REMARK 5.25. For curves in the plane, possibly with self–intersections, such that the initial curvature is never zero, this result was obtained via a different method by Angenent [**11**] (see also [**4**]), studying directly the parabolic equation satisfied by the curvature function.

In view of these results and the discussion about the classification of translating solution in Section 1, the strongest conjecture in this context is that all the blow up limits via the Hamilton's *modified* procedure at a type II singularity of the evolution of an embedded hypersurface with $\mathrm{H} \geq 0$ is the only rotationally symmetric, strictly convex, translating solution.
In [**93**], White was able to exclude the possibility to get as a blow up limit the product of a grim reaper with $\mathbb{R}^{n-1}$.

More in general, also without assuming the condition $\mathrm{H} > 0$, one can conjecture that blow up limits like the minimal catenoid surface M in $\mathbb{R}^3$ given by

$$\Omega = \big\{(x, y) \in \mathbb{R}^2 \times \mathbb{R} \mid \cosh |y| = |x|\big\}$$

or products of lower dimensional convex translating solutions with some $\mathbb{R}^k$, cannot happen.
This seems to be supported by the recent paper by Ecker [**27**].

5. The Special Case of Embedded Closed Curves in the Plane

In the special case of the evolution of an embedded closed curve in the plane, it is possible to exclude at all type II singularities. This, together with the case of convex, compact, hypersurfaces (as we have seen in the proof of Theorem 4.39 and 4.40) are the only cases in which this can be done.

By the previous section and embeddedness, any blow up limit must be translating and with unit multiplicity, that is, a grim reaper. We apply now a very geometric argument by Huisken in [**56**] in order to exclude also such possibility (see also [**51**] for another similar quantity).

Given the smooth flow γ_t of an initial embedded closed curve γ_0 on some interval $[0, T)$, we know that the curve stay embedded during the flow, so we can refer to every curve γ_t as a subset of $\mathbb{R}^2$. At every time $t \in [0, T)$, for every pair of points p and q in γ_t we define $d_t(p, q)$ to be the *geodesic* distance in γ_t of p and q, $|p - q|$ the standard distance in $\mathbb{R}^2$ and L_t the length of γ_t.
We consider the function $\Phi_t : \gamma_t \times \gamma_t \to \mathbb{R}$ defined as

$$\Phi_t(p, q) = \begin{cases} \frac{\pi|p-q|}{L_t} / \sin \frac{\pi d_t(p,q)}{L_t} & \text{if } p \neq q, \\ 1 & \text{if } p = q\,, \end{cases}$$

which is a perturbation of the quotient between the extrinsic and the intrinsic distance of a pair of points on γ_t.
Since γ_t is smooth and embedded for every time, the function Φ_t is well defined and positive. Moreover, it is easy to check that even if d_t is not C^1 at the pairs of points such

that $d_t(p,q) = L_t/2$, the function Φ_t is C^1 in the open set $\{p \neq q\} \subset \gamma_t \times \gamma_t$ and continuous on $\gamma_t \times \gamma_t$.
By compactness, for every $t \in [0,T)$, the following infimum is actually a minimum in this case,

$$E(t) = \inf_{p,q \in \gamma_t} \Phi_t(p,q)\,. \tag{5.6}$$

As the curve γ_t has no self–intersections we have $0 < E(t) \leq 1$, the converse is clearly also true. Finally, since the evolution is smooth it is easy to see that the function $E : [0,T) \to \mathbb{R}$ is continuous.

