$; since $u_0-B(0,\cdot)\in\mathcal{C}^{2+\alpha}(\Omega)$ and has vanishing Neumann boundary condition, it follows that $u_0-B(0,\cdot)\in D_A(\alpha/2+1,\infty)$. One then applies [**83**, Corollary 4.3.9 (iii)] to gain the required regularity of u_2 and the fact that $\frac{\partial u_2}{\partial\nu}=0$ on $\partial\Omega\times[0,T]$.

Notes

6.3.2. Inclusion principles in the presence of the forcing term. A version of Theorem 7.3 can be proved also in presence of the forcing term.

THEOREM 6.3.2. *Let $f_1, f_2\in\mathcal{KF}_g$ be two smooth compact mean curvature flows with forcing term g in a common time interval $[a,b]$. Define the function*

$$\delta(t) := \mathrm{dist}(f_1(t), \mathbb{R}^n\setminus f_2(t)), \qquad t\in[a,b]. \tag{6.86}$$

Assume that

$$\delta(a) > 0. \tag{6.87}$$

Then for any $t\in[a,b]$

$$\text{there exists } \lim_{\tau\to0^+}\frac{1}{\tau}\Big(\delta(t+\tau)-\delta(t)\Big) \geq -L_g\delta(t). \tag{6.88}$$

Hence the function $t\in[a,b]\to\delta(t)e^{L_g(t-a)}$ is nondecreasing.

PROOF. We can repeat the proof of Theorem 7.3 up to formula (6.19) included. Let us now show (6.88) for $t\in[a,\sigma)$. We can repeat the computations in (6.22), (6.23) and use that $f_1, f_2\in\mathcal{KF}_g$ to obtain

$$\langle\nu,\frac{\partial\varphi_2}{\partial t}(\hat{s}(t),t)\rangle = -H^{f_2(t)}(\hat{x}_t)+g(\hat{x}_t,t) \qquad \langle\nu,\frac{\partial\varphi_1}{\partial t}(s(t),t)\rangle = -H^{f_1(t)}(x_t)+g(x_t,t). \tag{6.89}$$

Therefore

$$\lim_{\tau\to0^+}\frac{1}{\tau}(\delta(t+\tau)-\delta(t)) = -H^{f_2(t)}(\hat{x}_t)+g(\hat{x}_t,t)+H^{f_1(t)}(x_t)-g(x_t,t). \tag{6.90}$$

Since (6.28) is still valid we obtain

$$\lim_{\tau\to0^+}\frac{1}{\tau}(\delta(t+\tau)-\delta(t)) \geq g(\hat{x}_t,t)-g(x_t,t) \geq -G|\hat{x}_t-x_t| = -G\delta(t). \tag{6.91}$$

Let us now show that from (6.90) it follows that $\delta e^{L_g(t-a)}$ is nondecreasing in $[a,b]$. Indeed, suppose by contradiction that we can find a time $t_1 \in \,]a,b]$ such that $\delta(t_1) < \delta(a)\exp(-L_g(t_1-a))$. Let $\mu(s) = P(s)\exp(-L_g(s-a))$, where P is a linear decreasing polynomial such that $\mu(a) = \delta(a)$ and $\mu(t_1) > \delta(t_1)$. Define

$$s^\star = \inf\{s \in [a,b] : \delta(s) \le \mu(s)\}.$$

Then $\mu(s^\star) = \delta(s^\star)$, hence $s^\star < b$, and by definition of $s^\star$

$$\liminf_{\tau\to 0^+} \frac{\delta(s^\star+\tau)-\delta(s^\star)}{\tau} \le \mu'(s^\star) < -G\delta(s^\star),$$

a contradiction. □

Again, the conclusion of Theorem **??** is equivalent to $\delta(t) \ge \delta(a)e^{-L_g(t-a)}$ for any $t \in [a,b]$.

In presence of the forcing term the distance between $f_1(t)$ and $\mathbb{R}^n \setminus f_2(t)$ can decrease, as shown by the following example.

