CHAPTER 4

Smooth flows: preliminary version

This version is in progress: please, take this into account. All corrections and comments are welcome.

In what follows we denote by $\mathcal{P}(\mathbb{R}^n)$ the class of all subsets of $\mathbb{R}^n$.

Definition 4.0.10. *We say that f is a smooth flow if there exist $a, b \in \mathbb{R}$, $a < b$ such that $f : [a, b] \to \mathcal{P}(\mathbb{R}^n)$ and*

(i) *the set $\{(z, t) : t \in [a, b], z \in f(t)\}$ is closed;*

(ii) *letting*[1]

$$d(z, t) := d(z, f(t)) = \mathrm{dist}(z, f(t)) - \mathrm{dist}(z, \mathbb{R}^n \setminus f(t)), \qquad z \in \mathbb{R}^n, \ t \in [a, b], \tag{4.1}$$

there exists an open set $A \subseteq \mathbb{R}^n$ such that $A \supset \partial f(t)$ for any $t \in [a, b]$, and $d \in \mathcal{C}^\infty(A \times [a, b])$.

We say that f is a smooth compact flow if in addition $\partial f(t)$ is compact for any $t \in [a, b]$.

Note that f is a smooth flow if and only if

$$f^c(t) := \overline{\mathbb{R}^n \setminus f(t)}, \qquad t \in [a, b],$$

is a smooth flow. Note also that $\partial f(t) \in \mathcal{C}^\infty$ for any $t \in [a, b]$, and if f is a smooth compact flow then $\partial f(t) \in \mathcal{C}^\infty \cap \mathcal{K}(\mathbb{R}^n)$ for any $t \in [a, b]$.

As usual, for $x \in \partial f(t)$, $N_x(\partial f(t))$ and $T_x(\partial f(t))$ denote the normal line and the tangent space, respectively, to $\partial f(t)$ at x.

Notation: When no confusion is possible, we sometimes use the notation

$$\Sigma(t) = \partial f(t).$$

Definition 4.0.11. *Let $\partial E \in \mathcal{C}^\infty$ and let $f : [a, b] \to \mathcal{P}(\mathbb{R}^n)$ be a smooth flow. If $f(a) = E$ we say that f starts from E at time a.*

Let us recall the definition of normal velocity vector. Let as usual $\nabla = (\frac{\partial}{\partial z^1}, \ldots, \frac{\partial}{\partial z^n})$.

Definition 4.0.12. *Let $f : [a, b] \to \mathcal{P}(\mathbb{R}^n)$ be a smooth flow and let $t \in [a, b]$. The normal velocity vector of the flow at $x \in \partial f(t)$ is defined as*

$$-\frac{\partial d}{\partial t}(x, t) \ \nabla d(x, t). \tag{4.2}$$

[1]Even if the time variable is present, for simplicity of notation we still use here the symbol d, as in (2.4).

Note that the normal velocity vector is unchanged if we replace d with d_{f^c} in (4.2).

REMARK 4.0.13. If we define

$$\eta := \frac{1}{2}(d)^2 \qquad \text{in } \mathbb{R}^n \times [a,b], \tag{4.3}$$

then $\eta \in \mathcal{C}^\infty(A \times [a,b])$ and

$$\frac{\partial d}{\partial t}\ \nabla d = \frac{\partial \nabla \eta}{\partial t} \qquad \text{on } \partial f(t).$$

4.0.1.1. *Normal velocity using parametrizations.* Using also the results in Chapter 2, one can prove that $f : [a,b] \to \mathcal{P}(\mathbb{R}^n)$ is a smooth compact flow (resp. a smooth flow) if and only if there exist a smooth compact (resp. smooth) $(n-1)$-dimensional manifold $\mathcal{S}$ without boundary and a map $\varphi \in \mathcal{C}^\infty(\mathcal{S} \times [a,b]; \mathbb{R}^n)$ such that

(i) for any $t \in [a,b]$ the map $\varphi(\cdot,t)$ is a bijection between $\mathcal{S}$ and

$$\partial f(t) = \varphi(\mathcal{S}, t);$$

(ii) for any $s \in \mathcal{S}$ and any $t \in [a,b]$ the differential $d\varphi(s,t)$ with respect to s is injective.

Hence for any $t \in [a,b]$ the map $\varphi(\cdot,t)$ is a smooth embedding of the manifold $\mathcal{S}$ in $\mathbb{R}^n$, and $\partial f(t)$ is the image of the embedding; in addition, φ depends smoothly on the variable t.

DEFINITION 4.0.14. *Let $s \in \mathcal{S}$, $t \in [a,b]$, $x = \varphi(s,t)$. We define $\mathbf{V}(s,t)$ as the orthogonal projection of $\frac{\partial \varphi}{\partial t}(s,t)$ on $N_x(\partial f(t))$, that is,*

$$\mathbf{V}(s,t) := \langle \nu(s,t), \frac{\partial \varphi}{\partial t}(s,t) \rangle \nu(s,t), \tag{4.4}$$

where $\nu(s,t) := \nabla d(x,t)$ denotes the unit normal to $\partial f(t)$ at $x = \varphi(s,t)$, pointing toward $\mathbb{R}^n \setminus f(t)$.