LEMMA 5.26 (Huisken [**56**]). *The function $E(t)$ is monotone increasing in every interval where $E(t) < 1$.*

PROOF. We start differentiating in time $\Phi_t(p,q)$,

$$\begin{aligned}
\frac{d}{dt}\Phi_t(p,q) =& \frac{\pi}{L_t}\frac{\langle p-q\,|\,k(p)\nu(p)-k(q)\nu(q)\rangle}{|p-q|}\Big/\sin\frac{\pi d_t(p,q)}{L_t}\\
&+\left(\frac{\pi|p-q|}{L_t^2}\int_{\gamma_t}k^2\,ds\right)\Big/\sin\frac{\pi d_t(p,q)}{L_t}\\
&-\frac{\pi^2|p-q|}{L_t^2}\cos\frac{\pi d_t(p,q)}{L_t}\left(\frac{d_t(p,q)}{L_t}\int_{\gamma_t}k^2\,ds-\int_q^p k^2\,ds\right)\Big/\sin^2\frac{\pi d_t(p,q)}{L_t}\\
=&\left[\frac{\langle p-q\,|\,k(p)\nu(p)-k(q)\nu(q)\rangle}{|p-q|^2}+\frac{1}{L_t}\int_{\gamma_t}k^2\,ds\right.\\
&\left.-\frac{\pi}{L_t}\cot\frac{\pi d_t(p,q)}{L_t}\left(\frac{d_t(p,q)}{L_t}\int_{\gamma_t}k^2\,ds-\int_q^p k^2\,ds\right)\right]\Phi_t(p,q)\\
=&\left[\frac{\langle p-q\,|\,k(p)\nu(p)-k(q)\nu(q)\rangle}{|p-q|^2}+\frac{1}{L_t}\left(1-\frac{\pi d_t(p,q)}{L_t}\cot\frac{\pi d_t(p,q)}{L_t}\right)\int_{\gamma_t}k^2\,ds\right.\\
&\left.+\frac{\pi}{L_t}\cot\frac{\pi d_t(p,q)}{L_t}\int_q^p k^2\,ds\right]\Phi_t(p,q)
\end{aligned}$$

where s is the arclength and k the curvature of γ_t. It is then easy to see that being the function E the infimum of a family of locally uniformly Lipschitz functions, it is also locally Lipschitz, hence differentiable almost everywhere. Then, to prove the statement it is enough to show that $\frac{dE(t)}{dt} > 0$ for every time t such that this derivative exists. We will do that as usual, by Hamilton's trick, Lemma 3.3.
Let (p,q) a minimizing pair at a differentiability time t and suppose that $E(t) < 1$. By the very definition of Φ_t, it must be $p \neq q$.
We set $\alpha = \pi d_t(p,q)/L_t$ and notice that $\alpha\cot\alpha < 1$ as $\alpha \in (0,\pi/2]$. Moreover, $\int_{\gamma_t}k^2\,ds \geq \left(\int_{\gamma_t}k\,ds\right)^2/L_t \geq 4\pi^2/L_t$. Then, we have

$$\frac{d}{dt}E(t) \geq \left[\frac{\langle p-q\,|\,k(p)\nu(p)-k(q)\nu(q)\rangle}{|p-q|^2}+\frac{4\pi^2}{L_t^2}(1-\alpha\cot\alpha)+\frac{\pi}{L_t}\cot\alpha\int_q^p k^2\,ds\right]E(t)$$

that is,

$$\frac{d}{dt}\log E(t) \geq \frac{\langle p-q \,|\, k(p)\nu(p) - k(q)\nu(q)\rangle}{|p-q|^2} + \frac{4\pi^2}{L_t^2}(1-\alpha\cot\alpha) + \frac{\pi}{L_t}\cot\alpha\int_q^p k^2\,ds\,, \quad (5.7)$$

at any minimizing pair (p, q).
Assume that the curve is parametrized counterclockwise in arclength, that $d_t(p, q) < L_t/2$ and that the geodesic connecting p and q is the counterclockwise oriented part of the curve from q to p, like in the figure.

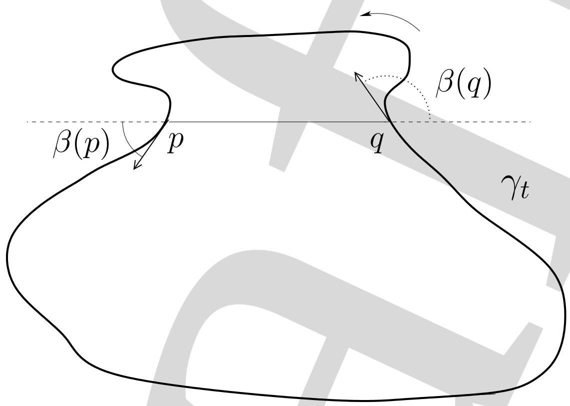

FIGURE 1.

