CHAPTER 2

Definition of the Flow and Small Time Existence

1. Notations and Preliminaries

We devote this section to introduce the basic notations and facts about Riemannian manifolds and their submanifolds we need in the paper, a good reference is **[39]**.

The main objects we will consider are n–dimensional connected and complete hypersurfaces immersed in $\mathbb{R}^{n+1}$, that is, pairs (M, φ) where M is an n–dimensional smooth manifold, connected with empty boundary, and $\varphi : M \to \mathbb{R}^{n+1}$ is a smooth immersion, that is, the rank of $d\varphi$ is everywhere equal to n.

The manifold M gets in a natural way a metric tensor g turning it in a Riemannian manifold (M, g), by pulling back the standard scalar product of $\mathbb{R}^{n+1}$ with the immersion map φ.

Taking local coordinates around $p \in M$, we have local bases of T_pM and T_p^*M, respectively given by vectors $\left\{\frac{\partial}{\partial x_i}\right\}$ and covectors $\{dx_j\}$.

We will denote vectors on M by $X = X^i$, which means $X = X^i \frac{\partial}{\partial x_i}$, covectors by $Y = Y_j$, that is, $Y = Y_j dx_j$ and a general mixed tensor with $T = T^{i_1 \dots i_k}_{j_1 \dots j_l}$, where the indices refer to the local basis.

Often, we will consider tensors along M, viewing it as a submanifold of $\mathbb{R}^{n+1}$ via the map φ, in that case we will use the Greek indices to denote the components of such tensors in the canonical basis $\{e_\alpha\}$ of $\mathbb{R}^{n+1}$, for instance, given a vector field X along M, not necessarily tangent, we will have $X = X^\alpha e_\alpha$.

In all the paper the convention to sum over repeated indices will be adopted.

The inner product on M, extended to tensors, is given by

$$g(T, S) = g_{i_1 s_1} \dots g_{i_k s_k} g^{j_1 z_1} \dots g^{j_l z_l} T^{i_1 \dots i_k}_{j_1 \dots j_l} S^{s_1 \dots s_k}_{z_1 \dots z_l}$$

where g_{ij} is the matrix of the coefficients of the metric tensor in local coordinates and g^{ij} is its inverse. Clearly, the norm of a tensor is

$$|T| = \sqrt{g(T, T)}\,.$$

The scalar product in $\mathbb{R}^{n+1}$ will be denoted with $\langle \cdot \,|\, \cdot \rangle$. As the metric g is obtained pulling it back with φ, we have

$$g_{ij} = g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = (d\varphi^* \langle \cdot \,|\, \cdot \rangle)\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = \left\langle \frac{\partial \varphi}{\partial x_i} \,\middle|\, \frac{\partial \varphi}{\partial x_j} \right\rangle .$$

The canonical measure induced by the metric g is given in a coordinate chart by $\mu = \sqrt{G}\,\mathcal{L}^n$ where $G = \det(g_{ij})$ and $\mathcal{L}^n$ is the standard Lebesgue measure on $\mathbb{R}^n$.

The induced covariant derivative on (M, g) of a vector field X and of a 1–form ω are given by

$$\nabla_j X^i = \frac{\partial X^i}{\partial x_j} + \Gamma^i_{jk} X^k\,, \qquad \nabla_j \omega_i = \frac{\partial \omega_i}{\partial x_j} - \Gamma^k_{ji}\omega_k\,,$$

where the Christoffel symbols $\Gamma = \Gamma^i_{jk}$ are expressed by the following formula,

$$\Gamma^i_{jk} = \frac{1}{2} g^{il}\left(\frac{\partial}{\partial x_j} g_{kl} + \frac{\partial}{\partial x_k} g_{jl} - \frac{\partial}{\partial x_l} g_{jk}\right).$$

For a general tensor the covariant derivative is defined by means of Leibniz rule.

In all the paper the covariant derivative ∇T of a general tensor $T = T^{i_1\dots i_k}_{j_1\dots j_l}$ will be denoted by $\nabla_s T^{i_1\dots i_k}_{j_1\dots j_l} = (\nabla T)^{i_1\dots i_k}_{s j_1\dots j_l}$ (we recall that such covariant derivative is defined uniquely on the tensor algebra by imposing Leibniz rule and commutativity with contractions).
With $\nabla^m T$ we will mean the m–th iterated covariant derivative of T.

We recall that the gradient ∇f of a function and the divergence $\operatorname{div} X$ of a vector field at a point $p \in M$ are defined respectively by

$$g(\nabla f(p), v) = df_p(v) \qquad \forall v \in T_p M$$

and

$$\operatorname{div} X = \operatorname{tr}\nabla X = \nabla_i X^i = \frac{\partial}{\partial x_i} X^i + \Gamma^i_{ik} X^k\,.$$

The (rough) Laplacian ΔT of a tensor T is

$$\Delta T = g^{ij}\nabla_i \nabla_j T\,.$$

If X is a smooth vector field with compact support on M, as $\partial M = \emptyset$, the usual *divergence theorem* holds

$$\int_M \operatorname{div} X\, d\mu = 0\,,$$

which clearly implies

$$\int_M \Delta f\, d\mu = 0$$

for every smooth function $f : M \to \mathbb{R}$ with compact support.

Since φ is locally an embedding in $\mathbb{R}^{n+1}$, at every point $p \in M$ we can define up to a sign a unit normal vector $\nu(p)$. Locally, we can always choose ν in order to be smooth, if M is orientable, this choice can be done globally.
If the hypersurface M is embedded, that is, the map φ is one–to–one, as M is compact, there will be an *inside* of the hypersurface, in this case we will consider ν to be the *inner pointing* unit normal vector at every point of M.

The *second fundamental form* $\mathrm{A} = h_{ij}$ of M is the symmetric 2–form defined as follows:

$$h_{ij} = \left\langle \nu \,\middle|\, \frac{\partial^2 \varphi}{\partial x_i \partial x_j} \right\rangle ,$$

the *mean curvature* H is the trace of A, that is $\mathrm{H} = g^{ij} h_{ij}$.

REMARK 2.1. Notice that since ν is defined up to a sign, the same is true for A. Instead, the *vector valued second fundamental form* $h_{ij}\nu$ which is a 2–form with values in $\mathbb{R}^{n+1}$ is uniquely defined.
With our choice of ν as the inner pointing unit normal, the sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ has a positive definite second fundamental form and positive mean curvature, the same holds for every strictly *convex* hypersurface in $\mathbb{R}^{n+1}$.

The linear map $W_p : T_pM \to T_pM$ given by $W_p(v) = h^i_j(p) v^j \frac{\partial}{\partial x_i}$ is called Weingarten operator and its eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$ the *principal curvatures* at the point $p \in M$. It is easy to see that $\mathrm{H} = \lambda_1 + \cdots + \lambda_n$ and $|\mathrm{A}|^2 = \lambda_1^2 + \cdots + \lambda_n^2$.

EXERCISE 2.2. Show that if the hypersurface $M \subset \mathbb{R}^{n+1}$ is defined locally as the graph of a function $f : \mathbb{R}^n \to \mathbb{R}$ we have, $\varphi(x) = (x, f(x))$

$$g_{ij} = \delta_{ij} + f_i f_j \,, \qquad \nu = -\frac{(\nabla f, -1)}{\sqrt{1 + |\nabla f|^2}}$$

$$h_{ij} = \frac{\mathrm{Hess}_{ij} f}{\sqrt{1 + |\nabla f|^2}}$$

$$\mathrm{H} = \frac{\Delta f}{\sqrt{1 + |\nabla f|^2}} - \frac{\mathrm{Hess} f(\nabla f, \nabla f)}{(\sqrt{1 + |\nabla f|^2})^3} = \mathrm{div}\left(\frac{\nabla f}{\sqrt{1 + |\nabla f|^2}} \right)$$

where $f_i = \partial_{x_i} f$ and $\mathrm{Hess} f$ is the Hessian of the function f.