$\mathbf{V}(s,t)$ depends only on the set $\partial f(t)$ and not on the way $\partial f(t)$ is parameterized, since reparameterizations add only tangential components to the velocity. Precisely, let $\psi \in \mathcal{C}^\infty(\mathcal{S} \times [a,b]; \mathcal{S})$ be such that for any $t \in [a,b]$ the map $\psi(\cdot,t)$ is a smooth diffeomorphism of $\mathcal{S}$, and set $\widetilde{\varphi}(s,t) := \varphi(\psi(s,t),t)$. Then the orthogonal projections of $\frac{\partial \widetilde{\varphi}}{\partial t}(s,t)$ and of $\frac{\partial \varphi}{\partial t}(\psi(s,t),t)$ on $N_x(\partial f(t))$, $x = \varphi(\psi(s,t),t)$, are equal, since $\frac{\partial \widetilde{\varphi}}{\partial t} = d\varphi \frac{\partial \psi}{\partial t} + \frac{\partial \varphi}{\partial t}$, and $d\varphi(\psi(s,t),t)\frac{\partial \psi(s,t)}{\partial t} \in T_x(\partial f(t))$. On the other hand, orthogonal projections of $\frac{\partial \varphi}{\partial t}$ on lines different from the normal line may depend in general on parameterizations.

PROPOSITION 4.0.15. *For any $s \in \mathcal{S}$ and any $t \in [a,b]$ we have*

$$-\frac{\partial d}{\partial t}(x,t)\ \nabla d(x,t) = \mathbf{V}(s,t), \qquad x := \varphi(s,t) \in \partial f(t). \tag{4.5}$$

PROOF. We know that $d(\varphi(s,t),t) = 0$ for any $s \in \mathcal{S}$ and any $t \in [0,T]$. Hence, differentiating with respect to t and setting $x := \varphi(s,t)$, we get

$$\langle \frac{\partial \varphi}{\partial t}(s,t), \nabla d(x,t) \rangle + \frac{\partial d}{\partial t}(x,t) = 0. \tag{4.6}$$

Then (4.5) follows from (4.4). □

4.0.1.2. *The diffeomorphism Φ between $\mathcal{S}\times(-\rho,\rho)\times[a,b]$ and $A\times[a,b]$.* If $f:[a,b]\to\mathcal{P}(\mathbb{R}^n)$ is a smooth compact flow, in general for two different $t_1,t_2\in[a,b]$ it may happen that $\Sigma(t_1)\cap\Sigma(t_2)\neq\emptyset$[2]. On the other hand,

$$t_1\neq t_2\Rightarrow\{(x,t_1):x\in\Sigma(t_1)\}\cap\{(x,t_2):x\in\Sigma(t_2)\}=\emptyset. \tag{4.7}$$

We let $s:A\times[a,b]\to\mathcal{S}$ be the map defined as follows: given $(z,t)\in A\times[a,b]$, the point $\varphi(s(z,t),t)\in\partial E(t)$ is the unique projection point of z on $\partial E(t)$, namely

$$z-\varphi(s(z,t),t)=d(z,t)\nabla d(z,t). \tag{4.8}$$

We define the map $\Phi\in\mathcal{C}^\infty(A\times[a,b];\mathcal{S}\times(-\rho,\rho)\times[a,b])$ as

$$\Phi(z,t):=(s(z,t),d(z,t),t).$$

The map Φ can be inverted, so that $\Phi^{-1}\in\mathcal{C}^\infty(\mathcal{S}\times(-\rho,\rho)\times[a,b];A\times[a,b])$,

$$\Phi^{-1}(s,d,t)=(z,t),\qquad z(s,d,t)=\varphi(s,t)+dn(s,t).$$

EXAMPLE 4.0.16. Let $n=2$, $e_1=(1,0)$, and let $f:[0,1]\to\mathcal{P}(\mathbb{R}^2)$ be the smooth flow consisting of the initial disk $f(0)=\{z\in\mathbb{R}^2:|z|\leq1\}$ which translates in the e_1-direction with constant scalar speed $v>0$, i.e., $f(t)=\{z\in\mathbb{R}^2:|z-tve_1|\leq1\}$ for $t\in[0,1]$. We have $d(z,t)=|z-tve_1|-1$, and the normal velocity vector at $z\in\partial f(0)$ equals $\langle z,ve_1\rangle z$.

DEFINITION 4.0.17. *The quantity $\langle\frac{\partial\varphi}{\partial t}(s,t),\nabla d(x,t)\rangle$ is called normal velocity of the flow and equals $-\frac{\partial d}{\partial t}(x,t)$.*

Finally, let $e\in\mathbb{S}^{n-1}$ be a unit vector of $\mathbb{R}^n$. The velocity vector of the flow in the direction e at $x\in\partial f(t)$ is defined as

$$-\langle\nabla d(x,t),e\rangle^{-1}\frac{\partial d(x,t)}{\partial t}\,e,$$

and it is such that its orthogonal projection on $N_x(\partial f(t))$ is the normal velocity vector at x. The velocity of the flow at x in the direction e is defined as $-\langle\nabla d(x,t),e\rangle^{-1}\frac{\partial d}{\partial t}(x,t)$.

REMARK 4.0.18. The normal velocity vector can also be expressed as follows. Let $u:\mathbb{R}^n\times[a,b]\to\mathbb{R}$ be a continuous function which is smooth in $A\times[a,b]$, where $A\subset\mathbb{R}^n$ is an open set containing $\cup_{t\in[a,b]}\{u(\cdot,t)=0\}$, and such that $u^2+|\nabla u|^2>0$ in $A\times[a,b]$. Then $f:[a,b]\to\mathcal{P}(\mathbb{R}^n)$ defined as $f(t):=\{z\in\mathbb{R}^n:u(z,t)\leq0\}$ is a smooth flow, and $\partial f(t)=\{z\in\mathbb{R}^n:u(z,t)=0\}$. Letting $u_t:=\frac{\partial u}{\partial t}$, the normal velocity vector equals $-\frac{u_t}{|\nabla u|}\frac{\nabla u}{|\nabla u|}$. If in addition there exists $v\in\mathcal{C}^\infty(\mathbb{R}^{n-1}\times[a,b])$ such that $u(s,z_n,t):=v(s,t)-z_n$, we can parametrize the flow as $(s,t)\to\varphi(s,t):=(s,v(s,t))$. Therefore $\frac{\partial\varphi}{\partial t}=(0,\frac{\partial v}{\partial t})$, and the normal velocity can be written as

$$\langle\frac{\partial\varphi}{\partial t}(s,t),\nabla d\rangle\nabla d=\frac{\frac{\partial v}{\partial t}}{1+|\nabla v|^2}(-\nabla v,1), \tag{4.9}$$

where ∇v on the right hand side is the gradient with respect to s.