EXAMPLE 6.3.3. Fix $0 < \overline{\lambda} < 1$ sufficiently close to 1, in such a way that $\dot R_{\overline{\lambda}}(0) = -1/2$, $\dot R_{\overline{\lambda}}(\tau) < -1$, for a suitable $\tau \in \,]0,T[$, where $T = t^{\overline{\lambda}}$. We have $\ddot R_{\overline{\lambda}} \le -\sigma$ on $[0,\tau]$ for a suitable $\sigma > 0$. Choose $\overline{\mu} > 1$ large enough in such a way that $\dot R_{\overline{\mu}}(0) \ge 3/4$, and $\ddot R_{\overline{\mu}} < \sigma$ on $[0,\tau]$. Setting $f = R_{\overline{\lambda}} + R_{\overline{\mu}}$, we have $\dot f(0) > 0$, $\dot f(\tau) < 0$, and $\ddot f < 0$ on $[0,\tau]$.

Hence f has a unique strict local maximum $t^\star \in \,]0,\tau[$ on $[0,\tau]$.

Set $R^\star = R_{\overline{\mu}}(t^\star)$, $r^\star = R_{\overline{\lambda}}(t^\star)$, and $F = B_{R_{\overline{\mu}}(0)}(-R^\star,0) \cup B_{R_{\overline{\lambda}}(0)}(r^\star,0)$. Observe that F is the union of two disjoint balls.

6.3.3. Continuity with respect to the initial data. We discuss here the local well-posedness, in particular the continuity with respect to initial data, of the initial value problem for a second order fully nonlinear parabolic equation with first order nonlinear boundary condition. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with boundary of class $\mathcal{C}^{2+\alpha}$ and let $u_0 \in \mathcal{C}^{2+\alpha}(\overline{\Omega})$. Consider the problem

$$\begin{cases} u_t = F(z,t,u,\nabla u,\nabla^2 u) & \text{in } \overline{\Omega}\times(0,t_0),\\ g(z,t,u,\nabla u) = 0 & \text{on } \partial\Omega\times[0,t_0]\\ u(\cdot,0) = u_0(\cdot) & \text{in } \Omega \end{cases} \tag{6.92}$$

where

- $F : Q := \overline{\Omega}\times[0,t_0]\times B_{R_0}((\overline{\mathrm{u}},\overline{p},\overline{q})) \to \mathbb{R}$, $(\overline{\mathrm{u}},\overline{p},\overline{X}) \in \mathbb{R}\times\mathbb{R}^n\times\mathrm{Sym}_n$ is differentiable with respect to $\zeta = (\mathrm{u},p,X)$, F, F_{p_i}, $F_{X_{ij}}$ are locally Lipschitz continuous with respect to ζ and locally $C^{\alpha,\alpha/2}$ with respect to (z,t), uniformly with respect to the other variables: i.e., for every $\overline{t} \in [0,t_0]$ and $\beta \in \{0,1\}$

$$\sup\{\|\nabla_\zeta^\beta F(\cdot,\cdot,\zeta)\|_{C^{\alpha,\alpha/2}(\overline{\Omega}\times[0,\overline{t}])} : \zeta \in B_{R_0}((\overline{\mathrm{u}},\overline{p},\overline{X}))\} < +\infty, \tag{6.93}$$

and

$$\exists L > 0 \;:\; |\nabla_\zeta^\beta F(z,t,\zeta_1) - \nabla_\zeta^\beta F(z,t,\zeta_2)| \le L|\zeta_1-\zeta_2|, \tag{6.94}$$

for $(z,t) \in \overline{\Omega}\times[0,\overline{t}], \zeta_1,\zeta_2 \in B_{R_0}((\overline{u},\overline{p},\overline{q}))$, an the following ellipticity condition holds:

$$\sum_{i,j=1}^n \frac{\partial F}{\partial X_{ij}}(z,t,\mathrm{u},p,X)\xi_i\xi_j > 0, \;\; (z,t,\mathrm{u},p,X) \in Q, \;\; (\xi_1,\dots,\xi_n) \in \mathbb{R}^n\setminus\{0\}, \tag{6.95}$$

