CHAPTER 4

Monotonicity Formula and Type I Singularities

In all this chapter $\varphi : M \times [0,T) \to \mathbb{R}^{n+1}$ is the mean curvature flow of an n–dimensional compact and connected hypersurface in $\mathbb{R}^{n+1}$, defined by $\partial_t \varphi = \mathrm{H}\nu$, in the maximal interval of smooth existence $[0,T)$.

With $\widetilde{\mathcal{H}}^n$ we will denote the n–dimensional Hausdorff measure *counting multiplicities*.

1. The Monotonicity Formula for Mean Curvature Flow

We show now the important monotonicity formula for mean curvature flow, discovered by Huisken in [**54**], then generalized by Hamilton in [**46, 47**]. Such formula will be the main tool to analyze type I singularities in the next sections.

LEMMA 4.1. *Let $f : \mathbb{R}^{n+1} \times I \to \mathbb{R}$ be a smooth function. By a little abuse of notation, we denote with $\int_M f\, d\mu_t$ the integral $\int_M f(\varphi(p,t),t)\, d\mu_t(p)$.*
Then the following formula holds

$$\frac{d}{dt}\int_M f\, d\mu_t = \int_M (f_t - \mathrm{H}^2 f + \mathrm{H}\langle \nabla f \,|\, \nu\rangle)\, d\mu_t\,.$$

PROOF. Straightforward computation. □

If $u_t = -\Delta^{\mathbb{R}^{n+1}} u$ is a positive solution of backward heat equation in $\mathbb{R}^{n+1}$, we have

$$\begin{aligned}\frac{d}{dt}\int_M u\, d\mu_t &= \int_M (u_t - \mathrm{H}^2 u + \mathrm{H}\langle \nabla u \,|\, \nu\rangle)\, d\mu_t \\ &= -\int_M (\Delta^{\mathbb{R}^{n+1}} u + \mathrm{H}^2 u - \mathrm{H}\langle \nabla u \,|\, \nu\rangle)\, d\mu_t\,.\end{aligned} \tag{4.1}$$

LEMMA 4.2. *If $\varphi : M \to \mathbb{R}^{n+1}$ is an isometric immersion, for every smooth function u defined in a neighborhood of $\varphi(M)$ we have,*

$$\Delta^M(u(\varphi)) = (\Delta^{\mathbb{R}^{n+1}} u)(\varphi) - (\nabla^\perp \nabla^\perp u)(\varphi) + \mathrm{H}\langle \nu \,|\, (\nabla u)(\varphi)\rangle\,.$$

PROOF. Let $p \in M$ and we choose normal coordinates at p. Set $\widetilde{u} = u \circ \varphi$, then

$$
\begin{aligned}
\Delta^M \widetilde{u} &= \nabla^2_{ii}(u \circ \varphi) \\
&= \nabla_i \left(\frac{\partial u}{\partial y_\alpha} \frac{\partial \varphi^\alpha}{\partial x_i} \right) \\
&= \frac{\partial^2 u}{\partial y_\alpha \partial y_\beta} \frac{\partial \varphi^\alpha}{\partial x_i} \frac{\partial \varphi^\beta}{\partial x_i} + \frac{\partial u}{\partial y_\alpha} \frac{\partial^2 \varphi^\alpha}{\partial x_i^2} \\
&= \frac{\partial^2 u}{\partial y_\alpha \partial y_\beta} \frac{\partial \varphi^\alpha}{\partial x_i} \frac{\partial \varphi^\beta}{\partial x_i} + \frac{\partial u}{\partial y_\alpha} h_{ii} \nu^\alpha \\
&= (\Delta^{\mathbb{R}^{n+1}} u)(\varphi) - (\nabla^\perp \nabla^\perp u)(\varphi) + \mathrm{H} \langle \nu \,|\, (\nabla u)(\varphi) \rangle \,,
\end{aligned}
$$

where we used the Gauss–Weingarten relations (2.1). $\square$

It follows that, substituting $\Delta^{\mathbb{R}^{n+1}} u$ in formula (4.1), by means of this lemma, we get

$$
\begin{aligned}
\frac{d}{dt} \int_M u \, d\mu_t &= - \int_M (\Delta^M (u(\varphi)) + \nabla^\perp \nabla^\perp u + \mathrm{H}^2 u - 2\mathrm{H} \langle \nabla u \,|\, \nu \rangle) \, d\mu_t \\
&= - \int_M (\nabla^\perp \nabla^\perp u + \mathrm{H}^2 u - 2\mathrm{H} \langle \nabla u \,|\, \nu \rangle) \, d\mu_t \\
&= - \int_M u \left| \mathrm{H} - \frac{\langle \nabla u \,|\, \nu \rangle}{u} \right|^2 d\mu_t + \int_M \left(\frac{|\nabla^\perp u|^2}{u} - \nabla^\perp \nabla^\perp u \right) d\mu_t \,.
\end{aligned}
$$

Then the following theorem follows.

THEOREM 4.3 (Huisken's Monotonicity Formula – Hamilton's Extension). *Suppose that we have a positive smooth solution of the backward heat equation $u_t = -\Delta u$ in $\mathbb{R}^{n+1} \times [0, \tau)$. The generalization of Huisken's monotonicity formula by Hamilton read (see* **[46, 47]***)*

$$
\begin{aligned}
\frac{d}{dt} \Big[\sqrt{4\pi(\tau - t)} \int_M u \, d\mu_t \Big] = &- \sqrt{4\pi(\tau - t)} \int_M u \, |\mathrm{H} - \langle \nabla \log u \,|\, \nu \rangle|^2 \, d\mu_t \\
&- \sqrt{4\pi(\tau - t)} \int_M \Big(\nabla^\perp \nabla^\perp u - \frac{|\nabla^\perp u|^2}{u} + \frac{u}{2(\tau - t)} \Big) \, d\mu_t
\end{aligned}
\tag{4.2}
$$

in the time interval $[0, \min\{\tau, T\})$, where $\nabla^\perp$ denotes the covariant derivative along the normal direction in $\mathbb{R}^{n+1}$.

As we can see, the right side of the formula consists of a non positive term and a term which is nonpositive if $\frac{\nabla^\perp \nabla^\perp u}{u} - \frac{|\nabla^\perp u|^2}{u^2} + \frac{1}{2(\tau - t)} = \nabla^2_{\nu\nu} \log u + \frac{1}{2(\tau - t)}$ is nonnegative. Setting $v(x, t) = u(x, \tau - t)$, the function $v : \mathbb{R}^{n+1} \times (0, \tau] \to \mathbb{R}$ is a positive solution of the standard *forward* heat equation in all $\mathbb{R}^{n+1}$ and $\nabla^2_{\nu\nu} \log u + \frac{1}{2(\tau - t)} = \nabla^2_{\nu\nu} \log v + \frac{1}{2t}$. This last expression is exactly the Li–Yau–Hamilton 2–form $\nabla^2 \log v + g/(2t)$ for positive solutions of the heat equation on a compact manifold, evaluated on $\nu \otimes \nu$ (see **[46]**).
In the paper **[46]** (see also **[74]**) Hamilton generalized Li–Yau differential Harnack inequality in **[70]** (concerning the nonnegativity of $\Delta \log v + \frac{\dim M}{2t}$) showing that, under the

hypotheses on (M, g) of parallel Ricci tensor ($\nabla \mathrm{Ric} = 0$) and of non negative sectional curvatures, the 2–form $\nabla^2 \log v + g/(2t)$ is nonnegative definite (Hamilton's matrix Harnack inequality). In particular, in $\mathbb{R}^{n+1}$ with the canonical flat metric such hypotheses clearly hold and $\nabla^2_{\nu\nu} \log u + \frac{1}{2(\tau - t)} = \Big(\nabla^2 \log v + g^{\mathbb{R}^{n+1}}_{\mathrm{can}}/(2t)\Big)(\nu \otimes \nu) \geq 0$. Hence, assuming boundedness in space of v and its derivatives (equivalently of u), the monotonicity formula implies that $\sqrt{4\pi(\tau - t)} \int_M u \, d\mu_t$ is nonincreasing in time.

REMARK 4.4. We asked for boundedness in space of u and its derivatives as $\mathbb{R}^{n+1}$ lacks of the compactness which is required for the ambient space in the original paper of Hamilton (the proof is based on the maximum principle), but his result can be extended to $\mathbb{R}^{n+1}$ (and other noncompact spaces), by localization, under such hypothesis. See Appendix D for details.

Choosing in particular a *backward* heat kernel of $\mathbb{R}^{n+1}$, that is, $u(x, t) = \rho_{x_0,\tau}(x, t) = \frac{e^{-\frac{|x-x_0|^2}{4(\tau-t)}}}{[4\pi(\tau-t)]^{(n+1)/2}}$, we get the standard Huisken's monotonicity formula, as the Li–Yau–Hamilton expression is identically zero in this case.