[2]If f is a smooth compact mean curvature flow it happens that $\Sigma(t_1)\cap\Sigma(t_2)=\emptyset$ if, for instance, $\Sigma(a)$ has nonnegative mean curvature.

The definition of smooth flows can be generalized as follows[3].

Definition 4.0.19. *We say that f is a generalized smooth flow in $[a, b]$ if there exist $a, b \in \mathbb{R}$, $a < b$ such that $f : [a, b] \to \mathcal{P}(\mathbb{R}^n)$, if (i) of Definition 4.0.10 holds, and if for any $t \in [a, b]$ there exists an open set and $A_t \subseteq \mathbb{R}^n$ such that $A_t \supseteq \partial f(t)$, and $d \in \mathcal{C}^\infty\Big(\bigcup_{t\in[a,b]} (A_t \times \{t\})\Big)$. We say that f is a generalized smooth compact flow if in addition the set $\{(z, t) : t \in [a, b], z \in f(t)\}$ has compact boundary.*

Definition (4.0.19) can be given in the same way if $a = -\infty$ and/or $b = +\infty$.

4.1. Smooth mean curvature flows with forcing term

We are now in a position to define classical mean curvature flow of boundaries using the signed distance function d defined in (4.1).

From now on the function g (that stands for a driving force) will be assumed to satisfy the following properties:

$g \in \mathcal{C}^\infty(\mathbb{R}^n \times [0, +\infty)) \cap L^\infty(\mathbb{R}^n \times [0, +\infty))$;
there exists a constant $L_g > 0$ such that

$$|g(z, t) - g(y, t)| \le L_g|z - y|, \qquad z, y \in \mathbb{R}^n,\ t \in [0, +\infty). \tag{4.10}$$

***queste ipotesi su g non sono necessarie tutte, perche' i flussi sono compatti ***

Definition 4.1.1. *Let $f : [a, b] \to \mathcal{P}(\mathbb{R}^n)$ be a smooth flow. We say that f is a smooth mean curvature flow with forcing term g (in $[a, b]$), if*

$$\frac{\partial d}{\partial t}(x, t)\nabla d(x, t) = (\Delta d(x, t) + g(x, t))\nabla d(x, t), \qquad t \in [a, b],\ x \in \partial f(t). \tag{4.11}$$

If in addition f is a smooth compact flow we say that f is a smooth compact mean curvature flow with forcing term g in $[a, b]$, and we write $f \in \mathcal{KF}_g$. When $g \equiv 0$ we say that f is a smooth mean curvature flow; moreover, we write $f \in \mathcal{KF}$ in place of $f \in \mathcal{KF}_0$.

Remark 4.1.2. If $f : [a, b] \to \mathcal{P}(\mathbb{R}^n)$, $f \in \mathcal{KF}_g$, then the map $f^c : [a, b] \to \mathcal{P}(\mathbb{R}^n)$ is a smooth compact mean curvature flow with forcing term $-g$, so that $f^c \in \mathcal{KF}_{-g}$.

Let $f : [a, b] \to \mathcal{P}(\mathbb{R}^n)$ be a smooth flow. Recall our notation:

$$\nu(s, t) = \nabla d(x, t) =: \mathrm{n}(x, t), \qquad t \in [a, b], x = \varphi(s, t) \in \Sigma(t),$$
$$H(x, t) = \Delta d(x, t) =: \mathrm{H}(s, t), \qquad t \in [a, b], x = \varphi(s, t) \in \Sigma(t).$$

We also set

$$\mathbf{H}(s, t) := -\Delta d(x, t)\nabla d(x, t),$$

Then $f \in \mathcal{KF}$ if and only if

$$\mathbf{V}(s, t) = -\mathbf{H}(s, t), \qquad s \in \mathcal{S}, t \in [a, b].$$

[3]As we will see, the barriers' theory remains unchanged if one uses smooth compact mean curvature flows or generalized smooth compact mean curvature flows: see Remark 9.0.27 in Chapter 9.

REMARK 4.1.3. Let $g \equiv 0$. With the notation of Remark 4.0.13, recalling (2.18), we have that (4.11) can be written equivalently as

$$\frac{\partial \nabla \eta}{\partial t}(x,t) = \Delta \nabla \eta(x,t), \qquad t \in [a,b],\ x \in \partial f(t). \tag{4.12}$$

REMARK 4.1.4. Recalling that $|\nabla d(z,t)| = 1$ for any $(z,t) \in A \times [a,b]$, the system in (4.11) is equivalent to

$$\frac{\partial d}{\partial t}(x,t) = \Delta d(x,t) + g(x,t), \qquad t \in [a,b],\ x \in \partial f(t) \tag{4.13}$$

which, in turn, is equivalent to the system

$$\begin{cases} \dfrac{\partial d}{\partial t} = \Delta d + g, \\ \\ d(\cdot,t) = 0, \end{cases} \qquad t \in [a,b]. \tag{4.14}$$

We conclude this section with the definition of smooth sub/supersolutions of mean curvature flow, which will be useful in the sequel.