- $g : S := \overline{\Omega} \times [0, t_0] \times B_{R_0}((\overline{\mathrm{u}}, \overline{p}))$ satisfies the nontangentiality condition

$$\frac{\partial g}{\partial p_i}(z, t, \mathrm{u}, p)\nu_i(x) \neq 0, \qquad (z, t, \mathrm{u}, p) \in S,\ z \in \partial\Omega, \tag{6.96}$$

it is twice differentiable with respect to (u, p), each derivative up to the second order is locally Lipschitz continuous with respect to (u, p) and locally $C^{(1+\alpha)/2,1+\alpha}$ with respect to z, t, uniformly with respect to the other variables: i.e., for every $\overline{t} \geq 0$ we have

$$\sup\{\|D^{\beta}_{(\mathrm{u},p)} g(\cdot, \cdot, w)\|_{C^{(1+\alpha)/2,1+\alpha}([0,\overline{t}]\times\overline{\Omega})} :\ (\mathrm{u}, p) \in B((\overline{u}, \overline{p}), R_0),\ |\beta| = 0, 1, 2\} < +\infty \tag{6.97}$$

and there exists $M > 0$ such that

$$\begin{aligned} &|D^{\beta}_{(\mathrm{u},p)} g(z, t, \mathrm{u}_1, p_1) - D^{\beta}_{(\mathrm{u},p)} g(z, t, \mathrm{u}_2, p_2)| \leq M|(\mathrm{u}_1, p_1) - (\mathrm{u}_2, p_2)|, \\ &\forall (z, t) \in [0, \overline{t}] \times \overline{\Omega},\ (\mathrm{u}_1, p_1),\ (\mathrm{u}_2, p_2) \in B_{R_0}((\overline{u}, \overline{p})),\ |\beta| = 0, 1, 2. \end{aligned} \tag{6.98}$$

Assumptions (6.93) and (6.94) are satisfied if f is twice continuously differentiable with respect to all its arguments, and assumptions (6.97), (6.98) are satisfied if g is thrice continuously differentiable with respect to all its arguments.

Theorem 6.3.4. *Assume tuat u_0 verifies the compatibility condition*

$$g(0, z, u_0(z), \nabla u_0(z)) = 0, \qquad z \in \partial\Omega, \tag{6.99}$$

and that the range of $(u_0, \nabla u_0, \nabla^2 u_0)$ is contained in $B_{R_0/2}(\overline{\mathrm{u}}, \overline{p}, \overline{X})$. Then there exist $t_0 > 0$ and a unique $u \in C^{2+\alpha,1+\alpha/2}(\overline{\Omega} \times [0, t_0])$ satisfying (6.92). Moreover, for any $\rho > 0$, there exist $\delta_0 > 0$ and $K_0 > 0$ such that for every $u_1, u_2 \in C^{2+\alpha}(\overline{\Omega})$ satisfying

$$g(0, z, u_1(z), \nabla u_1(z)) = g(0, z, u_2(z), \nabla u_2(z)) = 0, \qquad z \in \partial\Omega \tag{6.100}$$

and

$$\|u_i - u_0\|_{C^{2+\alpha}(\overline{\Omega})} \leq \rho, \tag{6.101}$$

the solutions $u(\cdot, u_i)$ of problems (6.92) with initial data u_i satisfy

$$\|u(\cdot, u_1) - u(\cdot, u_2)\|_{C^{2+\alpha,1+\alpha/2}(\overline{\Omega}\times[0,\delta_0])} \leq K_0 \|u_1 - u_2\|_{C^{2+\alpha}(\overline{\Omega})}. \tag{6.102}$$