EXERCISE 2.3. Show that if the hypersurface $M \subset \mathbb{R}^{n+1}$ is defined locally as the *zero set* of a smooth function $f : \mathbb{R}^{n+1} \to \mathbb{R}$, with $\nabla f \neq 0$, we have

$$\mathrm{H} = \frac{\Delta f}{|\nabla f|} - \frac{\mathrm{Hess} f(\nabla f, \nabla f)}{|\nabla f|^3} = \mathrm{div}\left(\frac{\nabla f}{|\nabla f|} \right) .$$

The following Gauss–Weingarten relations will be fundamental,

$$\frac{\partial^2 \varphi}{\partial x_i \partial x_j} = \Gamma^k_{ij} \frac{\partial \varphi}{\partial x_k} + h_{ij}\nu \,, \qquad \frac{\partial}{\partial x_j}\nu = -h_{jl} g^{ls} \frac{\partial \varphi}{\partial x_s} \,. \tag{2.1}$$

In other words, they express the fact that $\nabla^M = \nabla^{\mathbb{R}^{n+1}} - \mathrm{A}\nu$. We recall that, considering M locally as a regular submanifold of $\mathbb{R}^{n+1}$ we have $\nabla^M_X Y = (\nabla^{\mathbb{R}^{n+1}}_X \widetilde{Y})^M$ where ^{M} denotes the projection on the tangent space to M and $\widetilde{Y}$ is a local extension of the field Y in a local neighborhood $\Omega \subset \mathbb{R}^{n+1}$ of $\varphi(M)$.
Notice that, by these relations, it follows

$$\Delta \varphi = g^{ij} \nabla^2_{ij} \varphi = g^{ij} \left(\frac{\partial^2 \varphi}{\partial x_i \partial x_j} - \Gamma^k_{ij} \frac{\partial \varphi}{\partial x_k} \right) = g^{ij} h_{ij} \nu = \mathrm{H}\nu \,, \tag{2.2}$$

component by component.
By a straightforward computation, we can see that the Riemann tensor, the Ricci tensor and the scalar curvature can be expressed via the second fundamental form as follows,

$$\begin{aligned} \mathrm{R}_{ijkl} &= g(\nabla^2_{ji}\partial_k - \nabla^2_{ij}\partial_k, \partial_l) = h_{ik}h_{jl} - h_{il}h_{jk}\,, \\ \mathrm{Ric}_{ij} &= g^{kl}\mathrm{R}_{ikjl} = \mathrm{H}\,h_{ij} - h_{il}g^{lk}h_{kj}\,, \\ \mathrm{R} &= g^{ij}\mathrm{Ric}_{ij} = g^{ij}g^{kl}\mathrm{R}_{ikjl} = \mathrm{H}^2 - |\mathrm{A}|^2\,. \end{aligned}$$

Hence, the formulas for the interchange of covariant derivatives, which involve the Riemann tensor, become

$$\begin{aligned} \nabla_i\nabla_j X^s - \nabla_j\nabla_i X^s &= \mathrm{R}_{ijkl}g^{ks}X^l = (h_{ik}h_{jl} - h_{il}h_{jk})\,g^{ks}X^l\,, \\ \nabla_i\nabla_j\omega_k - \nabla_j\nabla_i\omega_k &= \mathrm{R}_{ijkl}g^{ls}\omega_s = (h_{ik}h_{jl} - h_{il}h_{jk})\,g^{ls}\omega_s\,. \end{aligned}$$

The symmetry properties of the covariant derivative of A are expressed by the Codazzi equations,

$$\nabla_i h_{jk} = \nabla_j h_{ik} = \nabla_k h_{ij}$$

which imply the following *Simon's identity* (see [**83**]),

$$\Delta h_{ij} = \nabla_i\nabla_j\mathrm{H} + \mathrm{H}\,h_{il}g^{ls}h_{sj} - |\mathrm{A}|^2 h_{ij}\,. \tag{2.3}$$

We will write $T * S$, following Hamilton [**43**], to denote a tensor formed by contraction on some indices of the tensors T and S using g_{ij} and g^{ij}.
A very useful property of this $*$–product is that

$$|T * S| \leq C|T|\,|S|$$

where the constant C depends only on the "structure" of the tensors T and S.

Sometimes we will need the n–dimensional Hausdorff measure in $\mathbb{R}^{n+1}$, we will denote it with $\mathcal{H}^n$.

We advise the reader that in all the paper the constants could vary between different formulas and from a line to another.

2. First Variation of the Area Functional

Given an immersion $\varphi : M \to \mathbb{R}^{n+1}$ of a smooth hypersurface in $\mathbb{R}^{n+1}$, we consider the Area functional

$$\mathrm{Area}(\varphi) = \int_M d\mu$$

where μ is the canonical measure associated to the Riemannian metric g which is induced on M by the scalar product of $\mathbb{R}^{n+1}$ via the immersion φ.

In this section we are going to analyze the first variation of such functional which is nothing else than the volume of the hypersurface.

We consider a smooth one parameter family of immersions $\varphi_t : M \to \mathbb{R}^{n+1}$, with $t \in (-\varepsilon, \varepsilon)$ and $\varphi_0 = \varphi$, such that, outside of a compact set $K \subset M$, we have $\varphi_t(p) = \varphi(p)$ for every $t \in (-\varepsilon, \varepsilon)$.
Defining the field $X = \frac{\partial\varphi_t}{\partial t}\big|_{t=0}$ along M as a submanifold of $\mathbb{R}^{n+1}$, we see that it is zero

outside K, we call such field the *infinitesimal generator* of the variation φ_t.
We compute

$$\begin{aligned}\frac{\partial}{\partial t}g_{ij}\Big|_{t=0} &= \frac{\partial}{\partial t}\left\langle \frac{\partial\varphi_t}{\partial x_i}\,\Big|\,\frac{\partial\varphi_t}{\partial x_j}\right\rangle\Big|_{t=0}\\ &= \left\langle \frac{\partial X}{\partial x_i}\,\Big|\,\frac{\partial\varphi}{\partial x_j}\right\rangle + \left\langle \frac{\partial X}{\partial x_j}\,\Big|\,\frac{\partial\varphi}{\partial x_i}\right\rangle\\ &= \frac{\partial}{\partial x_i}\left\langle X\,\Big|\,\frac{\partial\varphi}{\partial x_j}\right\rangle + \frac{\partial}{\partial x_j}\left\langle X\,\Big|\,\frac{\partial\varphi}{\partial x_i}\right\rangle - 2\left\langle X\,\Big|\,\frac{\partial^2\varphi}{\partial x_i\partial x_j}\right\rangle\\ &= \frac{\partial}{\partial x_i}\left\langle X^M\,\Big|\,\frac{\partial\varphi}{\partial x_j}\right\rangle + \frac{\partial}{\partial x_j}\left\langle X^M\,\Big|\,\frac{\partial\varphi}{\partial x_i}\right\rangle - 2\Gamma_{ij}^k\left\langle X^M\,\Big|\,\frac{\partial\varphi}{\partial x_k}\right\rangle - 2h_{ij}\langle X\,|\,\nu\rangle\,,\end{aligned}$$

where X^M is the tangent component of the field X and we used the Gauss–Weingarten relations (2.1) in the last step.
Calling ω the 1–form defined by $\omega(Y) = g(d\varphi^*(X^M), Y)$, this formula can be rewritten as

$$\frac{\partial}{\partial t}g_{ij}\Big|_{t=0} = \frac{\partial\omega_j}{\partial x_i} + \frac{\partial\omega_i}{\partial x_j} - 2\Gamma_{ij}^k\omega_k - 2h_{ij}\langle X\,|\,\nu\rangle = \nabla_i\omega_j + \nabla_j\omega_i - 2h_{ij}\langle X\,|\,\nu\rangle\,.$$