DEFINITION 4.1.5. *Let $f : [a,b] \to \mathcal{P}(\mathbb{R}^n)$ be a smooth compact flow. We write $f \in \mathcal{KF}_g^{\geq}$ if*

$$\frac{\partial d}{\partial t}(x,t) \geq \Delta d(x,t) + g(x,t), \qquad t \in [a,b],\ x \in \partial f(t). \tag{4.15}$$

Similarly, we write $f \in \mathcal{KF}_g^{>}$ (resp. $f \in \mathcal{KF}_g^{\leq}$, $f \in \mathcal{KF}_g^{<}$) if the inequality $>$ (resp. $\leq$, $<$) holds in (4.15).

4.2. Examples

In this section we give some examples.

EXAMPLE 4.2.1. Let $n = 1$, $f : [a,b] \to \mathcal{P}(\mathbb{R})$, $f \in \mathcal{KF}_g$, and let d, A be as in Definition 4.0.10. Since $d_f \in \mathcal{C}^\infty(A)$, it follows that $\partial f(t)$ is a finite union of points, so that $f(t)$ is a finite union of intervals for $t \in [a,b]$, evolving in a smooth way. Then $d(\cdot,t)$ is linear in a neighbourhood of each extremum of the intervals, and hence $\Delta d = 0$ in this neighbourhood. Assume that $[x^-(t), x^+(t)]$ is one of the intervals composing $f(t)$ for $t \in [a,b]$. Note that

$$\frac{\partial d}{\partial t}(x^-(t),t) = \frac{dx^-}{dt}(t), \qquad \frac{\partial d}{\partial t}(x^+(t),t) = -\frac{dx^+}{dt}(t), \qquad t \in [a,b].$$

Hence by (4.13) we get

$$\frac{dx^-}{dt}(t) = -g(x^-(t),t), \qquad \frac{dx^+}{dt}(t) = g(x^+(t),t), \qquad t \in [a,b]. \tag{4.16}$$

EXAMPLE 4.2.2. Let $v \in C^\infty(\mathbb{R}^{n-1})$ and assume that $E := \{(s,z_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : z_n \geq v(s)\}$ is such that ∂E has zero mean curvature. Then, given $T > 0$, the map $f : [0,T] \to \mathcal{P}(\mathbb{R}^n)$, $f(t) := E$ for any $t \in [0,T]$, is a smooth mean curvature flow starting from E. Hence smooth graphs with vanishing mean curvature are stationary solutions to mean curvature flow.

Example 4.2.3. Let $R_0 > 0$ and $y \in \mathbb{R}^n$; a smooth compact mean curvature flow starting from the ball $B_{R_0}(z_0)$ is the ball $f(t) = B_{R(t)}(z_0)$, where

$$R(t) = \sqrt{R_0^2 - 2(n-1)t}, \qquad t \in [0,T], \qquad T < t^\dagger := \frac{R_0^2}{2(n-1)}.$$

Indeed $d(z,t) = |z - z_0| - R(t)$, hence $d_f \in \mathcal{C}^\infty((\mathbb{R}^n \setminus \{z_0\}) \times [0,T])$, and $\frac{\partial d}{\partial t}(z,t) = -\dot{R}(t)$,

$$\nabla d(z,t) = \frac{z - z_0}{|z - z_0|}, \quad \nabla^2 d(z,t) = \frac{1}{|z - z_0|}\left(\mathrm{Id} - \frac{z - z_0}{|z - z_0|} \otimes \frac{z - z_0}{|z - z_0|}\right), \quad \Delta d(z,t) = \frac{n-1}{|z - z_0|}.$$

Hence (4.13) becomes

$$\dot{R}(t) = -\frac{n-1}{R(t)}. \tag{4.17}$$

Coupled with $R(0) = R_0$, the solution is $R(t) = \sqrt{R_0^2 - 2(n-1)t}$. Observe that

$$B_{R(t)}(z_0) = \sqrt{1 - \frac{t}{t^\dagger}}\, B_{R_0}(z_0).$$

Note that

$$\lim_{t \uparrow t^\dagger} \int_{\partial B_{R(t)}(z_0)} H^2 \, d\mathcal{H}^{n-1} = \begin{cases} +\infty & \text{if } n = 2, \\ 16\pi & \text{if } n = 3, \\ 0 & \text{if } n \geq 4, \end{cases} \qquad \int_0^{t^\dagger} \int_{\partial B_{R(t)}} H^2 \, d\mathcal{H}^{n-1} < +\infty.$$

Note also that since H is constant, no informations can be inferred from the $L^2_{\mathcal{H}^{n-1}}(\partial B_{R(t)}(z_0))$-norms of the various derivatives of H.

Example 4.2.4. Let $m \in \{1, \dots, n-1\}$, $R_0 > 0$, and let $C := \{(\sigma, y) \in \mathbb{R}^{n-m} \times \mathbb{R}^m : |\sigma| \leq R_0\}$. Then a smooth mean curvature flow starting from C is given by the cylinder $f(t) = C(t) = \{(\sigma, y) \in \mathbb{R}^{n-m} \times \mathbb{R}^m : |\sigma| \leq R(t)\}$, where

$$R(t) = \sqrt{R_0^2 - 2(n-m-1)t}, \qquad t \in [0,T], \qquad T < t^\dagger := \frac{R_0^2}{2(n-m-1)}.$$

Observe that $C(t) = \sqrt{1 - \frac{t}{t^\dagger}}\, C$.

Definition 4.2.5. *We say that f is a smooth self-similar evolution if there exist $E \subset \mathbb{R}^n$ with $\partial E \in \mathcal{C}^\infty$, an interval $I \subseteq \mathbb{R}$, and a smooth function $\alpha : I \to (0, +\infty)$ such that*

$$f(t) = \alpha(t) E, \tag{4.18}$$

for any $t \in I$.

Observe that if E and α are as in Definition 4.2.5, then λE and $\frac{\alpha(t)}{\lambda}$ give raise to the same self-similar evolution, for any $\lambda > 0$. We denote by $I_{\max}$ the maximal open interval where we can smoothly extend the self-similar solution α.