Proof. The proof of the existence and uniqueness part of the statement is given in [**83**, Theorem 8.5.4]. Let us show (6.102). We first check that for every $u_1 \in C^{2+\alpha}(\overline{\Omega})$ satisfying (6.100) and (6.101) the solution of problem (6.92) is defined in a common time interval $[0, \delta_0]$. This is done revisiting the proof of [**83**, Theorem 8.5.4], taking u_1 instead of u_0 as initial datum. We have to check *** that $C(R)$ in [**83**, formula 8.5.18] may be taken independent of u_1. We have $C(R) = C(C_7(R) + C_8(R))$, where C is the constant in (6.74), and $C_7(R)$, $C_8(R)$ appear in the two estimates on [**83**, pag. 325]. Looking at the proof of the estimate involving $C_7(R)$, we see that $C_7(R)$ is bounded by $a(R) + b(R)\|u_1\|_{C^{2+\alpha}(\overline{\Omega})}$ where $a(R)$, $b(R)$ do not depend on u_1. So, $C_7(R) \leq a(R) + b(R)(\|u_0\|_{C^{2+\alpha}} + \rho)$, and a similar estimate holds for $C_8(R)$. Therefore, $C(R)$ can be taken independent of u_1.

It follows that for every u_1 as above, the map Γ (see [**83**, pag. 321]) defined as $\Gamma(u) := w$, where w is the solution[7] of

$$\begin{cases} w_t(z,t) - \mathcal{A}w &= F(z,t,u,\nabla u,\nabla^2 u) - \mathcal{A}u, \\ & \quad 0 \le t \le \delta,\ z \in \overline{\Omega}, \\ \\ \mathcal{B}w(z,t) &= -g(z,t,u(z,t),\nabla u(z,t)) + \mathcal{B}u(z,t),\ \ 0 \le t \le \delta,\ z \in \partial\Omega, \\ w(0,z) &= u_1(z),\ \ z \in \overline{\Omega}. \end{cases}$$

is a 1/2-contraction in the set

$$Y_{u_1} := \{u \in \mathcal{C}^{2+\alpha,1+\alpha/2}(\overline{\Omega}\times[0,\delta]) : u(0,\cdot) = u_1,\ \|u - u_1\|_{\mathcal{C}^{2+\alpha,1+\alpha/2}} \le R\}$$

provided that $C(R)\delta^{\alpha/2} \le 1/2$. Moreover, Γ maps Y_{u_1} into itself, provided that the constant C' defined by

$$C' := C(\|F(\cdot,\cdot,u_1,\nabla u_1,\nabla^2 u_1)\|_{C^{\alpha/2,\alpha}} + \|g(\cdot,\cdot,u_1,\nabla u_1)\|_{C^{(1+\alpha)/2,1+\alpha}})$$

(C is still the constant in (6.74)) satisfies $C' \le R/2$. The sum of the norms $\|F(\cdot,\cdot,u_1,Du_1,D^2u_1)\|_{C^{\alpha/2,\alpha}}$ and $\|g(\cdot,\cdot,u_1,Du_1)\|_{C^{(1+\alpha)/2,1+\alpha}}$ does not exceed $a + b\|u_1\|_{C^{2+\alpha}}$ for suitable constants a, $b > 0$. Therefore,

$$C' \le C(a + b(\|u_0\|_{C^{2+\alpha}} + \rho)).$$

In turn, C depends on u_1 through the $C^{\alpha/2,\alpha}$-norm of the coefficients of $\mathcal{A}$ and through the $C^{(1+\alpha)/2,1+\alpha}$-norm of the coefficients of $\mathcal{B}$. The coefficients of $\mathcal{A}$ are the derivatives of F with respect to X_{ij}, p_i, u, evaluated at $(0,z,u_1(z),\nabla u_1(z),\nabla^2 u_1(z))$. Their $C^{\alpha/2,\alpha}$-norm (which coincides with their $C^{0,\alpha}$-norm, since they do not depend on time) does not exceed $a_1 + b_1\|u_1\|_{C^{2+\alpha}}$ for suitable constants a_1, $b_1 > 0$, hence it does not exceed $a_1 + b_1(\|u_0\|_{C^{2+\alpha}} + \rho)$. Similarly, the $C^{(1+\alpha)/2,1+\alpha}$-norm of the coefficients of $\mathcal{B}$ does not exceed $a_2 + b_1(\|u_0\|_{C^{2+\alpha}} + \rho)$, for suitable a_2, $b_2 > 0$. Therefore, C is bounded by a constant independent of u_1. Hence we can choose R large enough, in such a way that $C' \le R/2$, and then $\delta \le \delta_0 := (2C(R))^{-2/\alpha}$. For this choice, Γ is a 1/2-contraction that maps Y_{u_1} into itself, for every u_1 as above.