THEOREM 4.5 (Huisken's Monotonicity Formula). *For every $\tau > 0$ we have (see* **[54]***)*

$$\frac{d}{dt} \int_M \frac{e^{-\frac{|x-x_0|^2}{4(\tau-t)}}}{[4\pi(\tau - t)]^{n/2}} \, d\mu_t = -\int_M \frac{e^{-\frac{|x-x_0|^2}{4(\tau-t)}}}{[4\pi(\tau - t)]^{n/2}} \left| \mathrm{H} + \frac{\langle x - x_0 \,|\, \nu \rangle}{2(\tau - t)} \right|^2 d\mu_t \tag{4.3}$$

in the time interval $[0, \min\{\tau, T\})$.

2. Type I Singularities and the Rescaling Procedure

In the previous chapter we showed that the curvature must blow up at the maximal time T with the following lower bound

$$\max_{p \in M} |\mathrm{A}(p, t)| \geq \frac{1}{\sqrt{2(T - t)}} \,.$$

DEFINITION 4.6. Let T the maximal time of existence of a mean curvature flow. If there exist a constant $C > 1$ such that we have the upper bound

$$\max_{p \in M} |\mathrm{A}(p, t)| \leq \frac{C}{\sqrt{2(T - t)}} \,,$$

we say that the flow is developing, at time T, a *type I singularity*.
If such constant does not exist, that is,

$$\limsup_{t \to T} \max_{p \in M} |\mathrm{A}(p, t)| \sqrt{T - t} = +\infty$$

we say that we have a *type II singularity*.

In the rest of this chapter we will deal with the first case, the next one will be devoted to type II singularities.
Thus, from now on, we suppose that there is some $C_0 > 1$ such that

$$\frac{1}{\sqrt{2(T-t)}} \leq \max_{p\in M} |\mathrm{A}(p,t)| \leq \frac{C_0}{\sqrt{2(T-t)}}\,, \tag{4.4}$$

for every $t \in [0, T)$.

Let $p \in M$ and $0 \leq t \leq s < T$, then

$$|\varphi(p,s)-\varphi(p,t)| = \left|\int_t^s \frac{\partial\varphi(p,\xi)}{\partial t}\,d\xi\right| \leq \int_t^s |\mathrm{H}(p,\xi)|\,d\xi \leq \int_t^s \frac{C_0\sqrt{n}}{\sqrt{2(T-\xi)}}\,d\xi \leq C_0\sqrt{n(T-t)}$$

which implies that the sequence of functions $\varphi(\cdot,t)$ converges as $t \to T$ to some function $\varphi_T : M \to \mathbb{R}^{n+1}$. Moreover, as the constant C_0 is independent of $p \in M$, such convergence is uniform and the limit function φ_T is continuous. Finally, passing to the limit in the above inequality, we get

$$|\varphi(p,t) - \varphi_T(p)| \leq C_0\sqrt{n(T-t)}\,. \tag{4.5}$$

Often we will denote $\varphi_T(p) = \widehat{p}$.

DEFINITION 4.7. Let $\mathcal{S}$ be the set of points $x \in \mathbb{R}^{n+1}$ such that there exists a sequence of pairs (p_i, t_i) with $t_i \nearrow T$ and $\varphi(p_i, t_i) \to x$.
We call $\mathcal{S}$ the set of *reachable* points.

We have seen in Proposition 3.13 that $\mathcal{S}$ is compact and $x \in \mathcal{S}$ if and only if, for every $t \in [0, T)$ the closed ball of radius $\sqrt{2n(T-t)}$ and center x intersects $\varphi(M, t)$. We show now that $\mathcal{S} = \{\widehat{p}\,|\, p \in M\}$.
Clearly $\{\widehat{p}\,|\, p \in M\} \subset \mathcal{S}$, suppose that $x \in \mathcal{S}$ and $\varphi(p_i, t_i) \to x$, then, by inequality (4.5) we have $|\varphi(p_i,t_i) - \widehat{p}_i| \leq C_0\sqrt{T-t_i}$, hence, $\widehat{p}_i \to x$ as $i \to \infty$. As the set $\{\widehat{p}\,|\, p \in M\}$ is closed we have that x must belong to it.

We define now a tool which will be fundamental in the sequel.

DEFINITION 4.8. For every $p \in M$, we define the *heat density* function

$$\theta(p,t) = \int_M \frac{e^{-\frac{|x-\widehat{p}|^2}{4(T-t)}}}{[4\pi(T-t)]^{n/2}}\,d\mu_t$$

and the *limit heat density* function as

$$\Theta(p) = \lim_{t\to T} \theta(p,t)\,.$$

As M is compact, we can also define the following *maximal heat density*,

$$\sigma(t) = \max_{x_0\in\mathbb{R}^{n+1}} \int_M \frac{e^{-\frac{|x-x_0|^2}{4(T-t)}}}{[4\pi(T-t)]^{n/2}}\,d\mu_t \tag{4.6}$$

and its limit $\Sigma = \lim_{t\to T} \sigma(t)$.

Clearly, $\theta(p,t) \leq \sigma(t)$, for every $p \in M$ and $t \in [0,T)$ and $\Theta(p) \leq \Sigma$ for every $p \in M$. The function Θ is well defined as the limit exists since $\theta(p,t)$ is monotone nonincreasing in t and positive. Moreover, the functions $\theta(\cdot,t)$ are all continuous and monotonically converging to Θ, hence this latter is upper semicontinuous and nonnegative.

The function $\sigma : [0,T) \to \mathbb{R}$ is also positive and monotone nonincreasing, being the maximum of a family of nonincreasing smooth functions, hence the limit Σ is well defined. Moreover, such family is uniformly locally Lipschitz (look at the right hand side of the monotonicity formula), hence also σ is Lipschitz, then by Hamilton's trick 3.3, at every differentiability time $t \in [0,T)$ of σ, we have

$$\sigma'(t) = -\int_M \frac{e^{-\frac{|x-x_t|^2}{4(T-t)}}}{[4\pi(T-t)]^{n/2}} \left| \mathrm{H} + \frac{\langle x - x_t \,|\, \nu \rangle}{2(T-t)} \right|^2 d\mu_t \tag{4.7}$$

where $x_t \in \mathbb{R}^2$ is any *maximum* point such that

$$\sigma(t) = \int_M \frac{e^{-\frac{|x-x_t|^2}{4(T-t)}}}{[4\pi(T-t)]^{n/2}} \, d\mu_t \, .$$

REMARK 4.9. Notice that we did not define $\sigma(t)$ as the maximum

$$\max_{p \in M} \int_M \frac{e^{-\frac{|x-\widehat{p}|^2}{4(T-t)}}}{[4\pi(T-t)]^{n/2}} \, d\mu_t$$

which is *taken among* $p \in M$. Clearly, this latter maximum can be smaller than $\sigma(t)$.

We rescale now the moving hypersurfaces around $\widehat{p} = \lim_{t \to T} \varphi(p,t)$ as follows, following Huisken **[54]**,

$$\widetilde{\varphi}(q,s) = \frac{\varphi(q,t(s)) - \widehat{p}}{\sqrt{2(T-t(s))}} \qquad s = s(t) = -\frac{1}{2}\log(T-t)\, .$$

We now compute the evolution equation of $\widetilde{\varphi}(q,s)$ in the time interval $\left[-\frac{1}{2}\log T, +\infty\right)$,

$$\begin{aligned}
\frac{\partial \widetilde{\varphi}(q,s)}{\partial s} &= \left(\frac{ds}{dt}\right)^{-1} \frac{\partial}{\partial t} \frac{\varphi(q,t) - \widehat{p}}{\sqrt{2(T-t)}} \\
&= \sqrt{2(T-t)} \frac{\partial \varphi(q,t)}{\partial t} + \frac{\varphi(q,t) - \widehat{p}}{\sqrt{2(T-t)}} \\
&= \sqrt{2(T-t)}\,\mathrm{H}(q,t)\nu(q,t) + \widetilde{\varphi}(q,s) \\
&= \widetilde{\mathrm{H}}(q,s)\widetilde{\nu}(q,s) + \widetilde{\varphi}(q,s)\, ,
\end{aligned}$$

where $\widetilde{\mathrm{H}}$ is the mean curvature of the rescaled hypersurfaces $\widetilde{\varphi}$.
As $|\widetilde{\mathrm{A}}| = \sqrt{2(T-t)}|\mathrm{A}| \leq C_0 < +\infty$, all the hypersurfaces $\widetilde{\varphi}(\cdot,s)$ have equibounded

curvatures, moreover,

$$|\widetilde{\varphi}(p,s)| = \left|\frac{\varphi(p,t(s)) - \widehat{p}}{\sqrt{2(T-t(s))}}\right| \leq \frac{C_0\sqrt{2n(T-t(s))}}{\sqrt{2(T-t(s))}} = C_0\sqrt{n}$$

which means that at every time s the ball of radius $C_0\sqrt{2n}$ centered at the origin of $\mathbb{R}^{n+1}$ intersects the hypersurface $\widetilde{\varphi}(\cdot, s)$, to be precise, the point $\widetilde{\varphi}(p,s)$ belongs to the interior of such ball.