The following proposition describes a class of special solutions to mean curvature flow.

PROPOSITION 4.2.6. *Let $f : I \to \mathcal{P}(\mathbb{R}^n)$ be a smooth self-similar evolution. If f is a smooth mean curvature flow then one of the following three conditions hold: setting $d(\cdot) := d(\cdot, E)$,*

(i) *there exist $t_0 \in \mathbb{R}$ and $T > 0$ such that $I_{\max} = (-\infty, t_0)$, $\alpha(t) = \sqrt{\frac{t_0}{T}}\sqrt{1 - \frac{t}{t_0}}$ for any $t \in I_{\max}$, and*

$$\Delta d(x) = \frac{1}{2T}\langle x, \nabla d(x)\rangle, \qquad x \in \partial E; \tag{4.19}$$

(ii) *$I_{\max} = \mathbb{R}$, $\alpha'(t) = 0$ and*

$$\Delta d(x) = 0, \qquad x \in \partial E; \tag{4.20}$$

(iii) *there exist $t_0 \in \mathbb{R}$ and $T > 0$ such that $I_{\max} = (t_0, +\infty)$, $\alpha(t) = \sqrt{\frac{t_0}{T}}\sqrt{\frac{t}{t_0} - 1}$ for any $t \in I_{\max}$, and*

$$\Delta d(x) = -\frac{1}{2T}\langle x, \nabla d(x)\rangle, \qquad x \in \partial E. \tag{4.21}$$

Conversely, assume that one of the conditions (i)-(iii) holds. Define $f : I_{\max} \to \mathcal{P}(\mathbb{R}^n)$ as in (4.18). Then f is a smooth mean curvature flow.

PROOF. Assume that f in (4.18) is a smooth mean curvature flow. Let $z \in \mathbb{R}^n$. We have

$$\mathrm{dist}\,(z, f(t)) = \inf_{y \in f(t)} |y - z| = \alpha(t) \inf_{y/\alpha(t) \in E} |y/\alpha(t) - z/\alpha(t)| = \alpha(t)\mathrm{dist}(z/\alpha(t), E).$$

Similarly, $\mathrm{dist}(z, \mathbb{R}^n \setminus f(t)) = \alpha(t)\mathrm{dist}(z/\alpha(t), \mathbb{R}^n \setminus E)$. Hence, if d is the function defined in (4.1) and $d = d$ is the one defined in (2.4), we have $d(z,t) = \alpha(t)d(z/\alpha(t))$. Then we compute:

$$\nabla d(z) = \nabla d(z/\alpha(t)), \qquad \Delta d(z) = \frac{1}{\alpha(t)}\Delta d(z/\alpha(t)), \tag{4.22}$$

$$\frac{\partial d}{\partial t}(z,t) = \alpha'(t)d(z/\alpha(t)) - \frac{\alpha'(t)}{\alpha(t)}\langle z, \nabla d(z/\alpha(t))\rangle, \tag{4.23}$$

where $'$ denotes differentiation with respect to t. Since $\partial f(t) = \{z \in \mathbb{R}^n : d(z,t) = 0\} = \alpha(t)\partial E = \{z \in \mathbb{R}^n : d(z/\alpha(t)) = 0\}$, from (4.23) we deduce

$$\frac{\partial d}{\partial t}(x,t) = -\frac{\alpha'(t)}{\alpha(t)}\langle x, \nabla d(x/\alpha(t))\rangle, \qquad x \in \partial f(t). \tag{4.24}$$

Using (4.22) and (4.24), equation (4.13) (with $g \equiv 0$) expressing mean curvature flow of $f(t)$ becomes an equation for the function d on ∂E which reads as

$$-\alpha'(t)\langle x/\alpha(t), \nabla d(x/\alpha(t))\rangle = \frac{1}{\alpha(t)}\Delta d(x/\alpha(t)), \qquad x/\alpha(t) \in \partial E,$$

i.e.,

$$\Delta d(x) = -\alpha'(t)\alpha(t)\langle x, \nabla d(x)\rangle, \qquad x \in \partial E.$$

Since the left hand side does not depend on t, we deduce that

$$\alpha'(t)\alpha(t) \equiv \alpha \in \mathbb{R}, \qquad t \in I.$$

We now distinguish the three cases $\alpha < 0$, $\alpha = 0$ and $\alpha > 0$. If $\alpha < 0$, writing $\alpha = -\frac{1}{2T}$ for $T > 0$, we have $\alpha(t) = \sqrt{-\alpha}\sqrt{2(t_0 - t)}$ for any $t \in I = (-\infty, t_0)$. If $\alpha = 0$ then (ii) immediately follows. If $\alpha > 0$ we have $\alpha(t) = \sqrt{\alpha}\sqrt{2(t - t_0)}$ for any $t \in I = (t_0, +\infty)$.

Conversely, let $E \subset \mathbb{R}^n$ be such that $\partial E \in \mathcal{C}^\infty$ and (4.19) holds for some $T > 0$. Repeating the previous computations in reverse order, one checks that the map f in (**??**) is a smooth mean curvature flow on I. Similar reasonings apply in cases (ii) and (iiii). □

In case (i) we say that f is a self-similar contracting mean curvature flow, and in case (iii) we say that f is a self-similar expanding mean curvature flow.

Remark 4.2.7. In view of Example 3.2.8, equation (4.19) expresses the stationarity condition of ∂E for the functional in (3.28), and (4.21) expresses the stationarity condition of ∂E for the functional in (3.29).