Let now u_1, u_2 be two initial data as in the statement, and set $w(\cdot) = u(\cdot,u_1) - u(\cdot,u_2)$. Then w satisfies

$$\begin{cases} w_t = a_{ij}\nabla_{ij}w + b_i\nabla_i w + cw & \text{on } \overline{\Omega}\times[0,\delta_0] \\ 0 = \beta_i D_i w + \gamma w, & \text{on } \partial\Omega\times[0,\delta_0] \\ w(0,z) = u_1(z) - u_2(z) & z \in \overline{\Omega}, \end{cases}$$

where

$$a_{ij}(z,t) := \int_0^1 F_{X_{ij}}(z,t,\xi_\sigma(z,t))\,d\sigma, \qquad b_i(z,t) := \int_0^1 F_{p_i}(z,t,\xi_\sigma(z,t))\,d\sigma,$$

$$c(z,t) := \int_0^1 F_{\mathrm{u}}(z,t,\xi_\sigma(z,t))\,d\sigma,$$

$$\xi_\sigma(z,t) := \sigma(u(z,t,u_1),\nabla u(z,t,u_1),\nabla^2 u(z,t,u_1)) + (1-\sigma)(u(z,t,u_2),\nabla u(z,t,u_2),\nabla^2 u(z,t,u_2)),$$

$$\beta_i(z,t) := \int_0^1 g_{p_i}(z,t,\eta_\sigma(z,t))\,d\sigma,\ \ \gamma(z,t) = \int_0^1 g_u(z,t,\eta_\sigma(z,t))d\sigma,$$

---

[7] $\mathcal{A}v := F_{X_{ij}}\nabla_{ij}v + F_{p_i}\nabla_i v + F_{\mathrm{u}}v$, $\mathcal{B}v = \sum_{i=1}^n g_{p_i}D_i v + g_u v$ where the derivatives of F are evaluated at $(z,0,u_1(z),\nabla u_1(z),\nabla^2 u_1(z))$ **c'e' il solito problema dell'estensione di F fuori dalle simmetriche***, and the derivatives of g are evaluated at $(0,z,u_1(z),\nabla u_1(z))$. *** corretto? ***

$$\eta_\sigma(z,t) = \sigma(u(z,t,u_1),\nabla u(z,t,u_1)) + (1-\sigma)(u(z,t,u_2),\nabla u(z,t,u_2))$$

Since the ranges of $(u(\cdot,u_1), Du(\cdot,u_1),\nabla^2 u(\cdot,u_1))$ and of $(u(\cdot,u_2),\nabla u(\cdot,u_2),\nabla^2 u(\cdot,u_2))$ are contained (see [**83**, pag. 321]) in $B_{R_0}((\overline{u},\overline{p},\overline{X}))$, then $\xi_\sigma(z,t) \in B_{R_0}((\overline{u},\overline{p},\overline{X}))$ and $\eta_\sigma(z,t) \in B_{R_0}((\overline{u},\overline{p}))$ for every t and x. Therefore, the ellipticity condition (6.70) and the nontangentiality condition (6.71) are satisfied by constants C, ν_0 independent of u_1 and u_2. If we prove that the $\mathcal{C}^{\alpha,\alpha/2}$-norm of the coefficients a_{ij}, b_i, c and the $\mathcal{C}^{1+\alpha,(1+\alpha)/2}$-norm of the coefficients β_i, γ are bounded by constants independent of u_1, u_2 we may apply Theorem 6.3.1 to obtain

$$\|w\|_{C^{1+\alpha/2,2+\alpha}([0,\delta_0]\times\overline{\Omega})} \leq C\|u_1-u_2\|_{C^{2+\alpha}(\overline{\Omega})}$$

and (6.102) follows.