Hence, using the formula $\partial_t \det A(t) = \det A(t)\mathrm{Trace}[A^{-1}(t)\partial_t A(t)]$, we get

$$\begin{aligned}\frac{\partial}{\partial t}\sqrt{\det(g_{ij})}\Big|_{t=0} &= \frac{\sqrt{\det(g_{ij})}\,g^{ij}\frac{\partial}{\partial t}g_{ij}\big|_{t=0}}{2}\\ &= \frac{\sqrt{\det(g_{ij})}\,g^{ij}(\nabla_i\omega_j + \nabla_j\omega_i - 2h_{ij}\langle X\,|\,\nu\rangle)}{2}\\ &= \sqrt{\det(g_{ij})}(\operatorname{div} X^M - \mathrm{H}\langle X\,|\,\nu\rangle)\,.\end{aligned}$$

If the Area of the immersion φ is finite, the same holds for all the maps φ_t, as they are compact deformations. Supposing that the compact K is contained in a single coordinate chart, we have

$$\begin{aligned}\frac{\partial}{\partial t}\mathrm{Area}(\varphi_t)\Big|_{t=0} &= \frac{\partial}{\partial t}\int_K d\mu_t\Big|_{t=0}\\ &= \frac{\partial}{\partial t}\int_K \sqrt{\det(g_{ij})}\,dx\Big|_{t=0}\\ &= \int_K \frac{\partial}{\partial t}\sqrt{\det(g_{ij})}\Big|_{t=0}\,dx\\ &= \int_K (\mathrm{div}\,X^M - \mathrm{H}\langle X\,|\,\nu\rangle)\sqrt{\det(g_{ij})}\,dx\\ &= \int_M (\mathrm{div}\,X^M - \mathrm{H}\langle X\,|\,\nu\rangle)\,d\mu\\ &= -\int_M \mathrm{H}\langle X\,|\,\nu\rangle\,d\mu\end{aligned}$$

where we used the fact that X is zero outside K and in the last step we applied the divergence theorem. Notice that the integrals are all well defined because actually we are integrating only on the compact K.
In the case that K is contained in several charts, the same conclusion follows from a standard argument using a partition of unity.

PROPOSITION 2.4. *The first variation of the* Area *functional depends only on the normal component of the infinitesimal generator* $X = \frac{\partial\varphi_t}{\partial t}\big|_{t=0}$ *of the variation, precisely*

$$\frac{\partial}{\partial t}\mathrm{Area}(\varphi_t)\Big|_{t=0} = -\int_M \mathrm{H}\langle X\,|\,\nu\rangle\,d\mu\,.$$

Moreover, such dependence is linear.

Given any immersion $\varphi : M \to \mathbb{R}^{n+1}$ and a field X along M, with compact support, we can always construct a variation with infinitesimal generator X as $\varphi_t(p) = \varphi(p) + tX(p)$. It is easy to see that for $|t|$ small the map φ_t is still a smooth immersion.
Hence, as the hypersurfaces which are critical point of the Area functional must satisfy $\int_M \mathrm{H}\langle X\,|\,\nu\rangle\,d\mu = 0$ for every X with compact support, they must have $\mathrm{H} = 0$ everywhere, that is, zero mean curvature (and clearly viceversa). This is the well known definition of the so called *minimal surfaces*.

As the quantity $-\mathrm{H}\nu$ can be interpreted as the *gradient* of the Area functional (be careful here, the measure μ is varying with φ, we are not taking the gradient with respect a to a canonical L^2 structure on the space of immersions of M in $\mathbb{R}^{n+1}$), one is interested in the motion of a hypersurface by this gradient, that is the *mean curvature flow*. So, one looks for hypersurfaces moving with velocity $\mathrm{H}\nu$ at every point of space and time. This means choosing, among all the velocity functions with fixed $L^2(\mu)$ norm, the one such that the area decreases most rapidly.

This idea is quite natural and arises often in studying the dynamics of models of physical situations where the energy is given by the "volume" of the interfaces between the phases of a system. Moreover, as the Area functional is the simplest (in terms of derivatives of the parametrization) geometric functional, that is, invariant by isometries of $\mathbb{R}^{n+1}$ and diffeomorphisms of M, the motion by mean curvature is the simplest *variational* geometric flow for immersed hypersurfaces. Any other geometric functional, for instance depending on the next simpler geometric invariant, the curvature, produces a first variation of order higher than two (actually four at least) in the derivatives of the parametrization, and a relative higher order PDE's system.
One can clearly consider other second order flows where the velocity of the motion is related to different functions of the curvature, like the Gauss flow of surfaces, for instance, where the velocity is given by $\mathrm{G}\nu$ (G is the Gauss curvature of M, that is, $\mathrm{G} = \det \mathrm{A}$), or more complicated flows, but these evolutions are not variational, they do not arise as "gradients" of geometric functionals (see Section 6).

3. The Mean Curvature Flow – Definition

DEFINITION 2.5. Let $\varphi_0 : M \to \mathbb{R}^{n+1}$ be a smooth immersion of a connected n–dimensional manifold. The mean curvature flow of φ_0 (or of M) is a family of smooth immersions $\varphi_t : M \to \mathbb{R}^{n+1}$ for $t \in [0, T)$ such that setting $\varphi(p, t) = \varphi_t(p)$ the map $\varphi : M \times [0, T) \to \mathbb{R}^{n+1}$ is a smooth solution of the following system of PDE's

$$\begin{cases} \frac{\partial}{\partial t}\varphi(p,t) = \mathrm{H}(p,t)\nu(p,t) \\ \varphi(p,0) = \varphi_0(p) \end{cases} \tag{2.4}$$

where $\mathrm{H}(p,t)$ and $\nu(p,t)$ are respectively the mean curvature and the normal of the hypersurface φ_t at the point $p \in M$.

REMARK 2.6. Notice that even if the unit normal vector is defined up to a sign, the field $\mathrm{H}(p,t)\nu(p,t)$ is independent of such choice.

Using equation (2.2), this system can be rewritten in the appealing form

$$\frac{\partial \varphi}{\partial t} = \Delta \varphi$$

but, despite of its formal analogy with the heat equation, actually, it is a second order quasilinear degenerate parabolic system, as the Laplacian is the one associated to the evolving hypersurfaces at time t,

$$\Delta\varphi(p,t) = \Delta_{g(p,t)}\varphi(p,t) = g^{ij}(p,t)\nabla^g_i\nabla^g_j\varphi(p,t)$$

and both g and ∇^g depends on the first derivatives of φ and on time t.
Moreover, this operator is degenerate, as its symbol (or the symbol of the linearized operator) admits zero eigenvalues, see **[38]** for details, due to the invariance of the Laplacian by diffeomorphisms.

hypothesis on x, we have $d^2_{t_i}(x) \to 0$, hence

$$d^2(t) = \lim_{i\to\infty} d^2_t(x) - d^2_{t_i}(x) \leq \lim_{i\to\infty} 2n(t_i - t) = 2n(T - t)$$

which is the thesis of the proposition.
The closure of $\mathcal{S}$ is obvious, boundedness is a consequence of sphere comparison in Corollary 3.10. □

A very important fact about hypersurfaces moving by mean curvature is the following.

PROPOSITION 3.14. *Suppose that the initial hypersurface is compact and embedded, then it remains embedded during the flow.*

PROOF. Let $\varphi : M \times [0, T] \to \mathbb{R}^{n+1}$ be the evolving hypersurface. It is clear that if φ_0 is an embedding it remains so for a small positive time, otherwise we will have a sequence of points and times, with $\varphi(p_i, t_i) = \varphi(q_i, t_i)$ and $t_i \to 0$, then, extracting a subsequence (not relabeled) such that $p_i \to p$ and $q_i \to q$, either $p \neq q$ so $\varphi(p, 0) = \varphi(q, 0)$, which is a contradiction, or $p = q$. By the smooth existence of the flow, in particular by the nonsingularity of the differential of $\partial_x\varphi(p, t)$ there exists a ball $B \subset M$ around p such that for $t \in [0, \varepsilon)$ the map $\varphi_t|_B$ is one–to–one, which is in contradiction with the hypotheses.
This small time embeddedness property is immediate by revisiting the proof of the small time existence theorem, representing the moving hypersurfaces as graphs on the initial one, for small time.