We set $p(s) = \gamma_t(s_0 + s)$ with $p = \gamma_t(s_0)$, then, by minimality we have

$$0 = \frac{d}{ds}\Phi_t(p(s), q)\bigg|_{s=0} = \frac{\pi}{L_t}\frac{\langle p-q \,|\, \tau(p)\rangle}{|p-q|}\Big/\sin\frac{\pi d_t(p,q)}{L_t} - \frac{\pi|p-q|}{L_t\sin^2\frac{\pi d_t(p,q)}{L_t}}\frac{\pi\cos\frac{\pi d_t(p,q)}{L_t}}{L_t}$$

where we denoted with $\tau(p)$ the oriented unit tangent vector to γ_t at p.
By this last equality we get

$$\cos\beta(p) = \frac{\langle p-q \,|\, \tau(p)\rangle}{|p-q|} = \frac{\pi|p-q|}{L_t\sin\frac{\pi d_t(p,q)}{L_t}}\cos\frac{\pi d_t(p,q)}{L_t} = E(t)\cos\alpha$$

where $\beta(p)$ is the angle between the vectors $p - q$ and $\tau(p)$.
Repeating this argument for the other point q we get

$$\cos\beta(q) = -E(t)\cos\alpha$$

where, as before, $\beta(q)$ is the angle between $q - p$ and $\tau(q)$, see Figure 1. Clearly, $\beta(q) = \pi - \beta(p)$.
Notice that if one of these intersection is tangential, we would have $E(t)\cos\alpha = 1$ which is impossible as we assumed that $E(t) < 1$. Moreover, by the relation $\cos\beta(p) = E(t)\cos\alpha < \cos\alpha$ it follows that $\beta > \alpha$.

We look now at the second variation of Φ_t, at the same minimizing pair of points (p, q). With the same notation, if $p = \gamma_t(s_1)$ and $q = \gamma_t(s_2)$ we set $p(s) = \gamma_t(s_1 + s)$ and $q(s) = \gamma_s(s_2 - s)$. After a straightforward computation one gets,

$$0 \leq \frac{d^2}{ds^2}\Phi_t(p(s), q(s))\bigg|_{s=0} = \frac{\pi}{L_t}\left(\frac{\langle p - q \,|\, k(p)\nu(p) - k(q)\nu(q)\rangle}{|p - q|} + \frac{4\pi^2|p - q|}{L_t^2}\right) / \sin\frac{\pi d_t(p, q)}{L_t}$$
$$= \left[\frac{\langle p - q \,|\, k(p)\nu(p) - k(q)\nu(q)\rangle}{|p - q|^2} + \frac{4\pi^2}{L_t^2}\right] E(t)\,.$$

Hence, getting back to inequality (5.7), we have

$$\frac{d}{dt}\log E(t) \geq \frac{\langle p - q \,|\, k(p)\nu(p) - k(q)\nu(q)\rangle}{|p - q|^2} + \frac{4\pi^2}{L_t^2}(1 - \alpha\cot\alpha) + \frac{\pi}{L_t}\cot\alpha\int_q^p k^2\,ds$$
$$\geq -\frac{4\pi^2}{L_t^2}\alpha\cot\alpha + \frac{\pi}{L_t}\cot\alpha\int_q^p k^2\,ds$$
$$= \frac{\pi\cot\alpha}{L_t}\left(\int_q^p k^2\,ds - \frac{4\pi}{L_t}\alpha\right),$$