We rescale also the monotonicity formula in order to get information on these hypersurfaces.

PROPOSITION 4.10 (Rescaled Monotonicity Formula). *We have*

$$\frac{d}{ds}\int_M e^{-\frac{|y|^2}{2}}\, d\widetilde{\mu}_s = -\int_M e^{-\frac{|y|^2}{2}} \left|\widetilde{\mathrm{H}} + \langle y \,|\, \widetilde{\nu}\rangle\right|^2 d\widetilde{\mu}_s \leq 0 \tag{4.8}$$

which integrated becomes

$$\int_M e^{-\frac{|y|^2}{2}}\, d\widetilde{\mu}_{s_1} - \int_M e^{-\frac{|y|^2}{2}}\, d\widetilde{\mu}_{s_2} = \int_{s_1}^{s_2}\int_M e^{-\frac{|y|^2}{2}} \left|\widetilde{\mathrm{H}} + \langle y \,|\, \widetilde{\nu}\rangle\right|^2 d\widetilde{\mu}_s\, ds\,.$$

In particular,

$$\int_{-\frac{1}{2}\log T}^{+\infty}\int_M e^{-\frac{|y|^2}{2}} \left|\widetilde{\mathrm{H}} + \langle y \,|\, \widetilde{\nu}\rangle\right|^2 d\widetilde{\mu}_s\, ds \leq \int_M e^{-\frac{|y|^2}{2}}\, d\widetilde{\mu}_{-\frac{1}{2}\log T} \leq C < +\infty\,,$$

for a uniform constant $C = C(\mathrm{Area}(\varphi_0), T)$, independent of s and $p \in M$.

PROOF. Keeping in mind that $y = \frac{x-\widehat{p}}{\sqrt{2(T-t)}}$ and $s = -\frac{1}{2}\log(T-t)$ we have,

$$\begin{aligned}
\frac{d}{ds}\int_M e^{-\frac{|y|^2}{2}}\, d\widetilde{\mu}_s &= \left(\frac{ds}{dt}\right)^{-1}\frac{d}{dt}\int_M e^{-\frac{|y|^2}{2}}\, d\widetilde{\mu}_s\\
&= 2(T-t)\frac{d}{dt}\int_M \frac{e^{-\frac{|x-\widehat{p}|^2}{4(T-t)}}}{[2(T-t)]^{n/2}}\, d\mu_t\\
&= -2(T-t)\int_M \frac{e^{-\frac{|x-\widehat{p}|^2}{4(T-t)}}}{[2(T-t)]^{n/2}}\left|\mathrm{H} + \frac{\langle x-\widehat{p}\,|\,\nu\rangle}{2(T-t)}\right|^2 d\mu_t\\
&= -2(T-t)\int_M e^{-\frac{|y|^2}{2}}\left|\frac{\widetilde{\mathrm{H}}}{\sqrt{2(T-t)}} + \frac{\langle y\,|\,\widetilde{\nu}\rangle}{\sqrt{2(T-t)}}\right|^2 d\widetilde{\mu}_s\\
&= -\int_M e^{-\frac{|y|^2}{2}}\left|\widetilde{\mathrm{H}} + \langle y\,|\,\widetilde{\nu}\rangle\right|^2 d\widetilde{\mu}_s\,.
\end{aligned}$$

The other two statements trivially follow. □

As a first consequence, we work out an upper estimate on the volume of the rescaled hypersurfaces in the balls of $\mathbb{R}^{n+1}$.
Fix a radius $R > 0$, if $B_R = B_R(0) \subset \mathbb{R}^{n+1}$, then we have

$$\begin{aligned}\widetilde{\mathcal{H}}^n(\widetilde{\varphi}(M,s) \cap B_R) &= \int_M \chi_{B_R}(y)\, d\widetilde{\mu}_s \\ &\leq \int_M \chi_{B_R}(y) e^{\frac{R^2-|y|^2}{2}}\, d\widetilde{\mu}_s \\ &\leq e^{R^2/2} \int_M e^{-\frac{|y|^2}{2}}\, d\widetilde{\mu}_s \\ &\leq e^{R^2/2} \int_M e^{-\frac{|y|^2}{2}}\, d\widetilde{\mu}_{-\frac{1}{2}\log T} \\ &\leq \widehat{C} e^{R^2/2}\end{aligned} \tag{4.9}$$

where the constant $\widehat{C}$ is independent of R and s.

REMARK 4.11. As

$$\int_M e^{-\frac{|y|^2}{2}}\, d\widetilde{\mu}_{-\frac{1}{2}\log T} = \int_M \frac{e^{-\frac{|x-\widehat{p}|^2}{4T}}}{(2T)^{n/2}}\, d\mu_0 \leq \frac{\mathrm{Area}(\varphi_0)}{(2T)^{n/2}}\,,$$

we can choose the constant $\widehat{C}$ to be also independent of $p \in M$.

Another consequence is the following key technical lemma which is necessary in order to take the limits in the integrals over the sequences of rescaled hypersurfaces.

LEMMA 4.12 (Stone **[91]**). *The following estimates hold.*

(1) *There is a uniform constant $C = C(n, \mathrm{Area}(\varphi_0), T)$ such that, for any $p \in M$, and for all s,*

$$\int_M e^{-|y|}\, d\widetilde{\mu}_s \leq C\,. \tag{4.10}$$

(2) *For any $\varepsilon > 0$ there is a uniform radius $R = R(\varepsilon, n, \mathrm{Area}(\varphi_0), T)$ such that, for any $p \in M$, and for all s,*

$$\int_{\widetilde{\varphi}_s(M)\setminus B_R(0)} e^{-|y|^2/2}\, d\widetilde{\mathcal{H}}^n \leq \varepsilon\,. \tag{4.11}$$

PROOF. By the rescaled monotonicity formula (4.8) we have that, for any $p \in M$, and for all s,

$$\int_M e^{-|y|^2/2}\, d\widetilde{\mu}_s \leq \int_M e^{-|y|^2/2}\, d\widetilde{\mu}_{-\frac{1}{2}\log T}\,.$$

By Remark 4.11, the right hand integral may be estimated by a constant depending only on T and $\mathrm{Area}(\varphi_0)$, and not on p. Hence, in particular, we have the following estimates, for all $p \in M$, and for all s,

$$\int_{\widetilde{\varphi}_s(M)\cap B_{n+1}(0)} e^{-|y|}\, d\widetilde{\mathcal{H}}^n \leq C_1 \tag{4.12}$$

and

$$\int_{\widetilde{\varphi}_s(M)\cap B_{2n+2}(0)} e^{-|y|}\, d\widetilde{\mathcal{H}}^n \leq C_2 \tag{4.13}$$

where C_1 and C_2 are constants determined only by n, T and $\mathrm{Area}(\varphi_0)$.
By the evolution equation for the rescaled hypersurfaces $\widetilde{\varphi}_s$, we can see that $\frac{d}{ds}\widetilde{\mu}_s = (n - \widetilde{\mathrm{H}}^2)\widetilde{\mu}_s$, then we compute, for any p and s,

$$\begin{aligned}
\frac{d}{ds}\int_M e^{-|y|}\, d\widetilde{\mu}_s &= \int_M \Big\{n - \widetilde{\mathrm{H}}^2 - \frac{1}{|y|}\langle y \,|\, \widetilde{\mathrm{H}}\widetilde{\nu} + y\rangle\Big\} e^{-|y|}\, d\widetilde{\mu}_s \\
&\leq \int_M \big\{n - \widetilde{\mathrm{H}}^2 - |y| + |\widetilde{\mathrm{H}}|\big\} e^{-|y|}\, d\widetilde{\mu}_s \\
&< \int_M \big\{n + 1 - |y|\big\} e^{-|y|}\, d\widetilde{\mu}_s \\
&\leq (n+1)\left\{\int_{\widetilde{\varphi}_s(M)\cap B_{n+1}(0)} e^{-|y|}\, d\widetilde{\mathcal{H}}^n - \int_{\widetilde{\varphi}_s(M)\setminus B_{2n+2}(0)} e^{-|y|}\, d\widetilde{\mathcal{H}}^n\right\}.
\end{aligned}$$

But then, noting (4.12), we see that we must have *either*

$$\frac{d}{ds}\int_M e^{-|x|}\, d\widetilde{\mu}_s < 0\,,$$

or

$$\int_{\widetilde{\varphi}_s(M)\setminus B_{2n+2}(0)} e^{-|y|}\, d\widetilde{\mathcal{H}}^n \leq C_1\,.$$

Hence, in view of inequality (4.13), we must have that, for any $p \in M$, and for all s, *either*

$$\frac{d}{ds}\int_M e^{-|y|}\, d\widetilde{\mu}_s < 0\,,$$

or

$$\int_M e^{-|y|}\, d\widetilde{\mu}_s \leq C_1 + C_2\,,$$

which immediately implies that, for any p and s,

$$\int_M e^{-|y|}\, d\widetilde{\mu}_s \leq \max\left\{(C_1 + C_2), \int_M e^{-|y|}\, d\widetilde{\mu}_{-\frac{1}{2}\log T}\right\} = C_3\,.$$

The proof of part (1) is now completed by noting that the integral quantity on the right hand side can clearly be estimated by a constant depending on T and $\mathrm{Area}(\varphi_0)$, but not on p.