Another class of solutions is given by translatory solutions. We say that $f : \mathbb{R} \to \mathcal{P}(\mathbb{R}^n)$ is a translatory evolution if there exist $E \subset \mathbb{R}^n$ with $\partial E \in C^\infty$ and $v \in \mathbb{R}^n$ such that

$$f(t) = E + tv, \qquad t \in \mathbb{R}.$$

In this case we have

$$d(z,t) = d(z - tv, E),$$

so that f is a translatory smooth mean curvature flow if and only if

$$\Delta d(x) = -\langle v, \nabla d(x)\rangle, \qquad x \in \partial E. \tag{4.25}$$

Note that (4.25) expresses the stationarity condition of ∂E for the functional

$$\int_{\partial E} e^{\langle v, \mathrm{n}\rangle t}\, d\mathcal{H}^{n-1}.$$

Example 4.2.8. Let u, A, f and v be as in Remark 4.0.18. Then (4.11) reads as

$$\frac{\frac{\partial u}{\partial t}}{|\nabla u|}\frac{\nabla u}{|\nabla u|} = \mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right)\frac{\nabla u}{|\nabla u|} \qquad \text{on } \{u = 0\}, \tag{4.26}$$

which is invariant under the transformation $u \to \lambda u$, where $\lambda \in \mathbb{R} \setminus \{0\}$. Note that if u is a solution to (4.26) which is smooth in a space time region around one of its level sets $\{u(\cdot, t) = \lambda\}$, then this level set flows smoothly by mean curvature. Equation (4.26) can be rewritten in the scalar form as

$$|\nabla u|^2\left(\frac{\partial u}{\partial t} - \Delta u\right) = -\nabla_i u \nabla_j u \nabla^2_{ij} u \qquad \text{on } \{u = 0\}. \tag{4.27}$$

If $|\nabla u|^2 = 1$ in a neighbourhood of $\{u = 0\}$ then problem (4.27) reduces to (4.14), i.e.,

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u, \\ u = 0. \end{cases} \tag{4.28}$$

Moreover, at the points of the graph of v we have that the mean curvature vector equals

$$-\mathrm{div}\left(\frac{\nabla v}{\sqrt{1+|\nabla v|^2}}\right)\frac{(\nabla v,-1)}{\sqrt{1+|\nabla v|^2}}. \tag{4.29}$$

The smooth mean curvature flow of the graph of v is therefore expressed using the equation

$$\frac{\partial v}{\partial t}=\sqrt{1+|\nabla v|^2}\,\mathrm{div}\left(\frac{\nabla v}{\sqrt{1+|\nabla v|^2}}\right) \tag{4.30}$$

in $\mathbb{R}^{n-1}\times[a,b]$. ***dire qualcosa sulla velocita' verticale? e sulla equazione senza la radice? ***

Observe that

- if $\pi_n(z) := z^n$ then, recalling also (2.14), we have $\frac{\partial}{\partial t}\overline{\pi_n} = \Delta\overline{\pi_n}$ on graph(v)
- the velocity of the flow in the direction e_n is given by $\frac{\partial v}{\partial t}$
- if $n=2$, equation (4.30) takes the form

$$\frac{\partial v}{\partial t}=f(v_x)_x,$$

where $f(p)=\mathrm{arctg}(p)$ for any $p\in\mathbb{R}$.

Example 4.2.9. If we look for special solutions to (4.30) of the form $v(s,t)=h(s)+t$, for some smooth real valued function h, we have to impose

$$\sqrt{1+|\nabla h|^2}\,\mathrm{div}\left(\frac{\nabla h}{\sqrt{1+|\nabla h|^2}}\right)=1. \tag{4.31}$$

If we assume $n=2$ then (4.31) reduces to the following ordinary differential equation:

$$\frac{h''}{1+h'^2}=(\mathrm{arctg}(h'))'=1. \tag{4.32}$$

A solution to (4.32) is given by $h(s)=-\log(\cos s)$ for $s\in(-\frac{\pi}{2},\frac{\pi}{2})$. The corresponding solution $v(s,t)=-\log(\cos s)+t$, for $s\in(-\frac{\pi}{2},\frac{\pi}{2})$ and $t\in[0,+\infty)$ (called grim reaper) is said to be a translating solution to curvature flow.

Example 4.2.10. Let $v\in C^\infty(\mathbb{R},(0,+\infty))$, and let E be as in Example 2.2.11. It follows that a smooth mean curvature flow starting from E is given by $f:[0,T]\to\mathcal{P}(\mathbb{R}^3)$, $f(t)=\{(z_1,z_2,z_3)\in\mathbb{R}^3 : (v(z_1,t))^2\le z_2^2+z_3^2\}$, for some $T>0$, and $w\in C^\infty(\mathbb{R}\times[0,T],(0,+\infty))$ is a solution to

$$\frac{\partial w}{\partial t}=\frac{w''}{1+(w')^2}-\frac{1}{w},\qquad w(\cdot,0)=v(\cdot). \tag{4.33}$$

Indeed, to obtain (4.33) it is enough to use (2.29) and use the equality $\frac{\partial h}{\partial t}/|\nabla h| = w\frac{\partial w}{\partial t}/(w^2(w')^2+z_2^2+z_3^2)^{1/2}$, where $h(z,t):=\frac{1}{2}((w(z,t))^2-z_2^2-z_3^2)$.