Let us consider the coefficients a_{ij}. For every $t\in[0,\delta_0]$ and $x\in\overline{\Omega}$, $|a_{ij}(z,t)|\leq \sup_Q |f_{q_{ij}}|$, and assumptions (6.93) and (6.94) imply that

$$\begin{aligned}
&|a_{ij}(z,t)-a_{ij}(y,s)|\\
\leq&\int_0^1 |F_{X_{ij}}(z,t,\xi_\sigma(z,t)) - F_{X_{ij}}(y,s,\xi_\sigma(z,t))|d\sigma + \int_0^1 |F_{X_{ij}}(y,s,\xi_\sigma(z,t)) - F_{X_{ij}}(y,s,\xi_\sigma(y,s))|\, d\sigma\\
\leq& K(|t-s|^{\alpha/2}+|x-y|^\alpha) + L\int_0^1 |\xi_\sigma(z,t)-\xi_\sigma(y,s)|d\sigma \leq K(|t-s|^{\alpha/2}+|x-y|^\alpha)\\
&+\frac{L}{2}\Big([u(\cdot,u_1)]_{C^{\alpha/2,\alpha}} + [u(\cdot,u_2)]_{C^{\alpha/2,\alpha}} + [Du(\cdot,u_1)]_{C^{\alpha/2,\alpha}}\\
&+[Du(\cdot,u_2)]_{C^{\alpha/2,\alpha}} + [D^2u(\cdot,u_1)]_{C^{\alpha/2,\alpha}} + [D^2u(\cdot,u_2)]_{C^{\alpha/2,\alpha}}\Big)(|t-s|^{\alpha/2}+|x-y|^\alpha).
\end{aligned}$$

Therefore,

$$[a_{ij}]_{C^{\alpha,\alpha/2}} \leq K + C(\|u(\cdot,u_1)\|_{C^{2+\alpha,1+\alpha/2}} + \|u(\cdot,u_2)\|_{C^{1+\alpha/2,2+\alpha}}) \leq K + 2C_0(R+\|u_0\|_{C^{2+\alpha}}+\rho).$$

Similar estimates are satisfied by the coefficients b_i and c.

Let us consider now the coefficients β_i. For every $t\in[0,\delta_0]$ and $z\in\partial\Omega$ we have $|\beta_i(z,t)|\leq \sup_S|g_{p_i}|$ and, arguing as above, with estimates (6.93) and (6.94) replaced by (6.97) and (6.98),

$$\begin{aligned}
&|\beta_i(z,t)-\beta_i(x,s)|\\
\leq&\int_0^1 |g_{p_i}(z,t,\eta_\sigma(z,t)) - g_{p_i}(x,s,\eta_\sigma(z,t))|\, d\sigma + \int_0^1 |g_{p_i}(x,s,\eta_\sigma(z,t)) - g_{p_i}(x,s,\eta_\sigma(x,s))|d\sigma\\
\leq& H(|t-s|^{(1+\alpha)/2}) + M\int_0^1 |\xi_\sigma(z,t)-\eta_\sigma(x,s)|\, d\sigma\\
\leq& H(|t-s|^{(1+\alpha)/2}) + \frac{M}{2}\Big([u(\cdot,u_1)]_{C^{(1+\alpha)/2,0}} + [u(\cdot,u_2)]_{C^{(1+\alpha)/2,0}} + [Du(\cdot,u_1)]_{C^{(1+\alpha)/2,0}} + [Du(\cdot,u_2)]_{C^{(1+\alpha)/2,0}}
\end{aligned}$$