Let again $p \in M$ and $s \in \left[-\frac{1}{2}\log T, +\infty\right)$ arbitrary. Now subdivide $\widetilde{\varphi}_s(M)$ into "annular pieces", $\{\widetilde{M}^k_s\}_{k=0}^{\infty}$, by setting

$$\widetilde{M}^0_s = \widetilde{\varphi}_s(M) \cap B_1(0)\,,$$

and, for each $k \geq 1$,

$$\widetilde{M}^k_s = \big\{y \in \widetilde{\varphi}_s(M) \,|\, 2^{k-1} \leq |y| < 2^k\big\}\,.$$

Then, by part (1) of the lemma, $\widetilde{\mathcal{H}}^n(\widetilde{M}^k_s) \leq C_3 e^{(2^k)}$, for each k, independently of the choice of p and s. Hence in turn, for each k, we have

$$\int_{\widetilde{M}^k_s} e^{-|y|^2/2}\, d\widetilde{\mathcal{H}}^n \leq C_3 e^{-\frac{1}{2}(2^{k-1})^2} e^{(2^k)} = C_3 e^{(2^k - 2^{2k-3})}$$

again independently of the choice of p and s.
For any $\varepsilon > 0$, we can find a $k_0 = k_0(\varepsilon, n, \mathrm{Area}(\varphi_0), T)$, such that

$$\sum_{k=k_0}^{\infty} C_3 e^{(2^k - 2^{2k-3})} \leq \varepsilon\,,$$

then, if $R = R(\varepsilon, n, \mathrm{Area}(\varphi_0), T)$ is simply taken to be equal to 2^{k_0-1}, we have

$$\int_{\widetilde{\varphi}_s(M)\setminus B_R(0)} e^{-|y|^2/2}\, d\widetilde{\mathcal{H}}^n = \sum_{k=k_0}^{\infty} \int_{\widetilde{M}^k_s} e^{-|y|^2/2}\, d\widetilde{\mathcal{H}}^n \leq \sum_{k=k_0}^{\infty} C_3 e^{(2^k - 2^{2k-3})} \leq \varepsilon$$

and we are done also with part (2) of the lemma. □

COROLLARY 4.13. *If a sequence of rescaled hypersurfaces $\widetilde{\varphi}(\cdot, s_i)$ locally smoothly converges (up to reparametrization) to some limit hypersurface $\widetilde{M}_\infty$, we have*

$$\int_{\widetilde{M}_\infty} e^{-|y|}\, d\widetilde{\mathcal{H}}^n \leq C \tag{4.14}$$

and

$$\int_{\widetilde{M}_\infty} e^{-\frac{|y|^2}{2}}\, d\widetilde{\mathcal{H}}^n = \lim_{i\to\infty} \int_M e^{-\frac{|y|^2}{2}}\, d\widetilde{\mu}_{s_i}\,,$$

where the constant C is the same of the previous lemma.

PROOF. Actually, it is only sufficient that the associated measures $\widetilde{\mathcal{H}}^n \llcorner \widetilde{\varphi}(M, s_i)$ weakly*–converge to the measure $\widetilde{\mathcal{H}}^n \llcorner \widetilde{M}_\infty$. Indeed, for every $R > 0$ we have,

$$\int_{\widetilde{M}_\infty \cap B_R(0)} e^{-|y|}\, d\widetilde{\mathcal{H}}^n \leq \liminf_{i\to\infty} \int_{\widetilde{\varphi}(M,s_i)\cap B_R(0)} e^{-|y|}\, d\widetilde{\mathcal{H}}^n \leq \liminf_{i\to\infty} \int_M e^{-|y|}\, d\widetilde{\mu}_{s_i} \leq C$$

by the first part of the lemma above. Sending R to $+\infty$, the first inequality follows.
The second statement is a consequence of the estimate of the second part of the lemma. □

Now we want to estimate the covariant derivatives of the rescaled hypersurfaces.

PROPOSITION 4.14 (Huisken [**54**]). *There exist constants C_k depending only on n, k, C_0 (see formula (4.4)) and the initial hypersurface such that $|\widetilde{\nabla}^k \widetilde{\mathrm{A}}|_{\widetilde{g}} \leq C_k$ for every time $s \in \left[-\frac{1}{2}\log T, +\infty\right)$.*

PROOF. By Proposition 3.19 we have for the original flow,

$$\frac{\partial}{\partial t}|\nabla^k \mathrm{A}|^2 = \Delta|\nabla^k \mathrm{A}|^2 - 2|\nabla^{k+1}\mathrm{A}|^2 + \sum_{p+q+r=k \,|\, p,q,r\in\mathbb{N}} \nabla^p \mathrm{A} * \nabla^q \mathrm{A} * \nabla^r \mathrm{A} * \nabla^k \mathrm{A}\,,$$

hence, with a straightforward computation, noticing that $|\widetilde{\nabla}^k \widetilde{\mathrm{A}}|^2_{\widetilde{g}} = |\nabla^k \mathrm{A}|^2_g [2(T-t)]^{k+1}$, we get

$$\begin{aligned}\frac{\partial}{\partial s}|\widetilde{\nabla}^k \widetilde{\mathrm{A}}|^2_{\widetilde{g}} \leq & -2(1+k)|\widetilde{\nabla}^k \widetilde{\mathrm{A}}|^2_{\widetilde{g}} + \widetilde{\Delta}|\widetilde{\nabla}^k \widetilde{\mathrm{A}}|^2_{\widetilde{g}} - 2|\widetilde{\nabla}^{k+1}\widetilde{\mathrm{A}}|^2_{\widetilde{g}}\\ &+ C(n,k) \sum_{p+q+r=k \,|\, p,q,r\in\mathbb{N}} |\widetilde{\nabla}^p \widetilde{\mathrm{A}}|_{\widetilde{g}} |\widetilde{\nabla}^q \widetilde{\mathrm{A}}|_{\widetilde{g}} |\widetilde{\nabla}^r \widetilde{\mathrm{A}}|_{\widetilde{g}} |\widetilde{\nabla}^k \widetilde{\mathrm{A}}|_{\widetilde{g}}\end{aligned}$$

As $|\widetilde{\mathrm{A}}|_{\widetilde{g}}$ is bounded by the constant C_0, supposing by induction that up to the order $(k-1)$ we have uniform bounds on all the covariant derivatives of $\widetilde{\mathrm{A}}$ with constants $C_i = C_i(n, C_0)$, we can conclude by means of Peter–Paul inequality,

$$\frac{\partial}{\partial s}|\widetilde{\nabla}^k \widetilde{\mathrm{A}}|^2_{\widetilde{g}} \leq \widetilde{\Delta}|\widetilde{\nabla}^k \widetilde{\mathrm{A}}|^2_{\widetilde{g}} + B_k|\widetilde{\nabla}^k \widetilde{\mathrm{A}}|^2_{\widetilde{g}} - 2|\widetilde{\nabla}^{k+1}\widetilde{\mathrm{A}}|^2_{\widetilde{g}} + C(n,k,C_0)$$