Note that the perimeter of E in $(a,b)\times\mathbb{R}^2$ is given by $\mathcal{F}(v) = 2\pi \int_{(a,b)} v\sqrt{1+(v')^2}\, dz_1$; the first variation of $\mathcal{F}$ is given by $\frac{d}{d\lambda}\mathcal{F}(v+\lambda\varphi)_{|\lambda=0} = 2\pi \int_{(a,b)} \varphi\left((1+(v')^2)^{1/2}) - v(\frac{v'}{(1+(v')^2)^{1/2}})'\right)\, dz_1$.
****

Example 4.2.11. Let $\lambda > 0$. Consider in $\mathbb{R}^2$ a disk of radius $\rho_\lambda(t)$ which evolves according to (4.13), with $\rho_\lambda(0) = \lambda$. Then

$$\begin{cases} \rho_\lambda'(t) = -\dfrac{1}{\rho_\lambda(t)} + 1, & t \in (0, t^\lambda), \\ \rho_\lambda(0) = \lambda, \end{cases}$$

where t^λ denotes the extinction time. One can verify that if $0 < \lambda < 1$ then $t^\lambda \in (0,+\infty)$, and ρ_λ is a nonnegative concave strictly decreasing function on $[0, t^\lambda]$ such that $\rho_\lambda(t^\lambda) = 0$. If $\lambda = 1$ then $\rho_\lambda \equiv 1$, so that there is no extinction time (so that $t^\lambda = +\infty$) and if $\lambda > 1$ then ρ_λ is a positive convex strictly increasing function on $[0,+\infty)$ such that $\lim_{t\to+\infty} \rho_\lambda'(t) = 1$ (and again $t^\lambda = +\infty$). Note that

$$\rho_\lambda(t) + \log(|\lambda - \rho_\lambda(t)|) = \lambda + \log(|1-\lambda|) + t.$$

Remark 4.2.12. Let $f : [a,b] \to \mathcal{P}(\mathbb{R}^n)$, $f \in \mathcal{KF}$. Then

- as a consequence of (3.7) and (4.11) we have

$$\frac{d}{dt}|f(t)| = -\int_{\partial f(t)} \Delta d(\cdot,t)\, d\mathcal{H}^{n-1}. \tag{4.34}$$

  Hence if $n = 2$ then $\frac{d}{dt}|f(t)| = -2\pi$.
- As a consequence of (3.23), (4.11) and (4.0.12), we have

$$\frac{d}{dt}\mathcal{H}^{n-1}(\partial f(t)) = -\int_{\partial f(t)} (\Delta d(\cdot,t))^2\, d\mathcal{H}^{n-1}, \qquad t \in [a,b], \tag{4.35}$$

  which shows how the perimeter of $\partial E(t)$ is decreasing along a smooth compact mean curvature flow.

Definition 4.2.13. *Let $f : [a,b] \to \mathcal{P}(\mathbb{R}^n)$ be a smooth flow, let d_f and A be as in Definition 4.0.10. Let $v \in \mathcal{C}^\infty(A \times [a,b])$. We define*

$$\frac{dv}{dt}(z,t) := -\langle \frac{\partial d}{\partial t}(z,t)\nabla d(z,t), \nabla v(z,t)\rangle + \frac{\partial v}{\partial t}(z,t), \qquad (z,t) \in A \times [a,b]. \tag{4.36}$$

Note that if $f \in \mathcal{KF}$ then

$$\frac{dv}{dt}(z,t) := -\Delta d(z,t)\langle \nabla d(z,t), \nabla v(z,t)\rangle + \frac{\partial v}{\partial t}(z,t), \qquad (z,t) \in A \times [a,b].$$

4.2.1. Extensions. Similarly to Section 2.1.1, we now the define various tangential operators that will be used in the sequel.

Let $f : [a, b] \to \mathcal{P}(\mathbb{R}^n)$ be a smooth flow, let d_f and A be as in Definition 4.0.10, set $\Sigma(t) = \partial f(t)$ and define

$$\Xi := \bigcup_{t\in[a,b]} (\Sigma(t) \times \{t\}).$$

Let $u \in \mathcal{C}^\infty(\Xi)$ and let u^e be any smooth extension of u in a neighbourhood U of Ξ. Given $t \in [a, b]$ and $x \in \Sigma(t)$, the tangential gradient $\nabla^{\Sigma(t)}u(x, t)$ of $u(\cdot, t)$ on $\Sigma(t)$, evaluated at (x, t), is the orthogonal projection on $\Sigma(t)$ of $\nabla u^e(x, t)$. The tangential gradient of $u(\cdot, t)$ on $\Sigma(t)$ depends only on the values of u on Ξ. We define $\overline{u} \in \mathcal{C}^\infty(U)$ as

$$\overline{u}(z, t) = u(\mathrm{pr}_{\Sigma(t)}(z), t) = u(z - d(z, t)\nabla d(z, t), t) \tag{4.37}$$

Observe that

$$\nabla^{\Sigma(t)}u(x, t) = \nabla\overline{u}(x, t), \qquad t \in [a, b], x \in \partial f(t). \tag{4.38}$$

Remark 4.2.14. Let $f \in \mathcal{KF}$ and $h \in \mathcal{C}^\infty(\Xi)$. Then

$$\frac{\partial\overline{h}}{\partial t}(z, t) = \frac{d\overline{h}}{dt}(z, t), \qquad z \in A, t \in [a, b].$$

Indeed, from Definition 4.2.13 we have for $(z, t) \in A \times [a, b]$

$$\frac{d\overline{h}}{dt}(z, t) = -\langle\frac{\partial d}{\partial t}(z, t)\nabla d(z, t), \nabla\overline{h}(z, t)\rangle + \frac{\partial\overline{h}}{\partial t}(z, t) = \frac{\partial\overline{h}}{\partial t}(z, t)$$

Lemma 4.2.15. Let $f \in \mathcal{KF}$, $h \in \mathcal{C}^\infty(\Xi)$ and define $\mathrm{h} \in \mathcal{C}^\infty(\mathcal{S} \times [a, b])$ as

$$\mathrm{h}(s, t) := h(\varphi(s, t), t), \qquad (s, t) \in \mathcal{S} \times [a, b]. \tag{4.39}$$

Then for any $t \in [a, b]$ we have

$$\mathrm{h}(s, t) = \overline{h}(z, t), \qquad z \in A,\ \mathrm{pr}_{\Sigma(t)}(z) = \varphi(s, t), \tag{4.40}$$

$$\frac{\partial\mathrm{h}}{\partial t}(s, t) = \frac{\partial\overline{h}}{\partial t}(x, t), \qquad x = \varphi(s, t) \in \Sigma(t). \tag{4.41}$$