so it remains to show that this last expression is positive. As

$$\int_p^q k^2\,ds \geq \left(\int_p^q k\,ds\right)^2 / d_t(p, q)$$

and noticing that $\int_p^q k\,ds$ is the angle between the tangent vectors $\tau(p)$ and $\tau(q)$, we have $\left(\int_p^q k\,ds\right)^2 = 4\beta(p)^2 > 4\alpha^2$, as we concluded above.
Thus,

$$\frac{d}{dt}\log E(t) \geq \frac{\pi\cot\alpha}{L_t}\left(\int_q^p k^2\,ds - \frac{4\pi}{L_t}\alpha\right)$$
$$> \frac{\pi\cot\alpha}{L_t}\left(\frac{4\alpha^2}{d_t(p, q)} - \frac{4\pi}{L_t}\alpha\right)$$
$$= 0$$

recalling that $\alpha = \pi d_t(p, q)/L_t$. □

REMARK 5.27. Clearly, by its definition and this lemma, the function E is always nondecreasing. Actually, to be more precise, by means of a simple geometric argument it can be proved that if $E(t) = 1$ the curve must be a circle. Hence, in any other case E is strictly increasing in time.

REMARK 5.28. This lemma clearly implies that an initial embedded closed curve cannot develop a self–intersection during mean curvature flow, otherwise E would get zero, which is impossible as $E(0) > 0$ and E is nondecreasing.

An immediate consequence of this lemma is that for every initial embedded, closed curve in $\mathbb{R}^2$, there exists a positive constant C depending on the initial curve such that on all $[0, T)$ we have $E(t) \geq C$. The same conclusion holds for any rescaling of such curves as the function E is scaling invariant by construction.

REMARK 5.29. This lemma also provide an alternative proof of the fact that an initial embedded, closed curve stays embedded, that is, it cannot develop a self–intersection during mean curvature flow, otherwise E would get zero.

We can then exclude *Type II* singularities, indeed, as γ^∞ is a grim reaper and it is the limit of rescalings of curves of the family γ_t, the function E for such grim reaper (which is constant in time, since it moves by translation) is not smaller, at any time, than the infimum of the corresponding functions for the approximating curves, hence, by the discussion above, following the lemma, it is bounded below by some positive constant C.
But, if we consider a pair of points p, q on any grim reaper Γ_t such that the segment $[p, q]$ is orthogonal to the velocity vector $w \in \mathbb{R}^2$ and we send such segment infinity, we can see that $\Phi_t(p, q) \to 0$, hence $E(\Gamma_t) = 0$, indeed, the distance $|p - q|$ is bounded by a constant (the width of the strip where the grim reaper lives) and the intrinsic distance $d_t(p, q)$ diverges.
This is in contradiction with the above conclusion.

PROPOSITION 5.30. Type II *singularities cannot develop during the mean curvature flow of an embedded, closed curve in* $\mathbb{R}^2$.

Collecting together the results of Chapter 4 about type I singularities and this last proposition, we obtain the following Theorem due to Grayson **[41]**, whose original proof is more geometric and direct, showing that the intervals of negative curvature vanish in finite time, before any singularity.

THEOREM 5.31. *Let γ_0 be a closed, smooth embedded curve in the plane and let γ_t, for $t \in [0, T)$ be its maximal evolution by mean curvature. There exists a time $\widehat{t} < T$ such that $\gamma_{\widehat{t}}$ is convex.*
As a consequence, the result of Gage and Hamilton 4.39 applies and subsequently the curve shrinks smoothly to a point $t \to T$.

PROOF. As we said no type II singularities are possible and the only type I singularities have a circle as limit of rescalings, see Section 5.
Hence, at some point the curve must have become convex. $\square$

We add a final remark in this case of embedded curves.
Letting $A(t)$ to be the area enclosed by γ_t which moves by mean curvature, we have

$$\frac{d}{dt} A(t) = -\int_{\gamma_t} k \, ds = -2\pi \, ,$$

hence, as the evolution is smooth till the curve shrinks to a point at time $T > 0$ and clearly $A(t)$ goes to zero, we have $A(0) = 2\pi T$. That is, the existence time is exactly equal to the initial enclosed area divided by 2π.