for some constant B_k depending only on n and k.
Then,

$$\begin{aligned}\frac{\partial}{\partial s}(|\widetilde{\nabla}^k \widetilde{\mathrm{A}}|^2_{\widetilde{g}} + B_k|\widetilde{\nabla}^{k-1} \widetilde{\mathrm{A}}|^2_{\widetilde{g}}) \leq & \,\widetilde{\Delta}|\widetilde{\nabla}^k \widetilde{\mathrm{A}}|^2_{\widetilde{g}} + B_k|\widetilde{\nabla}^k \widetilde{\mathrm{A}}|^2_{\widetilde{g}} - 2|\widetilde{\nabla}^{k+1}\widetilde{\mathrm{A}}|^2_{\widetilde{g}}\\ &+ B_k\widetilde{\Delta}|\widetilde{\nabla}^{k-1} \widetilde{\mathrm{A}}|^2_{\widetilde{g}} + B_kB_{k-1}|\widetilde{\nabla}^{k-1} \widetilde{\mathrm{A}}|^2_{\widetilde{g}} - 2B_k|\widetilde{\nabla}^k \widetilde{\mathrm{A}}|^2_{\widetilde{g}}\\ &+ C(n,k,C_0) + C(n,k-1,C_0)\\ \leq & \,\widetilde{\Delta}(|\widetilde{\nabla}^k \widetilde{\mathrm{A}}|^2_{\widetilde{g}} + B_k|\widetilde{\nabla}^{k-1} \widetilde{\mathrm{A}}|^2_{\widetilde{g}}) - B_k|\widetilde{\nabla}^k \widetilde{\mathrm{A}}|^2_{\widetilde{g}}\\ &+ C(n,k,C_0) + B_kB_{k-1}|\widetilde{\nabla}^{k-1} \widetilde{\mathrm{A}}|^2_{\widetilde{g}} + C(n,k-1,C_0)\\ \leq & \,\widetilde{\Delta}(|\widetilde{\nabla}^k \widetilde{\mathrm{A}}|^2_{\widetilde{g}} + B_k|\widetilde{\nabla}^{k-1} \widetilde{\mathrm{A}}|^2_{\widetilde{g}}) - B_k|\widetilde{\nabla}^k \widetilde{\mathrm{A}}|^2_{\widetilde{g}}\\ &+ C(n,k,C_0) + B_kB_{k-1}C^2_{k-1}(n,C_0) + C(n,k-1,C_0)\\ \leq & \,\widetilde{\Delta}(|\widetilde{\nabla}^k \widetilde{\mathrm{A}}|^2_{\widetilde{g}} + B_k|\widetilde{\nabla}^{k-1} \widetilde{\mathrm{A}}|^2_{\widetilde{g}}) - B_k(|\widetilde{\nabla}^k \widetilde{\mathrm{A}}|^2_{\widetilde{g}} + B_k|\widetilde{\nabla}^{k-1} \widetilde{\mathrm{A}}|^2_{\widetilde{g}})\\ &+ C(n,k,C_0)\end{aligned}$$

where we used the inductive hypothesis $|\widetilde{\nabla}^{k-1}\widetilde{\mathrm{A}}|_{\widetilde{g}} \leq C_{k-1}(n,C_0)$.
By maximum principle, the function $|\widetilde{\nabla}^k \widetilde{\mathrm{A}}|^2_{\widetilde{g}} + B_k|\widetilde{\nabla}^{k-1} \widetilde{\mathrm{A}}|^2_{\widetilde{g}}$ is then uniformly bounded in space and time by a constant depending on n, k, C_0 and the initial hypersurface. Again, by the inductive hypothesis, the thesis follows. □

We are ready to study the convergence of the rescaled hypersurfaces as $s \to +\infty$.

PROPOSITION 4.15. *For every point $p \in M$ and sequence of times $s_i \to +\infty$ there exists a subsequence (not relabeled) of times such that the hypersurfaces $\widetilde{\varphi}(\cdot, s_i)$, rescaled around $\widehat{p}$, locally smoothly converge (up to reparametrization) to some nonempty, smooth limit hypersurface*

$\widetilde{M}_\infty$ such that $\widetilde{\mathrm{H}} + \langle y \,|\, \widetilde{\nu}\rangle = 0$ at every $y \in \widetilde{M}_\infty$.
Any limit hypersurface satisfies $\widetilde{\mathcal{H}}^n(\widetilde{M}_\infty \cap B_R) \leq C_R$ for every ball of radius R in $\mathbb{R}^{n+1}$ and, for every $k \in \mathbb{N}$, there are constants C_k such that $|\widetilde{\nabla}^k \widetilde{\mathrm{A}}|_{\widetilde{g}} \leq C_k$.
Moreover, if the initial hypersurface was embedded, $\widetilde{M}_\infty$ is embedded.

PROOF. We give a sketch of the proof, following Huisken [**54**].
By estimate (4.9) there is a uniform bound on $\mathcal{H}^n(\widetilde{\varphi}(M,s) \cap B_R)$ for each R, independent of s. Moreover, by the uniform control on the norm of the second fundamental form of the rescaled hypersurfaces in Proposition 4.14, there is a number $r_0 > 0$ such that, for each s and each $q \in M$, if $U^s_{r_0,q}$ is the connected component of $\widetilde{\varphi}_s^{-1}(B_{r_0}(\widetilde{\varphi}_s(q)))$ in M containing q, then $\widetilde{\varphi}_s(U^s_{r_0,q})$ can be written as a graph of a smooth function f over the tangent hyperplane to $\widetilde{\varphi}_s(M) \subset \mathbb{R}^{n+1}$ at the point $\widetilde{\varphi}_s(q)$ in $B_{r_0}(\widetilde{\varphi}_s(q))$.
The estimates of Proposition 4.14 then imply that all the derivatives of such function f up to the order α are bounded by constants C_α independent of s.
Following now the method in [**69**] we can see that, for each $R > 0$, a subsequence of the hypersurfaces $\widetilde{\varphi}(M,s) \cap B_R(0)$ must converge smoothly to a limit hypersurface in $B_R(0)$. The existence of a limit hypersurface now follows from a diagonal argument, letting $R \to +\infty$, and recalling the fact that every rescaled hypersurface intersects the ball of radius $C_0\sqrt{2n}$ centered at $0 \in \mathbb{R}^{n+1}$, so the limit cannot be empty. The estimates on the volume and derivatives of the curvature follow from the analogous properties for the converging sequence.
The fact that any such limit hypersurface must satisfy $\widetilde{\mathrm{H}} + \langle y \,|\, \widetilde{\nu}\rangle = 0$ is a consequence of the rescaled monotonicity formula and of the uniform estimates on curvature and its covariant derivatives for the rescaled hypersurfaces. Indeed,

$$\begin{aligned}
\frac{\partial}{\partial s}\left|\widetilde{\mathrm{H}} + \langle y \,|\, \widetilde{\nu}\rangle\right|^2 &= \left(2\widetilde{\Delta}\widetilde{\mathrm{H}} + \widetilde{\mathrm{H}} + \widetilde{\mathrm{H}}|\widetilde{\mathrm{A}}|^2 + \langle \widetilde{\mathrm{H}}\widetilde{\nu} + y \,|\, \widetilde{\nu}\rangle - \langle y \,|\, \widetilde{\nabla}\widetilde{\mathrm{H}}\rangle\right)\left|\widetilde{\mathrm{H}} + \langle y \,|\, \widetilde{\nu}\rangle\right| \\
&\leq (|y| + C)(|y| + C) \\
&\leq |y|^2 + C
\end{aligned}$$

for a constant C as in Proposition 4.14, that is, independent of s.
Then,

$$\begin{aligned}
\frac{d}{ds}\int_M e^{-\frac{|y|^2}{2}}\left|\widetilde{\mathrm{H}} + \langle y \,|\, \widetilde{\nu}\rangle\right|^2 d\widetilde{\mu}_s & \qquad (4.15)\\
= \int_M e^{-\frac{|y|^2}{2}}\left[\left|\widetilde{\mathrm{H}} + \langle y \,|\, \widetilde{\nu}\rangle\right|^2\left(n - \widetilde{\mathrm{H}}^2 - \langle y \,|\, \widetilde{\mathrm{H}}\widetilde{\nu} + y\rangle\right) + \frac{\partial}{\partial s}\left|\widetilde{\mathrm{H}} + \langle y \,|\, \widetilde{\nu}\rangle\right|^2\right] d\widetilde{\mu}_s & \\
\leq \int_M e^{-\frac{|y|^2}{2}}\left[\left|\widetilde{\mathrm{H}} + \langle y \,|\, \widetilde{\nu}\rangle\right|^2(|y|^2 + C) + |y|^2 + C\right] d\widetilde{\mu}_s & \\
\leq \int_M e^{-\frac{|y|^2}{2}}(|y|^4 + C)\, d\widetilde{\mu}_s & \\
& \qquad (4.16)
\end{aligned}$$

and this last term is bounded uniformly in $s \in \left[-\frac{1}{2}\log T, +\infty\right)$ by a positive constant $C = C(n, \mathrm{Area}(\varphi_0), T)$, using the estimates in Stone's Lemma 4.12.
Supposing there is a sequence of times $s_i \to +\infty$ such that $\int_M e^{-\frac{|y|^2}{2}} \left|\widetilde{\mathrm{H}} + \langle y \,|\, \widetilde{\nu}\rangle\right|^2 d\widetilde{\mu}_{s_i} \geq \delta$ for some $\delta > 0$, then we have that in all the intervals $[s_i, s_i + \delta/(2C))$ such integral is larger than $\delta/2$. This is clearly in contradiction with the fact that

$$\int_{-\frac{1}{2}\log T}^{+\infty} \int_M e^{-\frac{|y|^2}{2}} \left|\widetilde{\mathrm{H}} + \langle y \,|\, \widetilde{\nu}\rangle\right|^2 d\widetilde{\mu}_s \, ds \leq C(\mathrm{Area}(\varphi_0), T) < +\infty\,,$$

stated in Proposition 4.10.
If $\widetilde{\varphi}_{s_i}$ is a (locally) smoothly converging subsequence of rescaled hypersurfaces, we have that for every ball B_R

$$\int_{\widetilde{\varphi}(M,s_i)\cap B_R} e^{-\frac{|y|^2}{2}} \left|\widetilde{\mathrm{H}} + \langle y \,|\, \widetilde{\nu}\rangle\right|^2 d\widetilde{\mathcal{H}}^n \leq \int_M e^{-\frac{|y|^2}{2}} \left|\widetilde{\mathrm{H}} + \langle y \,|\, \widetilde{\nu}\rangle\right|^2 d\widetilde{\mu}_{s_i} \to 0\,,$$

hence, the limit hypersurface satisfies $\widetilde{\mathrm{H}} + \langle y \,|\, \widetilde{\nu}\rangle = 0$ at every of its points.