This argument also implies that embeddedness holds in a open interval $[0, T)$, suppose then that T is the first time the hypersurface is not embedded. The set S of pairs (p, q) with $p \neq q$ and $\varphi(p, T) = \varphi(q, T)$ is a nonempty closed set disjoint from the diagonal in $M \times M$, otherwise φ_T fails to be an immersion at some point in M. Then, we can find a smooth open neighborhood Ω of the diagonal with $\overline{\Omega} \cap S = \emptyset$.
We consider the following quantity,

$$C = \inf_{t\in[0,T]} \inf_{(p,q)\in\partial\Omega} |\varphi(p, t) - \varphi(q, t)| ,$$

then C is positive, as $\overline{\Omega} \cap S = \emptyset$ and $\partial\Omega$ is compact. We claim that the following function

$$L(t) = \min_{(p,q)\in M\times M\setminus\Omega} |\varphi(p, t) - \varphi(q, t)| ,$$

is bounded below by $\min\{L(0), C\} > 0$ on $[0, T]$, this is clearly in contradiction with the fact that S is nonempty and contained in $M \times M \setminus \Omega$.
If at some time $L(t) < C$ then $L(t)$ is achieved by pairs (p, q) not belonging to $\partial\Omega$, then (p, q) are inner points in $M \times M \setminus \Omega$ and a geometric argument analogous to the one of the comparison Theorem 3.7 shows that $\frac{dL(t)}{dt} \geq 0$, hence $L(t)$ is nondecreasing in time.
This last fact clearly implies the claim. □

REMARK 3.15. Theorem 3.7 and Proposition 3.14 also hold if the involved hypersurfaces are not compact, with some assumptions on the behavior at infinity (for instance,

uniform bounds on the curvature).
The analysis is anyway more complicated, one has possibly to use the interior estimates of Ecker and Huisken in [**29**].

3. Evolution of Curvature

Now we derive the evolution equations for g, ν, Γ^i_{jk} and A. We already know that

$$\frac{\partial}{\partial t} g_{ij} = -2\mathrm{H}h_{ij}\,.$$

Differentiating the formula $g_{is}g^{sj} = \delta^j_i$ we get

$$\frac{\partial}{\partial t} g^{ij} = -g^{is}\frac{\partial}{\partial t} g_{sl} g^{lj} = 2\mathrm{H}g^{is}h_{sl}g^{lj} = 2\mathrm{H}h^{ij}\,.$$

The derivative of the normal ν is given by

$$\left\langle \frac{\partial \nu}{\partial t} \,\middle|\, \frac{\partial \varphi}{\partial x_i} \right\rangle = -\left\langle \nu \,\middle|\, \frac{\partial^2 \varphi}{\partial t \partial x_i} \right\rangle = -\left\langle \nu \,\middle|\, \frac{\partial (\mathrm{H}\nu)}{\partial x_i} \right\rangle = -\frac{\partial \mathrm{H}}{\partial x_i}\,.$$

Finally the derivative of the Christoffel symbols is

$$\begin{aligned}
\frac{\partial}{\partial t}\Gamma^i_{jk} = &\frac{1}{2} g^{il}\left\{\frac{\partial}{\partial x_j}\left(\frac{\partial}{\partial t} g_{kl}\right) + \frac{\partial}{\partial x_k}\left(\frac{\partial}{\partial t} g_{jl}\right) - \frac{\partial}{\partial x_l}\left(\frac{\partial}{\partial t} g_{jk}\right)\right\} \\
&+ \frac{1}{2}\frac{\partial}{\partial t} g^{il}\left\{\frac{\partial}{\partial x_j} g_{kl} + \frac{\partial}{\partial x_k} g_{jl} - \frac{\partial}{\partial x_l} g_{jk}\right\} \\
= &\frac{1}{2} g^{il}\left\{\nabla_j\left(\frac{\partial}{\partial t} g_{kl}\right) + \nabla_k\left(\frac{\partial}{\partial t} g_{jl}\right) - \nabla_l\left(\frac{\partial}{\partial t} g_{jk}\right)\right\} \\
&+ \frac{1}{2} g^{il}\left\{\frac{\partial}{\partial t} g_{kz}\Gamma^z_{jl} + \frac{\partial}{\partial t} g_{lz}\Gamma^z_{jk} + \frac{\partial}{\partial t} g_{jz}\Gamma^z_{kl} + \frac{\partial}{\partial t} g_{lz}\Gamma^z_{jk} - \frac{\partial}{\partial t} g_{jz}\Gamma^z_{kl} - \frac{\partial}{\partial t} g_{kz}\Gamma^z_{jl}\right\} \\
&- \frac{1}{2} g^{is}\frac{\partial}{\partial t} g_{sz} g^{zl}\left\{\frac{\partial}{\partial x_j} g_{kl} + \frac{\partial}{\partial x_k} g_{jl} - \frac{\partial}{\partial x_l} g_{jk}\right\} \\
= &\frac{1}{2} g^{il}\left\{\nabla_j\left(\frac{\partial}{\partial t} g_{kl}\right) + \nabla_k\left(\frac{\partial}{\partial t} g_{jl}\right) - \nabla_l\left(\frac{\partial}{\partial t} g_{jk}\right)\right\} \\
&+ g^{il}\frac{\partial}{\partial t} g_{lz}\Gamma^z_{jk} - g^{is}\frac{\partial}{\partial t} g_{sz}\Gamma^z_{jk} \\
= &\frac{1}{2} g^{il}\left\{\nabla_j\left(\frac{\partial}{\partial t} g_{kl}\right) + \nabla_k\left(\frac{\partial}{\partial t} g_{jl}\right) - \nabla_l\left(\frac{\partial}{\partial t} g_{jk}\right)\right\} \\
= &- g^{il}\left\{\nabla_j(\mathrm{H}h_{kl}) + \nabla_k(\mathrm{H}h_{jl}) - \nabla_l(\mathrm{H}h_{jk})\right\} \\
= &- h^i_k\nabla_j\mathrm{H} - h^i_j\nabla_k\mathrm{H} + h_{jk}\nabla^i\mathrm{H} - \mathrm{H}(\nabla_j h^i_k + \nabla_k h^i_j - \nabla^i h_{jk})\,.
\end{aligned}$$

Resuming, we have

$$\begin{aligned}
\frac{\partial}{\partial t} g_{ij} &= -2\mathrm{H}h_{ij}\\
\frac{\partial}{\partial t} g^{ij} &= 2\mathrm{H}h^{ij}\\
\frac{\partial}{\partial t} \nu &= -\nabla\mathrm{H}\\
\frac{\partial}{\partial t} \Gamma^{i}_{jk} &= \nabla\mathrm{H} * \mathrm{A} + \mathrm{H} * \nabla\mathrm{A} = \nabla\mathrm{A} * \mathrm{A}\,.
\end{aligned}$$

LEMMA 3.16. *The second fundamental form satisfies the evolution equation*

$$\frac{\partial}{\partial t} h_{ij} = \Delta h_{ij} - 2\mathrm{H}h_{il}g^{ls}h_{sj} + |\mathrm{A}|^2 h_{ij}\,. \tag{3.1}$$

It follows,

$$\frac{\partial}{\partial t}|\mathrm{A}|^2 = \Delta|\mathrm{A}|^2 - 2|\nabla\mathrm{A}|^2 + 2|\mathrm{A}|^4 \tag{3.2}$$

and

$$\frac{\partial}{\partial t}\mathrm{H} = \Delta\mathrm{H} + \mathrm{H}|\mathrm{A}|^2\,. \tag{3.3}$$