Proof. Equation (4.40) follows from (4.39) and (4.37). For any λ with $|\lambda|$ sufficiently small we have

$$\mathrm{h}(s, t) = \overline{h}\big(\varphi(s, t) + \lambda\nabla d(\varphi(s, t), t), t\big).$$

Hence

$$\frac{\partial\mathrm{h}}{\partial t}(s, t) = \langle\nabla\overline{h}(x + \lambda\nabla d(x, t), t), \frac{\partial\varphi}{\partial t}(s, t) + \lambda\frac{\partial}{\partial t}(\nabla d(\varphi(s, t), t))\rangle + \frac{\partial\overline{h}}{\partial t}(x + \lambda\nabla d(x, t), t).$$

Setting $\lambda = 0$ we have

$$\frac{\partial\mathrm{h}}{\partial t}(s, t) = \langle\nabla\overline{h}(x, t), \frac{\partial\varphi}{\partial t}(s, t)\rangle + \frac{\partial\overline{h}}{\partial t}(x, t) = \frac{\partial\overline{h}}{\partial t}(x, t)$$

since $\nabla\overline{h}(x, t) \in T_x(\Sigma(t))$ while $\frac{\partial\varphi}{\partial t}(s, t) \in N_x(\Sigma(t))$. □

If $X \in \mathcal{C}^\infty(\cup_{t\in[a,b]}(\Sigma(t)\times\{t\};\mathbb{R}^n))$, we define $\overline{X} : A \times [a,b] \to \mathbb{R}^n$ as

$$\overline{X}(z,t) := X(\mathrm{pr}_{\Sigma(t)}(z),t).$$

Given $t \in [a,b]$, the tangential divergence $\mathrm{div}_{\Sigma(t)}X$ is the trace of the orthogonal projection on $\Sigma(t)$ of the space gradient of any smooth extension of X in a neighbourhood of $\Sigma(t)$. Observe that

$$\mathrm{div}_{\Sigma(t)}X = \mathrm{div}\overline{X} \qquad \text{on } \Sigma(t).$$

We denote by $\Delta_{\Sigma(t)}u$ the tangential laplacian of u on $\Sigma(t)$, defined as $\Delta_{\Sigma(t)}u := \mathrm{div}_{\Sigma(t)}(\nabla^{\Sigma(t)}u)$. Recall that

$$\Delta_{\Sigma(t)}u = \Delta\overline{u} \qquad \text{on } \Sigma(t). \tag{4.42}$$

Setting $\mathrm{n} = \mathrm{n}^{f(t)}$ and $H = H^{\Sigma(t)}$, recall that $\overline{n} = \nabla d_f$ in $A\times[a,b]$, and that

$$\overline{H} = \mathrm{tr}(\nabla^2 d_f G), \qquad G := (\mathrm{Id} - d_f\nabla^2 d_f)^{-1} \qquad \text{in } A\times[a,b]. \tag{4.43}$$

4.3. Huisken's monotonicity formula

In this section we prove Huisken's monotonicity formula, which describes how the perimeter of a smooth hypersurface flowing by mean curvature changes when weighted with a suitable backward heat kernel. We begin with the following observation.

Lemma 4.3.1. *Let $f : [a,b] \to \mathcal{P}(\mathbb{R}^n)$, $f \in \mathcal{KF}$. Let $\psi \in C^\infty(\mathbb{R}^n \times [a,b])$. Then*

$$\frac{d}{dt}\int_{\partial f(t)} \psi \, d\mathcal{H}^{n-1} = \int_{\partial f(t)} \left(-\psi\,(\Delta d)^2 - \langle\nabla\psi,\nabla d\rangle\Delta d + \frac{\partial\psi}{\partial t}\right)\, d\mathcal{H}^{n-1}. \tag{4.44}$$

Proof. It follows from (3.27) with the choice $a = \psi$, and recalling that $V := -\Delta d\nabla d$ is the velocity field of $\partial f(t)$. □

Note that, using (3.2.3), and assuming $\psi > 0$, from (4.44) we deduce

$$\begin{aligned}\frac{d}{dt}\int_{\partial f(t)} \psi \, d\mathcal{H}^{n-1} = &-\int_{\partial f(t)} \psi\left(H + \frac{1}{\psi}\langle\nabla\psi,\mathrm{n}\rangle\right)^2 \, d\mathcal{H}^{n-1} \\ &+ \int_{\partial f(t)}\left(\frac{1}{\psi}\langle\nabla\psi,\mathrm{n}\rangle^2 + \frac{\partial\psi}{\partial t} + \mathrm{div}_{\Sigma(t)}\nabla\psi\right)\, d\mathcal{H}^{n-1}.\end{aligned} \tag{4.45}$$

In the particular case $\psi \equiv 1$, (4.45) coincides with (4.35).

Theorem 4.3.2. *Let $z_0 \in \mathbb{R}^n$, $t_0 \in [a,b]$ and set*

$$\rho(z,t) = \rho_{(z_0,t_0)}(z,t) := \frac{e^{-\frac{|z-z_0|^2}{4(t_0-t)}}}{(4\pi(t_0-t))^{\frac{n-1}{2}}}, \qquad z \in \mathbb{R}^n,\ t < t_0. \tag{4.46}$$

Then

$$\frac{d}{dt}\int_{\partial f(t)} \rho \, d\mathcal{H}^{n-1} = -\int_{\partial f(t)} \rho\left(H + \frac{1}{\rho}\langle\nabla\rho,\mathrm{n}\rangle\right)^2 \, d\mathcal{H}^{n-1} \le 0. \tag{4.47}$$