Therefore,

$$[\beta_i]_{C^{(1+\alpha)/2,0}} \leq H + C_1(\|u(\cdot,u_1)\|_{C^{1+\alpha/2,2+\alpha}} + \|u(\cdot,u_2)\|_{C^{1+\alpha/2,2+\alpha}}) \leq H + 2C_1(R+\|u_0\|_{C^{2+\alpha}}+\rho).$$

Moreover, for each k we have

$$\frac{\partial\beta_i}{\partial x_k}(z,t) = \int_0^1 \frac{\partial}{\partial x_k}(g_{p_i}(z,t,\eta_\sigma(z,t)))d\sigma$$

where

$$\frac{\partial}{\partial x_k}(g_{p_i}(z,t,\eta_\sigma(z,t))) = g_{p_i x_k}(z,t,\eta_\sigma(z,t)) + g_{p_i u}(z,t,\eta_\sigma(z,t))(\sigma u_{x_k}(z,t,u_1) + (1-\sigma)u_{x_k}(z,t,u_2))$$

$$+ \sum_{j=1}^{n} g_{p_i p_j}(z,t,\eta_\sigma(z,t))(\sigma u_{x_k x_j}(z,t,u_1) + (1-\sigma)u_{x_k x_j}(z,t,u_2)).$$

Therefore,

$$|\frac{\partial \beta_i}{\partial x_k}(z,t)| \leq \sup_S |g_{p_i x_k}| + \frac{1}{2}\sup_S |g_{p_i u}|(\sup_{0\leq t\leq \delta_0} \|u(t,\cdot,u_1)\|_{C^1(\overline{\Omega})} + \sup_{0\leq t\leq \delta_0} \|u(t,\cdot,u_2)\|_{C^1(\overline{\Omega})})$$

$$+\frac{1}{2}\sum_{j=1}^{n} \sup_S |g_{p_i p_j}|(\sup_{0\leq t\leq \delta_0} \|u(t,\cdot,u_1)\|_{C^2(\overline{\Omega})} + \sup_{0\leq t\leq \delta_0} \|u(t,\cdot,u_2)\|_{C^2(\overline{\Omega})})$$

$$\leq C_2(R + \|u_0\|_{C^{2+\alpha}} + \rho).$$

In a similar manner one estimates $[\frac{\partial \beta_i}{\partial x_k}]_{C^{0,\alpha}}$. □

Proposition 6.2.5 is proven in [**83**, Proposition 8.5.6]. The regularity of w in Theorem **??** is proven in [**83**, Theorem 8.5.6]. Theorem 6.3.1 is proven in [**83**, Corollary 5.1.22]. Proposition 6.2.5 is taken from [**83**, Proposition 8.5.6]. In [**69**, Section 2] the authors prove a short-time existence and uniqueness theorem for a manifold evolving by mean curvature inside another ambient manifold.

In [**11**] the author proves a short-time existence theorem for curves on surfaces evolving by a suitable geometric law (including curvature flow), allowing singular initial curves, namely curves with p-integrable curvature, and also curves that are locally graph of a Lipschitz function. See also [**12**, Theorem 3.2]. We also note a comment concerning the continuity of the curve solution with respect to parameters given in [**11**, p. 460].

A general short-time existence theorem for a large class of geometric evolution problems, including evolutions of higher order, can be found in [**75**, Theorem 7.17]: this theorem covers evolutions of the form (4.13), once one observes that an initially embedded hypersurface evolving under (4.13) remains embedded for short times [8]. An existence and uniqueness theorem for an evolution equation which is nonlocal, similarly to (5.5) where the nonlocality is due to the fact that g is evaluated on $\mathrm{pr}_{\Sigma(t)}(z)$, has been proved in [**33**, Theorem 3.1].

In [**57**, Section 4] the authors prove a short time existence result of a smooth solution of mean curvature flow starting from a locally Lipschitz initial manifold.