PROOF. Keeping in mind the Gauss–Weingarten relations (2.1) and the equations above, we compute

$$\begin{aligned}
\frac{\partial}{\partial t} h_{ij} &= \frac{\partial}{\partial t}\left\langle \nu \,\middle|\, \frac{\partial^2 \varphi}{\partial x_i \partial x_j}\right\rangle\\
&= \left\langle \nu \,\middle|\, \frac{\partial^2 (\mathrm{H}\nu)}{\partial x_i \partial x_j}\right\rangle - \left\langle \nabla\mathrm{H} \,\middle|\, \frac{\partial^2 \varphi}{\partial x_i \partial x_j}\right\rangle\\
&= \frac{\partial^2 \mathrm{H}}{\partial x_i \partial x_j} - \mathrm{H}\left\langle \nu \,\middle|\, \frac{\partial}{\partial x_i}\left(h_{jl}g^{ls}\frac{\partial \varphi}{\partial x_s}\right)\right\rangle\\
&\quad - \left\langle \frac{\partial \mathrm{H}}{\partial x_l}\cdot\frac{\partial \varphi}{\partial x_s}g^{ls} \,\middle|\, \Gamma^k_{ij}\frac{\partial \varphi}{\partial x_k} + h_{ij}\nu\right\rangle\\
&= \frac{\partial^2 \mathrm{H}}{\partial x_i \partial x_j} - \mathrm{H}h_{jl}g^{ls}\left\langle \nu \,\middle|\, \frac{\partial^2 \varphi}{\partial x_i \partial x_s}\right\rangle - \Gamma^k_{ij}\frac{\partial \mathrm{H}}{\partial x_k}\\
&= \nabla_i\nabla_j\mathrm{H} - \mathrm{H}h_{il}g^{ls}h_{sj}\,.
\end{aligned}$$

then using Simon's identity (2.3) we conclude

$$\frac{\partial}{\partial t} h_{ij} = \Delta h_{ij} - 2\mathrm{H}h_{il}g^{ls}h_{sj} + |\mathrm{A}|^2 h_{ij}\,.$$

The second equation follows from a straightforward computation as $\frac{\partial}{\partial t} g^{ij} = 2\mathrm{H}h^{ij}$. □

Now we deal with the covariant derivatives of A.

LEMMA 3.17. *The following formula for the interchange of time and covariant derivative of a tensor T holds*

$$\frac{\partial}{\partial t}\nabla T = \nabla\frac{\partial}{\partial t}T + T * \mathrm{A} * \nabla\mathrm{A}\,.$$

PROOF. We suppose that $T = T_{i_1\dots i_k}$ is a covariant tensor, the general case is analogous, as it will be clear by the following computation,

$$\begin{aligned}\frac{\partial}{\partial t}\nabla_j T_{i_1\dots i_k} &= \frac{\partial}{\partial t}\left(\frac{\partial T_{i_1\dots i_k}}{\partial x_j} - \sum_{s=1}^{k}\Gamma^l_{ji_s}T_{i_1\dots i_{s-1}li_{s+1}\dots i_k}\right)\\ &= \frac{\partial}{\partial x_j}\frac{\partial T_{i_1\dots i_k}}{\partial t} - \sum_{s=1}^{k}\Gamma^l_{ji_s}\frac{\partial T_{i_1\dots i_{s-1}li_{s+1}\dots i_k}}{\partial t} - \sum_{s=1}^{k}\frac{\partial}{\partial t}\Gamma^l_{ji_s}T_{i_1\dots i_{s-1}li_{s+1}\dots i_k}\\ &= \nabla_j\frac{\partial T_{i_1\dots i_k}}{\partial t} - \sum_{s=1}^{k}(\mathrm{A} * \nabla\mathrm{A})^l_{ji_s}T_{i_1\dots i_{s-1}li_{s+1}\dots i_k}\,,\end{aligned}$$

which is the formula we wanted. □

LEMMA 3.18. *We have for $k > 0$, denoting with ∇^k the k–th iterated covariant derivative,*

$$\frac{\partial}{\partial t}\nabla^k h_{ij} = \Delta\nabla^k h_{ij} + \sum_{p+q+r=k\,|\,p,q,r\in\mathbb{N}}\nabla^p\mathrm{A} * \nabla^q\mathrm{A} * \nabla^r\mathrm{A}$$

PROOF. We work by induction on $k \in \mathbb{N}$. The case $k = 0$ is given by equation (3.1), we then suppose that the formula holds for $(k-1)$. We have, by the previous lemma,

$$\begin{aligned}\frac{\partial}{\partial t}\nabla^k h_{ij} &= \nabla\frac{\partial}{\partial t}\nabla^{k-1}h_{ij} + \nabla^{k-1}\mathrm{A} * \nabla\mathrm{A} * \mathrm{A}\\ &= \nabla(\Delta\nabla^{k-1}h_{ij} + \sum_{p+q+r=k-1\,|\,p,q,r\in\mathbb{N}}\nabla^p\mathrm{A} * \nabla^q\mathrm{A} * \nabla^r\mathrm{A}) + \nabla^{k-1}\mathrm{A} * \nabla\mathrm{A} * \mathrm{A}\\ &= \nabla\Delta\nabla^{k-1}h_{ij} + \sum_{p+q+r=k\,|\,p,q,r\in\mathbb{N}}\nabla^p\mathrm{A} * \nabla^q\mathrm{A} * \nabla^r\mathrm{A}\end{aligned}$$

Interchanging now Laplacian and covariant derivative and recalling that $\mathrm{Riem} = \mathrm{A} * \mathrm{A}$, we have the conclusion, as all extra terms we get are of the form $\mathrm{A} * \mathrm{A} * \nabla^k\mathrm{A}$ and $\mathrm{A} * \nabla\mathrm{A} * \nabla^{k-1}\mathrm{A}$. □

PROPOSITION 3.19. *The following formula holds,*

$$\frac{\partial}{\partial t}|\nabla^k\mathrm{A}|^2 = \Delta|\nabla^k\mathrm{A}|^2 - 2|\nabla^{k+1}\mathrm{A}|^2 + \sum_{p+q+r=k\,|\,p,q,r\in\mathbb{N}}\nabla^p\mathrm{A} * \nabla^q\mathrm{A} * \nabla^r\mathrm{A} * \nabla^k\mathrm{A} \qquad (3.4)$$

PROOF. We compute

$$
\begin{aligned}
\frac{\partial}{\partial t}|\nabla^k \mathrm{A}|^2 &= 2g\left(\nabla^k \mathrm{A}, \frac{\partial}{\partial t}\nabla^k \mathrm{A}\right) + \nabla^k \mathrm{A} * \nabla^k \mathrm{A} * \mathrm{A} * \mathrm{A} \\
&= 2g\left(\nabla^k \mathrm{A}, \Delta\nabla^k \mathrm{A} + \sum_{p+q+r=k \,|\, p,q,r\in\mathbb{N}} \nabla^p \mathrm{A} * \nabla^q \mathrm{A} * \nabla^r \mathrm{A}\right) + \nabla^k \mathrm{A} * \nabla^k \mathrm{A} * \mathrm{A} * \mathrm{A} \\
&= 2g\left(\nabla^k \mathrm{A}, \Delta\nabla^k \mathrm{A}\right) + \sum_{p+q+r=k \,|\, p,q,r\in\mathbb{N}} \nabla^p \mathrm{A} * \nabla^q \mathrm{A} * \nabla^r \mathrm{A} * \nabla^k \mathrm{A} \\
&= \Delta|\nabla^k \mathrm{A}|^2 - 2|\nabla^{k+1} \mathrm{A}|^2 + \sum_{p+q+r=k \,|\, p,q,r\in\mathbb{N}} \nabla^p \mathrm{A} * \nabla^q \mathrm{A} * \nabla^r \mathrm{A} * \nabla^k \mathrm{A}\,.
\end{aligned}
$$