Suppose now that the initial hypersurface was embedded, then by Proposition 3.14, all the hypersurfaces $\widetilde{\varphi}_s$ are embedded and the only possibility for $\widetilde{M}_\infty$ not to be embedded is if two or more of its regions "touch" each other at some point $y \in \mathbb{R}^{n+1}$ with a common tangent space.
We consider the following set $\Omega \subset M \times M \times [0, T)$ given by $\{(p, q, t) \,|\, d_{g(t)}(p, q) \leq \varepsilon\sqrt{2(T-t)}\}$, where $d_{g(t)}$ is the geodesic distance in the Riemannian manifold $(M, g(t))$.
Let

$$C = \inf_{\partial\Omega} |\varphi(p,t) - \varphi(q,t)|/\sqrt{2(T-t)}$$

and suppose that $C = 0$, whatever small $\varepsilon > 0$ we take. This means that there exists a sequence of times $t_i \nearrow T$ and points p_i, q_i with $d_{g(t_i)}(p_i, q_i) = \varepsilon\sqrt{2(T-t_i)}$ and $|\varphi(p_i, t_i) - \varphi(q_i, t_i)|/\sqrt{2(T-t_i)} \to 0$, that is $|\widetilde{\varphi}(p_i, s_i) - \widetilde{\varphi}(q_i, s_i)| \to 0$ and $d_{\widetilde{g}(s_i)}(p_i, q_i) = \varepsilon$, where we rescaled the hypersurfaces around $\varphi(p_i)$. Reasoning like in the first part of this proof, by the uniform bound on the second fundamental form of the rescaled hypersurfaces, the connected component of $\widetilde{\varphi}_{s_i}(p_i)$ in $\widetilde{\varphi}_{s_i} \cap B_{r_0}(\widetilde{\varphi}_{s_i}(p_i))$ is locally the graph of some smooth function f_i on some hyperplane passing through $\widetilde{\varphi}_{s_i}(p_i)$.
As $d_{\widetilde{g}_{s_i}}(p_i, q_i) = \varepsilon$, if $\varepsilon > 0$ is small (depending on r_0 and C_0), the Lipschitz constants of these functions f_i are uniformly bounded by a constant depending on r_0 and C_0, moreover, for every $i \in \mathbb{N}$ the point $\widetilde{\varphi}_{s_i}(q_i)$ belongs to the graph of f_i.
It is then easy to see that there exists a uniform bound from below on $|\widetilde{\varphi}_{s_i}(p_i) - \widetilde{\varphi}_{s_i}(q_i)|$, hence the constant C cannot be zero.
Now, if we look at the function

$$L(p,q,t) = |\varphi(p,t) - \varphi(q,t)|/\sqrt{2(T-t)}$$

on $\complement\Omega \subset M \times M \times [0, T)$, we can see as before in the proof of Proposition 3.14, that if the minimum of L at time t is lower than ε, then such minimum is not taken on the boundary and the analogous geometric argument says that it is nondecreasing. Hence, there is a positive lower bound on

$$\inf_{\complement\Omega} |\varphi(p,t) - \varphi(q,t)|/\sqrt{2(T-t)}\,.$$

Now we are done, because if a limit hypersurface of the rescaling procedure is not embedded, representing the family of converging hypersurfaces locally around a couple of points with the same image in $\mathbb{R}^{n+1}$ by the limit immersion, we would have a contradiction with this argument. □

OPEN PROBLEM 4.16. The limit hypersurface $\widetilde{M}_\infty$ is unique? That is, independent of the sequence $s_i \to +\infty$?

This problem is the parabolic analog to the long–standing problem of uniqueness of the tangent cone in minimal surface theory.

OPEN PROBLEM 4.17. Classify all the hypersurfaces (compact or not) satisfying $\mathrm{H} + \langle y \,|\, \nu\rangle = 0$, or at least the ones arising from blow up of compact and embedded flows.
This problem is difficult, an equivalent formulation is to find the critical points of the Huisken's functional

$$\int_M e^{-\frac{|y|^2}{2}}\, d\mathcal{H}^n\,.$$

Besides the "standard" examples given by the hyperplanes for the origin, the sphere $\mathbb{S}^n(\sqrt{n})$ or one of the cylinders $\mathbb{S}^m(\sqrt{m}) \times \mathbb{R}^{n-m}$, we have the Angenent torus in **[13]**, moreover, the computations of Chopp **[19]** suggest that there could also exist higher genus surfaces in $\mathbb{R}^3$, see the notes of Ilmanen **[66]** (see also the works by Nguyen **[77, 52, 76]**).
As we will see in the next sections, the classification is possible under the extra hypothesis $\mathrm{H} \geq 0$.

REMARK 4.18. These hypersurfaces are often called "homothetic solutions of mean curvature flow", indeed, if a hypersurface $M = M_0 \subset \mathbb{R}^{n+1}$ satisfies $\mathrm{H} + \langle y \,|\, \nu\rangle = 0$, it is easy to see that the flow $M_t = M\sqrt{1-2t}$ is an ancient smooth flow by mean curvature in the time interval $(-\infty, 1/2)$, homothetically shrinking. The viceversa is also true, as a homothetically shrinking flow, collapsing at some time T, must satisfy $\mathrm{H} + \langle y \,|\, \nu\rangle = 0$ at time $t = T - 1/2$, by looking at the monotonicity formula (Exercise).
Notice that if a hypersurface moving by mean curvature satisfies $\mathrm{H} + \lambda\langle y \,|\, \nu\rangle = 0$ for some constant $\lambda > 0$ at some time, then it is homothetically shrinking, that is, such a condition is stable (with a time changing constant λ).

By looking at a slight modification of the function σ defined by formula 4.6 it is possible to exclude the existence of compact nonhomothetic *breathers* for mean curvature flow, that is, solutions such that $M_t = \lambda \mathrm{L}(M_s)$ for a couple of times $t > s$, a constant $\lambda > 0$ and an isometry L of $\mathbb{R}^{n+1}$.
It is useless to consider nonshrinking (steady or expanding) compact breathers with $\lambda = 1$ or $\lambda > 1$, indeed, by comparison with evolving spheres, they simply do not exist.

We can suppose that $s = 0$ and $t > 0$, then, for a compact hypersurface M and $\tau > 0$, we consider the function

$$\widetilde{\sigma}(M,\tau) = \max_{x_0\in\mathbb{R}^{n+1}} \int_M \frac{e^{-\frac{|x-x_0|^2}{4\tau}}}{(4\pi\tau)^{n/2}}\, d\widetilde{\mathcal{H}}^n\,.$$

It is easy to see that for every $\lambda > 0$ we have

$$\sigma(\lambda M, \lambda^2\tau) = \sigma(M,\tau) \tag{4.17}$$

and, by the same argument following Definition 4.8, for every $A > 0$, we have

$$\sigma(M_0,\tau(0)) - \sigma(M_t,\tau(t)) = \int_0^t \int_{M_s} \frac{e^{-\frac{|x-x_{\tau(s)}|^2}{4\tau(s)}}}{(4\pi\tau(s))^{n/2}} \left|\mathrm{H} - \frac{(x-x_{\tau(s)})^\perp}{2\tau(s)}\right|^2 d\widetilde{\mathcal{H}}^n, ds$$

where $\tau(s) = A - s$.
By the rescaling property of σ in formula 4.17, we have

$$\sigma(M_0, A) \geq \sigma(M_t, A-t) = \sigma(\lambda M_0, A-t) = \sigma(M_0, (A-t)/\lambda^2)$$

hence, if we choose $A = \frac{t}{1-\lambda^2} > t$ as $\lambda < 1$, we have $(A-t)/\lambda^2 = A$. It follows that $\sigma(M_0, A) = \sigma(M_t, A-t)$, hence, by the formula above, there exists at least one time $s \in (0,t)$ such that $\mathrm{H}(x,s) = \frac{(x-y)^\perp}{2(A-s)}$ for some $y \in \mathbb{R}^{n+1}$, which implies that we are dealing with a homothetically shrinking solution.