6.3.4. More on the inclusion principle. Using the continuity result with respect to initial data proved in Theorem **??**, we now want to improve the inclusion principles described in Sections **??**.

Proposition 6.3.5. *Let $E \in \mathcal{C}^\infty \cap \mathcal{K}$, and let $f : [0,t_0] \to \mathcal{P}(\mathbb{R}^n)$ be the smooth compact mean curvature flow given by Theorem* **??**. *Then there exist $\overline{\rho} > 0$ and $t_1 \in (0,t_0]$ such that*

- *for any $\rho \in [0,\overline{\rho}]$ we have $E_\rho^- \in \mathcal{C}_b^\infty \cap \mathcal{K}$ (resp. $E_\rho^+ \in \mathcal{C}_b^\infty \cap \mathcal{K}$)*
- *for any $\rho \in [0,\overline{\rho}]$ the set E_ρ^- (resp. E_ρ^+) has a unique smooth mean curvature flow $f_\rho^- : [0,t_1] \to \mathcal{P}(\mathbb{R}^n)$ (resp. $f_\rho^+ : [0,t_1] \to \mathcal{P}(\mathbb{R}^n)$).*

[8] See Section **??**

PROOF. The first assertion follows from the results in Chapter **??**. It remains to show that the existence time of the smooth flow starting from E_ρ^- does not tend to zero as $\rho \to 0^+$. This assertion can be proved as follows: it is possible to show that the maximal existence time of the smooth mean curvature flow starting from E is bounded from below by a constant times the L^∞ norm of the second fundamental form of ∂E[9]. Using Theorem 2.3.1, it is then enough to observe that the L^∞ norm of the second fundamental form of ∂E_ρ^- is uniformly bounded from above with respect to ρ, provided ρ is sufficiently small. □

PROPOSITION 6.3.6. *Let $E, f, t_0, \overline{\rho}, t_1, E_\rho^\pm$ and $f_\rho^\pm$ be as in Proposition 6.3.5. Then for any $t \in [a, a+t_0]$, we have $\cup_{\rho\in[0,\overline{\rho}]} f_\rho^-(t) = f(t)$ and $\cap_{\rho\in[0,\overline{\rho}]} f_\rho^+(t) = f(t)$.*

PROOF. From Theorem 6.2.7 it follows that $\cup_{\rho\in[0,\overline{\rho}]} f_\rho^-(t) \subseteq f(t)$. The opposite inclusion follows from (6.102). The proof of the assertion concerning f_ρ^+ is similar. □

THEOREM 6.3.7. *Let $f_1, f_2 : [a,b] \to \mathcal{P}(\mathbb{R}^n)$ be two smooth compact mean curvature flows. Assume that*

$$f_1(a) \subseteq f_2(a).$$

Then

$$f_1(t) \subseteq f_2(t), \qquad t \in [a,b]. \tag{6.103}$$

If moreover $\partial f_i(a)$ are connected for $i = 1,2$ and $f_1(a) \neq f_2(a)$, then

$$\partial f_1(t) \cap \partial f_2(t) = \emptyset, \qquad t \in (a,b]. \tag{6.104}$$

PROOF. Using the notation of Proposition 6.3.5 (with f_1 in place of f), we have that ${f_1}_\rho^-(t) \subseteq f_1(t)$ for any $t \in [a, a+t_0]$, and $\partial {f_1}_\rho^-(t) \cap \partial f_1(t) = \emptyset$ for any $t \in [a, a+t_0]$. ***ecc. ecc. From Proposition **??** □

Concerning the short time existence of $E \in \mathcal{C}^\infty(\mathbb{R}^n)$, in the paper [] it is considered the case when ∂E is an entire graph; see also ***chiedere referenze a cardaliaguet ***

Remark **??**: the maximal time of existence depends only on a bound of the second fundamental form of initial set ∂E: see i[**57**, Section 4], Ilmanen [], and Theorem **??**.

[9]This is shown, in the case of curves, in Chapter **??**. For the general case, see **??**.