□

4. Consequences of Evolution Equations

Let us see some consequences of the application of the maximum principle to the evolution equations for the curvature.
Suppose we have a mean curvature flow of a compact hypersurface M on the time interval $[0, T)$. We have seen that

$$\frac{\partial}{\partial t}|\mathrm{A}|^2 = \Delta|\mathrm{A}|^2 - 2|\nabla \mathrm{A}|^2 + 2|\mathrm{A}|^4 \leq \Delta|\mathrm{A}|^2 + 2|\mathrm{A}|^4$$

and

$$\frac{\partial}{\partial t}\mathrm{H} = \Delta\mathrm{H} + \mathrm{H}|\mathrm{A}|^2\,.$$

First we deal with the so called *mean convex* hypersurfaces that play a major role in the subject.
A hypersurface is mean convex if $\mathrm{H} \geq 0$ everywhere. We will see in the next proposition that this property is preserved by the mean curvature flow.
Mean convexity is a significant generalization of convexity, for instance, it is enough general to allow for the neckpinch behavior described in Section 4, in particular, mean convex hypersurfaces do not necessarily shrink to a point at the singular time.

PROPOSITION 3.20. *Suppose that the initial compact hypersurface satisfies* $\mathrm{H} \geq 0$*. Then, under the mean curvature flow, the minimum of* H *is increasing, hence* H *is positive for every positive time.*

PROOF. Arguing by contradiction, suppose that in an interval $(t_0, t_1) \subset \mathbb{R}^+$ we have $\mathrm{H}_{\min}(t) < 0$ and $\mathrm{H}_{\min}(t_0) = 0$ ($\mathrm{H}_{\min}$ is continuous).
Let $|\mathrm{A}|^2 \leq C$ in such interval, then

$$\frac{\partial \mathrm{H}}{\partial t} = \Delta\mathrm{H} + \mathrm{H}|\mathrm{A}|^2$$

implies

$$\frac{\partial \mathrm{H}_{\min}}{\partial t} \geq C\mathrm{H}_{\min}$$

for almost every $t \in (t_0, t_1)$.
Integrating this differential inequality in $[s,t] \subset (t_0,t_1)$ we get $\mathrm{H}_{\min}(t) \geq e^{C(t-s)}\mathrm{H}_{\min}(s)$, then sending $s \to t_0^+$ we conclude $\mathrm{H}_{\min}(t) = 0$ for every $t \in (t_0, t_1)$ which is a contradiction.

Since then $\mathrm{H} \geq 0$ we get

$$\frac{\partial \mathrm{H}}{\partial t} = \Delta\mathrm{H} + \mathrm{H}|\mathrm{A}|^2 \geq \Delta\mathrm{H} + \mathrm{H}^3/n\,.$$

With the notation of Theorem 3.1, we let $u = -\mathrm{H}$, $X = 0$ and $b(x) = x^3/n$, then if $\mathrm{H}_{\min}(0) = 0$ the ODE solution $h(t)$ is always zero, so if at some positive time $\mathrm{H}_{\min}(\tau) = 0$, we have that $\mathrm{H}(\cdot,\tau)$ is constant equal to zero on M, but there are no compact hypersurfaces with zero mean curvature. Hence, $\mathrm{H}_{\min}$ is always increasing during the flow and H is positive on all M at every positive time. □

Actually, this proposition can be slightly improved as follows.

PROPOSITION 3.21. *If the initial compact hypersurface satisfies $|\mathrm{A}| \leq \alpha\mathrm{H}$ for some constant α, then $|\mathrm{A}| \leq \alpha\mathrm{H}$ for every positive time.*

PROOF. We know that H is positive for every positive time, hence also $|\mathrm{A}| > 0$ for every positive time which implies that it is smooth as $|\mathrm{A}|^2$.
Let $[0,T)$ be the interval of smooth existence of the flow. Computing the evolution equation of the function $f = |\mathrm{A}| - \alpha\mathrm{H}$, we get

$$\begin{aligned}\frac{\partial f}{\partial t} &= \frac{1}{2|\mathrm{A}|}(\Delta|\mathrm{A}|^2 - 2|\nabla\mathrm{A}|^2 + 2|\mathrm{A}|^4) - \alpha(\Delta\mathrm{H} + \mathrm{H}|\mathrm{A}|^2)\\ &= \Delta|\mathrm{A}| + \frac{1}{2|\mathrm{A}|}(2|\nabla|\mathrm{A}||^2 - 2|\nabla\mathrm{A}|^2) + |\mathrm{A}|^3 - \alpha(\Delta\mathrm{H} + \mathrm{H}|\mathrm{A}|^2)\\ &= \Delta f + |\mathrm{A}|^2 f + \frac{1}{2|\mathrm{A}|}(2|\nabla|\mathrm{A}||^2 - 2|\nabla\mathrm{A}|^2)\\ &\leq \Delta f + |\mathrm{A}|^2|f|\,,\end{aligned}$$

as the term $|\nabla|\mathrm{A}||^2 - |\nabla\mathrm{A}|^2$ is nonpositive.
Hence, choosing any $T' < T$, if C is the maximum of $|\mathrm{A}|^2$ on $M \times [0,T']$, we have $\partial_t f \leq \Delta f + C|f|$ on $M \times [0,T']$. By maximum principle 3.1, as $f_{\max}(0) \leq 0$, we conclude $f \leq 0$ on $M \times [0,T']$. By the arbitrariness of $T' < T$, the thesis follows. □

COROLLARY 3.22. *If $H > 0$ for the initial compact n–dimensional hypersurface, then there exists $\alpha_0 > 0$ such that $\alpha_0|\mathrm{A}|^2 \leq \mathrm{H}^2 \leq n|\mathrm{A}|^2$ everywhere on M for every time.*
If the initial hypersurface has positive scalar curvature, then the same holds for all the positive times.

PROOF. The first claim is immediate by compactness of M and previous proposition (the right hand inequality is algebraic)
Recalling that the scalar curvature is equal to $\mathrm{H}^2 - |\mathrm{A}|^2$, positive scalar curvature implies that $\mathrm{H} > 0$ (H cannot change sign on M and there is always a point where it is positive, as M is compact) and $\mathrm{H}^2/|\mathrm{A}|^2 > 1$, the second part of this corollary is also consequence of Proposition 3.21. □

COROLLARY 3.23. *Suppose that the initial compact hypersurface has* $\mathrm{H} \geq 0$*, then, if* A *is not bounded as* $t \to T$*, the same holds for* H.

PROOF. Immediate consequence of Proposition 3.20 and the estimate of the previous corollary. □

Now we look at the evolution equation of $|\mathrm{A}|^2$, it implies

$$\frac{\partial}{\partial t}|\mathrm{A}|^2_{\max} \leq 2|\mathrm{A}|^4_{\max}\,.$$

Notice that $|\mathrm{A}|^2_{\max}$ is always positive, otherwise at some time t we would have $\mathrm{A} = 0$ identically on M, which would imply that M is a plane in $\mathbb{R}^{n+1}$ in contradiction with the compactness hypothesis. Hence, we can divide both members by $|\mathrm{A}|^2_{\max}$, obtaining the differential inequality for the locally Lipschitz function $1/|\mathrm{A}|^2_{\max}$, holding at almost every time $t \in [0, T)$, with $T < +\infty$,

$$-\frac{d}{dt}\frac{1}{|\mathrm{A}|^2_{\max}} \leq 2\,.$$

Integrating in time in any interval $[t, s] \subset [0, T)$, we get

$$\frac{1}{|\mathrm{A}(\cdot, t)|^2_{\max}} - \frac{1}{|\mathrm{A}(\cdot, s)|^2_{\max}} \leq 2(s - t)\,.$$

Suppose now that A is not bounded in $[0, T)$, that is, there exists a sequence of times $s_i \nearrow T$ such that $|\mathrm{A}(\cdot, s_i)|^2_{\max} \to +\infty$. Substituting these times s_i in the previous inequality and sending $i \to \infty$, we get

$$\frac{1}{|\mathrm{A}(\cdot, t)|^2_{\max}} \leq 2(T - t)\,.$$

EXERCISE 3.24. Show that the only compact hypersurfaces in $\mathbb{R}^{n+1}$ with constant mean curvature are the spheres.
What about a compact hypersurfaces in $\mathbb{R}^{n+1}$ with constant $|\mathrm{A}|$?