Suppose to fix a point $p \in M$ and consider a sequence of rescaled hypersurfaces, as above, $\widetilde{\varphi}(\cdot, s_i)$, locally smoothly converging (up to reparametrization) to some limit hypersurface $\widetilde{M}_\infty$ which satisfies $\widetilde{\mathrm{H}} + \langle y\,|\,\widetilde{\nu}\rangle = 0$ at every $y \in \widetilde{M}_\infty$.

We now relate the limit heat density $\Theta(p)$ in Definition 4.8 with the limits of rescaled hypersurfaces.

$$\begin{aligned}
\Theta(p) &= \lim_{t\to T} \theta(p,t)\\
&= \lim_{i\to\infty} \int_M \frac{e^{-\frac{|x-\widehat{p}|^2}{4(T-t(s_i))}}}{[4\pi(T-t(s_i))]^{n/2}}\, d\mu_{t(s_i)}\\
&= \lim_{i\to\infty} \int_M \frac{e^{-\frac{|y|^2}{2}}}{(2\pi)^{n/2}}\, d\widetilde{\mu}_{s_i}\\
&= \frac{1}{(2\pi)^{n/2}} \int_{\widetilde{M}_\infty} e^{-\frac{|y|^2}{2}}\, d\widetilde{\mathcal{H}}^n\,,
\end{aligned}$$

where in the last passage we applied the previous corollary.
In particular, if $\widetilde{M}_\infty$ is a unit multiplicity hyperplane for the origin of $\mathbb{R}^{n+1}$, then $\Theta(p) = \frac{1}{(2\pi)^{n/2}} \int_{\widetilde{M}_\infty} e^{-\frac{|y|^2}{2}}\, d\widetilde{\mathcal{H}}^n = 1$.

REMARK 4.19. If we choose a time $\tau > 0$ which is strictly less than the maximal time T of existence of the flow and we perform the rescaling procedure around the *nonsingular* point $\widehat{p} = \lim_{t\to\tau} \varphi(p,t) = \varphi(p,\tau)$, being the hypersurface regular around p at time τ, every limit of rescaled hypersurfaces must be flat, actually a union of hyperplanes through the origin. If moreover at $\varphi(p,\tau)$ the hypersurface has no self–intersections, such limit is a single hyperplane through the origin and

$$\lim_{t\to\tau} \int_M \frac{e^{-\frac{|x-\varphi(p,\tau)|^2}{4(\tau-t)}}}{[4\pi(\tau-t)]^{n/2}}\,d\mu_t = 1\,.$$

This clearly holds for every $p \in M$ if the initial hypersurface is embedded.

REMARK 4.20. By the previous remark, if $\tau \in (0,T)$ and $x_0 = \varphi_\tau(p)$, we have

$$\lim_{t\to\tau} \int_M \frac{e^{-\frac{|x-x_0|^2}{4(\tau-t)}}}{[4\pi(\tau-t)]^{n/2}}\,d\mu_t = 1$$

and

$$\int_M \frac{e^{-\frac{|x-x_0|^2}{4\tau}}}{[4\pi\tau]^{n/2}}\,d\mu_0 \geq 1$$

by monotonicity formula, for every $p \in M$.
Then,

$$\mathrm{Area}(\varphi_0) \geq \int_M e^{-\frac{|x-x_0|^2}{4\tau}}\,d\mu_0 \geq [4\pi\tau]^{n/2}$$

and $\tau \leq [\mathrm{Area}]^{2/n}/(4\pi)$. As this holds for every $\tau < T$, we conclude with the following estimate on the maximal time (independent of the type I singularity hypothesis) $T \leq [\mathrm{Area}]^{2/n}/(4\pi)$.

LEMMA 4.21 (White **[98]**). *Among all the smooth, complete, hypersurfaces M in $\mathbb{R}^{n+1}$, satisfying $\mathrm{H} + \langle y \,|\, \nu\rangle = 0$ and $\int_M e^{-|y|}\,d\widetilde{\mathcal{H}}^n < +\infty$, the hyperplanes (with unit multiplicity) through the origin are the only minimizers of the functional*

$$\frac{1}{(2\pi)^{n/2}} \int_M e^{-\frac{|y|^2}{2}}\,d\widetilde{\mathcal{H}}^n\,.$$

Hence, for all such hypersurfaces the value of this integral is at least 1.

PROOF. Suppose that there exists a smooth hypersurface $M = M_0$ such that

$$\frac{1}{(2\pi)^{n/2}} \int_M e^{-\frac{|y|^2}{2}}\,d\widetilde{\mathcal{H}}^n \leq 1$$

and satisfies $\mathrm{H} + \langle y \,|\, \nu\rangle = 0$, then the flow $M_t = M\sqrt{1-2t}$ is a smooth flow by mean curvature in the time interval $(-\infty, 1/2)$.
Choosing a point $y_0 \in \mathbb{R}^{n+1}$ and a time $\tau \leq 1/2$ we consider the limit

$$\lim_{t\to-\infty} \int_{M_t} \frac{e^{-\frac{|y-y_0|^2}{4(\tau-t)}}}{[4\pi(\tau-t)]^{n/2}}\,d\widetilde{\mathcal{H}}^n\,,$$

notice that the integral is well defined by the hypothesis $\int_M e^{-|y|}\,d\widetilde{\mathcal{H}}^n < +\infty$.
Changing variables, we have

$$\lim_{t\to-\infty}\int_{M_t}\frac{e^{-\frac{|y-y_0|^2}{4(\tau-t)}}}{[4\pi(\tau-t)]^{n/2}}\,d\widetilde{\mathcal{H}}^n(y)=\lim_{t\to-\infty}\int_{M}\frac{e^{-\frac{|x\sqrt{1-2t}-y_0|^2}{4(\tau-t)}}}{[4\pi(\tau-t)/(1-2t)]^{n/2}}\,d\widetilde{\mathcal{H}}^n(x)\,.$$

As $t\to-\infty$, the sequence of functions inside the integral pointwise converges to the function $e^{-\frac{|x|^2}{2}}/(2\pi)^{n/2}$ and they are definitely uniformly bounded above, outside some large fixed ball $B_R(0)\subset\mathbb{R}^{n+1}$, by the function $e^{-|x|}$. Since this last function is integrable on M by the hypothesis, by the dominated convergence theorem, we get

$$\lim_{t\to-\infty}\int_{M_t}\frac{e^{-\frac{|y-y_0|^2}{4(\tau-t)}}}{[4\pi(\tau-t)]^{n/2}}\,d\widetilde{\mathcal{H}}^n=\frac{1}{(2\pi)^{n/2}}\int_M e^{-\frac{|x|^2}{2}}\,d\widetilde{\mathcal{H}}^n\leq 1$$

By the monotonicity formula this implies that

$$\int_{M_t}\frac{e^{-\frac{|y-y_0|^2}{4(\tau-t)}}}{[4\pi(\tau-t)]^{n/2}}\,d\widetilde{\mathcal{H}}^n\leq 1$$

for every $y_0\in\mathbb{R}^{n+1}$ and $t<\tau\in(-\infty,1/2)$.
Choosing now $y_0\in M$ and $\tau=0$, repeating the argument in Remark 4.19 (in this noncompact case it can be carried on by means of the hypothesis $\int_M e^{-|y|}\,d\widetilde{\mathcal{H}}^n<+\infty$), we have

$$\lim_{t\to 0}\int_{M_t}\frac{e^{-\frac{|y-y_0|^2}{-4t}}}{[-4\pi t]^{n/2}}\,d\widetilde{\mathcal{H}}^n=1$$

hence, we conclude that the function

$$\int_{M_t}\frac{e^{-\frac{|y-y_0|^2}{-4t}}}{[-4\pi t]^{n/2}}\,d\widetilde{\mathcal{H}}^n$$

is constant equal to 1 for every $t\in(-\infty,0)$. Even if the evolving hypersurfaces M_t are not compact, by the hypothesis $\int_M e^{-|y|}\,d\widetilde{\mathcal{H}}^n<+\infty$, it is straightforward to check that the monotonicity formula still holds (writing every integral as an integral on M fixed). Hence, we must have that the right side is identically zero and $\mathrm{H}+\langle y-y_0\,|\,\nu\rangle=0$ for every $y\in M$.
Since this relation holds for every $y_0\in M$, it follows that for every $x,y\in M$ we have $\langle x\,|\,\nu(y)\rangle=0$, which easily implies that M is a hyperplane for the origin of $\mathbb{R}^{n+1}$. □

REMARK 4.22. The smoothness hypothesis can be weakened in this lemma, provided that the set M owns some definition of mean curvature to give sense to the condition $\mathrm{H}+\langle y\,|\,\nu\rangle=0$ and coherent with the monotonicity formula (for instance, one can allows integral varifolds with bounded variation, see **[17, 65]**).