In other words, we proved the following.

PROPOSITION 3.25. *If the second fundamental form* A *during the mean curvature flow of a compact hypersurface, is not bounded as* $t \to T < +\infty$*, then it must satisfy the following lower bound for the explosion rate*

$$\max_{p \in M} |\mathrm{A}(p, t)| \geq \frac{1}{\sqrt{2(T - t)}}\,.$$

Hence,

$$\lim_{t\to T}\max_{p\in M}|\mathrm{A}(p,t)| = +\infty\,.$$

Moreover, if the inequality is an equality at some time, then the hypersurface is a sphere at every time.

EXERCISE 3.26. Suppose that the initial compact hypersurface has $\mathrm{H} > 0$, then the maximal time of smooth existence of the flow can be estimated as $T_{\max} \leq n/2\mathrm{H}^2_{\min}(0)$.

PROPOSITION 3.27. *If the second fundamental form is bounded in the interval* $[0, T)$*,* $T < +\infty$*, then all its covariant derivatives are also bounded.*

PROOF. By Proposition 3.19 we have

$$\begin{aligned}\frac{\partial}{\partial t}|\nabla^k\mathrm{A}|^2 &= \Delta|\nabla^k\mathrm{A}|^2 - 2|\nabla^{k+1}\mathrm{A}|^2 + \sum_{p+q+r=k\,|\,p,q,r\in\mathbb{N}} \nabla^p\mathrm{A} * \nabla^q\mathrm{A} * \nabla^r\mathrm{A} * \nabla^k\mathrm{A}\\ &\leq \Delta|\nabla^k\mathrm{A}|^2 + P(|\mathrm{A}|,\dots,|\nabla^{k-1}\mathrm{A}|)|\nabla^k\mathrm{A}|^2 + Q(|\mathrm{A}|,\dots,|\nabla^{k-1}\mathrm{A}|)\,,\end{aligned}$$

where P and Q are smooth functions independent of time (actually they are polynomials in their arguments). Notice that in the arguments of P, Q there is not $\nabla^k\mathrm{A}$, indeed, in the terms $\nabla^p\mathrm{A} * \nabla^q\mathrm{A} * \nabla^r\mathrm{A} * \nabla^k\mathrm{A}$ there can be only one, or two occurrences of $\nabla^k\mathrm{A}$, by the condition $p + q + r = k$ and $p, q, r \in \mathbb{N}$. If there are two, suppose $r = k$, then necessarily $p = q = 0$ and we estimate $|\mathrm{A} * \mathrm{A} * \nabla^k\mathrm{A} * \nabla^k\mathrm{A}| \leq |\mathrm{A}|^2|\nabla^k\mathrm{A}|^2$, if there is only one, this means that $p, q, r < k$ and we again estimate $|\nabla^p\mathrm{A} * \nabla^q\mathrm{A} * \nabla^r\mathrm{A} * \nabla^k\mathrm{A}| \leq |\nabla^p\mathrm{A} * \nabla^q\mathrm{A} * \nabla^r\mathrm{A}|^2/2 + |\nabla^k\mathrm{A}|^2/2$.

Reasoning by induction on k, being the case $k = 0$ in the hypotheses, we suppose that all the covariant derivatives of A up to the order $(k-1)$ are bounded, hence also $P(|\mathrm{A}|,\dots,|\nabla^{k-1}\mathrm{A}|)$ and $Q(|\mathrm{A}|,\dots,|\nabla^{k-1}\mathrm{A}|)$ are bounded, thus

$$\frac{\partial}{\partial t}|\nabla^k\mathrm{A}|^2 \leq \Delta|\nabla^k\mathrm{A}|^2 + C|\nabla^k\mathrm{A}|^2 + D\,.$$

By maximum principle, this implies

$$\frac{d}{dt}|\nabla^k\mathrm{A}|^2_{\max} \leq C|\nabla^k\mathrm{A}|^2_{\max} + D\,,$$

and since the interval $[0, T)$ is bounded, the quantity $|\nabla^k\mathrm{A}|^2_{\max}$ is also bounded, as one can obtain an easy exponential estimate for the function $u(t) = |\nabla^k\mathrm{A}|^2_{\max}$, integrating the ordinary differential inequality $u' \leq Cu + D$, holding for almost every time $t \in [0, T)$. □

PROPOSITION 3.28. *If the second fundamental form is bounded in the interval* $[0, T)$*,* $T < +\infty$*, then* T *cannot be a singularity time for the mean curvature flow of a compact hypersurface* $\varphi(p, t) : M \times [0, T) \to \mathbb{R}^{n+1}$*.*

PROOF. By the previous proposition we know that all the covariant derivatives of A are bounded by constants depending on T and the geometry of the initial hypersurface.

We look at the map φ, as H is bounded,

$$|\varphi(p,t)-\varphi(p,s)| \leq \int_s^t |\mathrm{H}(p,\xi)|\, d\xi \leq C(t-s)$$

for every $0 \leq s \leq t < T$, then the maps $\varphi_t = \varphi(\cdot, t)$ uniformly converge to a continuous limit map $\varphi_T : M \to \mathbb{R}^{n+1}$ as $t \to T$.
We fix now a vector $v = \{v^i\} \in T_pM$,

$$\frac{d}{dt}\log|v|_g^2 = \frac{\frac{\partial g_{ij}}{\partial t}v^iv^j}{|v|_g^2} = \frac{-2\mathrm{H}h_{ij}v^iv^j}{|v|_g^2} \leq C\frac{|\mathrm{A}||v|_g^2}{|v|_g^2} \leq C$$

then, for every $0 \leq s \leq t < T$

$$\left|\log\frac{|v|_{g(t)}^2}{|v|_{g(s)}^2}\right| \leq \int_s^t \left|\frac{d}{d\xi}\log|v|_{g(\xi)}^2\right| d\xi \leq C(t-s)$$

which implies that the metrics $g(t)$ are all equivalent and the norms $|\cdot|_{g(t)}$ uniformly converge, as $t \to T$ to another equivalent norm $|\cdot|_T$, which is continuous. By the parallelogram identity, it also follows that this limit norm $|\cdot|_T$ comes from a metric tensor g_T which, since it is equivalent to all the other metrics, it is also positive definite. As a consequence, we are free to use any of these metrics in doing our estimates.

By the evolution equation for the Christoffel symbols, we see that

$$\left|\Gamma_{ij}^k(t)\right| \leq \left|\Gamma_{ij}^k(0)\right| + \int_0^t \left|\frac{\partial}{\partial t}\Gamma_{ij}^k\right| dt \leq C + \int_0^T |\mathrm{A} * \nabla \mathrm{A}|\, dt \leq C + DT\,,$$

for some constants depending only on the initial hypersurface. Thus, the Christoffel symbols are equibounded in time, after fixing a local chart. This implies that for every tensor S

$$\left|\,\left|\frac{\partial S}{\partial x_i}\right| - |\nabla_i S|\,\right| \leq C|S|$$

that is, the derivatives in coordinates differ by the relative covariant ones by equibounded terms.

In the following, by simplicity we will denote with ∂ the coordinate derivatives and with ∇ the covariant ones.

As the time derivative the Christoffel symbols is a tensor of the form $\mathrm{A} * \nabla\mathrm{A}$, we have

$$|\partial_t\partial_{l_1\dots l_s}^s\Gamma_{ij}^k| = |\partial_{l_1\dots l_s}^s\partial_t\Gamma_{ij}^k| = |\partial_{l_1\dots l_s}^s\mathrm{A} * \nabla\mathrm{A}|\,,$$

hence, by an induction argument on the order s and integration as above, one can show that $\left|\partial_{l_1\dots l_s}^s\Gamma_{ij}^k\right| \leq C$ for every $s \in \mathbb{N}$.
Then, again by induction, the following formula (where we avoid to indicate the indices) relating the iterated covariant and coordinate derivatives of a tensor S, holds

$$|\,|\nabla^s S| - |\partial^s S|\,| \leq \sum_{i=1}^{s}\sum_{j_1+\dots+j_i+k\leq s-1}\left|\partial^{j_1}\Gamma\dots\partial^{j_i}\Gamma\partial^k S\right| \leq C\sum_{k=1}^{s-1}\left|\partial^k S\right|\,.$$

This implies that if a tensor has all its covariant derivatives bounded, the same holds for the coordinates derivatives. In particular this holds for the tensor A, that is, $\left|\partial^k \mathrm{A}\right| \leq C_k$. Moreover, by induction, as $\nabla^k g = 0$ all the coordinate derivatives of the metric tensor g are equibounded.

We already know that $|\varphi|$ is bounded and $|\partial\varphi| = 1$, by the Gauss–Weingarten relations (2.1)

$$\partial^2\varphi = \Gamma\partial\varphi + \mathrm{A}\nu\,, \qquad \partial\nu = \mathrm{A} * \partial\varphi$$

we get

$$\begin{aligned}
|\partial^k\varphi| &= \left|\sum_{i=0}^{k-2}\binom{k-2}{i}\partial^{k-2-i}\Gamma\partial^{i+1}\varphi + \sum_{i=0}^{k-2}\binom{k-2}{i}\partial^{k-2-i}\mathrm{A}\partial^i\nu\right| \\
&\leq C\sum_{i=0}^{k-2}|\partial^{i+1}\varphi| + C\sum_{i=1}^{k-2}|\partial^{i-1}(\mathrm{A} * \partial\varphi)| + C \\
&= C\sum_{i=0}^{k-2}|\partial^{i+1}\varphi| + C\sum_{i=1}^{k-2}\left|\sum_{p+q+r=i-1}\partial^p\mathrm{A} * \partial^q g * \partial^{r+1}\varphi\right| + C \\
&\leq C\sum_{i=0}^{k-2}|\partial^{i+1}\varphi| + C\sum_{i=1}^{k-2}\sum_{r=0}^{i-1}|\partial^{r+1}\varphi| + C \\
&\leq C\sum_{i=0}^{k-2}|\partial^{i+1}\varphi| + C\sum_{i=1}^{k-2}|\partial^{i}\varphi| + C \\
&\leq C\sum_{i=0}^{k-1}|\partial^{i}\varphi|
\end{aligned}$$

where we estimated with a constant all the occurrences of $\partial^k\mathrm{A}$ and $\partial^k g$. Hence, we can conclude by induction that $|\partial^k\varphi| < C_k$ for constant C_k independent of time $t \in [0, T)$. By Ascoli–Arzelà theorem we can conclude that $\varphi_T : M \to \mathbb{R}^{n+1}$ is a smooth immersion and the convergence $\varphi(\cdot, t) \to \varphi_T$ is in C^∞.

Moreover, with the same argument, differentiating the evolution equation $\partial_t\varphi = \mathrm{H}\nu$ one gets also uniform boundedness of the time derivatives of the map φ, that is $|\partial_t^s\partial_x^k\varphi| \leq C_{s,k}$. Hence the map $\varphi : M \times [0, T) \to \mathbb{R}^{n+1}$ can be extended smoothly to the boundary of the domain of φ with the map φ_T.

By means of the small time existence Theorem 2.15 we can now "restart" the flow with the immersion φ, obtaining a smooth extension of the map φ which is in contradiction with the fact that T was the maximal existence time. □

We can so give a slightly improved version of Theorem 2.15 as follows.

THEOREM 3.29. *For any smooth compact hypersurface immersed in* $\mathbb{R}^{n+1}$*, there exists a unique mean curvature flow which is smooth on a maximal time interval* $[0, T_{\max})$.

Moreover, $T_{\max}$ *is finite and*

$$\max_{p\in M} |\mathrm{A}(p,t)| \geq \frac{1}{\sqrt{2(T_{\max}-t)}}$$

for every $t \in [0, T_{\max})$.

EXERCISE 3.30. Show that the maximal time of smooth existence of the flow can be estimated as $T_{\max} \geq 1/2|\mathrm{A}(\cdot,0)|^2_{\max}$.

5. Convexity Invariance

Corollary 3.22 is a consequence of a more general invariance property of the elementary symmetric polynomials of the curvatures, as we are going to to show.
We recall that the *elementary symmetric function* of degree k of $\lambda_1,\dots,\lambda_n$ is defined as

$$\mathrm{S}_k = \sum_{1\leq i_1<i_2<\dots<i_k\leq n} \lambda_{i_1}\lambda_{i_2}\dots\lambda_{i_k}$$

for $k=1,\dots,n$. In particular, if λ_i are the eigenvalues of the second fundamental form A we have $\mathrm{S}_1 = \mathrm{H}$, S_2 is the scalar curvature and $|\mathrm{A}|^2 = \mathrm{S}_1^2 + 2\mathrm{S}_2$.
It is not difficult to show that

$$\begin{aligned} \lambda_1 \geq 0,\dots,\lambda_n \geq 0 \quad &\Longleftrightarrow \quad \mathrm{S}_1 \geq 0,\dots,\mathrm{S}_n \geq 0\,, \\ \lambda_1 > 0,\dots,\lambda_n > 0 \quad &\Longleftrightarrow \quad \mathrm{S}_1 > 0,\dots,\mathrm{S}_n > 0\,. \end{aligned} \tag{3.5}$$

These polynomials enjoy various concavity properties, see **[71, 58]**.

PROPOSITION 3.31. *Let* $\Gamma_k \subset \mathbb{R}^n$ *denote the connected component of* $\mathrm{S}_k > 0$ *containing the positive cone. Then* $\mathrm{S}_l > 0$ *in* Γ_k *for all* $l = 1,\dots,k$ *and the quotient* $\mathrm{S}_{k+1}/\mathrm{S}_k$ *is concave on* Γ_k.

The above properties remain unchanged if we regard the polynomials S_k as functions of the Weingarten operator h^i_j, instead of the principal curvatures, as we have the following algebraic result, see [**8**, Lemma 2.22] or [**58**, Lemma 2.11].

PROPOSITION 3.32. *Let* $f(\lambda_1,\dots,\lambda_n)$ *be a symmetric convex (concave) function of its variables and let* $F(A) = f($*eigenvalues of* $A)$ *for any* $n\times n$ *symmetric matrix* A *whose eigenvalues belong to the domain of* f. *Then* F *is convex (concave).*

We are now ready to derive the evolution equation of relevant quantities and to apply the maximum principle to obtain invariance properties.

PROPOSITION 3.33. *Let* $F(h^i_j)$ *be a function homogeneous of degree one. Let* φ *be a mean curvature flow of compact* n*–dimensional hypersurfaces with* $\mathrm{H} > 0$ *and such that* h^i_j *belongs everywhere to the domain of* F. *Then,*

$$\frac{\partial}{\partial t}\frac{F}{\mathrm{H}} - \Delta\frac{F}{\mathrm{H}} = \frac{2}{\mathrm{H}}\left\langle \nabla \mathrm{H} \,\Big|\, \nabla\frac{F}{\mathrm{H}}\right\rangle - \frac{1}{\mathrm{H}}\frac{\partial^2 F}{\partial h^i_j \partial h^k_l}\nabla^p h^j_i \nabla_p h^k_l\,.$$