It is unknown to the author if the hypothesis $\int_M e^{-|y|}\,d\widetilde{\mathcal{H}}^n < +\infty$ can be removed. Anyway, it is satisfied by every limit hypersurface obtained by the rescaling procedure, by Corollary 4.13.

COROLLARY 4.23. *The function* $\Theta : M \to \mathbb{R}$ *satisfies* $\Theta \geq 1$ *on all* M. *Moreover, if* $\Theta(p) = 1$, *every converging sequence of rescaled hypersurfaces* $\widetilde{\varphi}(\cdot, s_i)$ *around* $\widehat{p}$ *converges to a unit multiplicity hyperplane for the origin of* $\mathbb{R}^{n+1}$.
It follows that $\Sigma \geq 1$.

REMARK 4.24. The fact that $\Theta \geq 1$ on all M can be also proved directly by the argument in Remark 4.20. Since for every $\tau < T$ and $x_0 = \varphi_\tau(p)$ we have

$$\lim_{t\to\tau} \int_M \frac{e^{-\frac{|x-x_0|^2}{4(\tau-t)}}}{[4\pi(\tau-t)]^{n/2}}\,d\mu_t = 1$$

we get

$$\int_M \frac{e^{-\frac{|x-x_0|^2}{4(\tau-t)}}}{[4\pi(\tau-t)]^{n/2}}\,d\mu_t \geq 1$$

for every $t < \tau$. Keeping $t < T$ fixed and sending $\tau \to T$, we have $\varphi_\tau(p) \to \widehat{p}$ and

$$\theta(p,t) = \int_M \frac{e^{-\frac{|x-\widehat{p}|^2}{4(T-t)}}}{[4\pi(T-t)]^{n/2}}\,d\mu_t = \lim_{\tau\to T} \int_M \frac{e^{-\frac{|x-x_0|^2}{4(\tau-t)}}}{[4\pi(\tau-t)]^{n/2}}\,d\mu_t \geq 1\,.$$

This clearly implies that $\Theta(p) = \lim_{t\to T} \theta(p,t) \geq 1$.

REMARK 4.25. Rescaling around some $\widehat{p}$, by the discussion after Definition 4.7, means rescaling around some *reachable* point. Actually, we could rescale around *any* point $x_0 \in \mathbb{R}^{n+1}$ but if $x_0 \notin \mathcal{S}$, as the distance from $\varphi(M,t)$ and x_0 is definitely positive, the limit hypersurface is empty. This would imply that

$$\int_M \frac{e^{-\frac{|x-x_0|^2}{4(T-t)}}}{[4\pi(T-t)]^{n/2}}\,d\mu_t \to 0\,,$$

as $t \to T$.
By this corollary, if instead we consider $x_0 \in \mathcal{S}$, that is, $x_0 = \widehat{p}$ for some $p \in M$, there holds $\Theta(p) \geq 1$. Hence, there is a dichotomy between the points of $\mathbb{R}^{n+1}$, according to the value of the limit heat density function which can be either zero or larger or equal to one.
Moreover, by looking carefully at the first part of the proof of Lemma 4.21, we can see that this fact is independent of the type I hypothesis (it is only a consequence of the upper semicontinuity of $\theta(p,t)$).

Actually, one can say more by the following result of White **[98]** (see also **[26**, Thm. 5.6, 5.7] and **[91]**, moreover compare with **[17**, Thm. 6.11]), which also gives a partial answer to Problem 4.16.

THEOREM 4.26 (White **[98]**). *There exist constants $\varepsilon = \varepsilon(n) > 0$ and $C = C(n)$ such that if $\Theta(p) < 1 + \varepsilon$, then $|\mathrm{A}| \leq C(n)$ in a ball of $\mathbb{R}^{n+1}$ around $\widehat{p}$, uniformly in time $t \in [0, T)$.*

If the limit of a subsequence of rescalings is a hyperplane through the origin, then $\Theta(p) = 1$ and (by the conclusion of the theorem) there is ball around $\widehat{p}$ where the curvature is bounded. Then in such a ball, the *unscaled* hypersurfaces φ_t (possibly after a reparametrization) converge locally uniformly in C^0 to some φ_T with uniformly bounded curvature, this implies that the convergence is actually C^∞, by the interior estimates of Ecker and Huisken in **[29]**. Hence, it follows easily that the tangent hyperplane to φ_T at the point $\widehat{p}$ (after translating it to the origin of $\mathbb{R}^{n+1}$) coincides with the limit of *any* sequence of rescaled hypersurfaces, that is, there is full convergence of the sequence of rescalings, hence, the limit hypersurface is unique, solving affirmatively Problem 4.16 in this special case.

REMARK 4.27. The strength of White's result is that it does not assume any condition on the blow up rate of the curvature. The theorem holds also without the type I hypothesis.

Another consequence is that there is a "gap" between the value 1 realized by the hyperplanes through the origin of $\mathbb{R}^{n+1}$ in the functional

$$\frac{1}{(2\pi)^{n/2}} \int_M e^{-\frac{|y|^2}{2}} \, d\widetilde{\mathcal{H}}^n$$

and any other smooth, complete, hypersurface M in $\mathbb{R}^{n+1}$, satisfying $\mathrm{H} + \langle y \,|\, \nu \rangle = 0$ and $\int_M e^{-|y|} \, d\widetilde{\mathcal{H}}^n < +\infty$.

3. Analysis of Singularities

DEFINITION 4.28. We say that $p \in M$ is a *singular point* if there exists a sequence of points $p_i \to p$ in M and times $t_i \to T$ such that for some constant $\delta > 0$ there holds

$$|\mathrm{A}(p_i, t_i)| \geq \frac{\delta}{\sqrt{2(T - t_i)}} \,.$$

We say that $p \in M$ is a *special singular point* if there exists a sequence of times $t_i \to T$ such that for some constant $\delta > 0$ there holds

$$|\mathrm{A}(p, t_i)| \geq \frac{\delta}{\sqrt{2(T - t_i)}} \,.$$

Suppose that $p \in M$ is a special singular point, then, after rescaling the hypersurface as before around $\widehat{p}$, we have for $s_i = -\frac{1}{2} \log{(T - t_i)}$,

$$|\widetilde{\mathrm{A}}(p, s_i)| = \sqrt{2(T - t_i)} |\mathrm{A}(p, t_i)| \geq \delta > 0$$

which implies that, taking a subsequence of $s_i \to +\infty$, any limit hypersurface obtained by Proposition 4.15 cannot be flat as $\widetilde{\mathrm{A}} \neq 0$ at some point in the ball $B_{C_0\sqrt{2n}}$.

If $p \in M$ is not a special singular point, clearly, for every sequence $s_i \to +\infty$,

$$|\widetilde{\mathrm{A}}(p, s_i)| = \sqrt{2(T - t_i)} |\mathrm{A}(p, t_i)| \to 0,$$

that is, any limit hypersurface satisfies $\widetilde{\mathrm{A}} = 0$ at some point in the ball $B_{C_0\sqrt{2n}}$.

OPEN PROBLEM 4.29. Any limit hypersurface associated to a nonspecial singular point is a union of hyperplanes through the origin?
This conclusion would follows if any nonflat hypersurface M satisfying $\mathrm{H} + \langle y \,|\, \nu\rangle = 0$ and $\int_M e^{-|y|}\,d\widetilde{\mathcal{H}}^n < +\infty$ cannot have a point where the second fundamental form is zero.

By means of a small variation of an argument by Stone, we have a good description when the limit hypersurface is a single hyperplane.

PROPOSITION 4.30 (Stone **[91]**). *If the limit of rescaled hypersurfaces around $\widehat{p}$, is a unit multiplicity hyperplane through the origin of $\mathbb{R}^{n+1}$, or equivalently by Lemma 4.21, $\Theta(p) = 1$, then p cannot be a singular point.*

PROOF. By Corollary 4.23, the point $p \in M$ is a minimum of $\Theta : M \to \mathbb{R}$ which is an upper semicontinuous function. Hence, p is actually a continuity point for Θ. We want to show that for every sequence $p_i \to p$ and $t_i \nearrow T$ we have $\theta(p_i, t_i) \to 1 = \Theta(p)$.
Suppose there exists $\delta > 0$ such that $\theta(p_i, t_i) \to 1 + \delta$, then, keeping fixed i, for every $j > i$ we have $\theta(p_j, t_j) \leq \theta(p_j, t_i)$ and sending $j \to \infty$ we get $1 + \delta \leq \theta(p, t_i)$. This is clearly a contradiction, as sending now $i \to \infty$, we have $\theta(p, t_i) \to \Theta(p) = 1$ (what we did is closely related to Dini's Theorem on monotone convergence of functions).
If p is a singular point with $p_i \to p$ and $t_i \to T$ such that for some constant $\delta > 0$ there holds $|\mathrm{A}(p_i, t_i)| \geq \frac{\delta}{\sqrt{2(T-t_i)}}$ we consider the families of rescaled hypersurfaces around $\widehat{p}_i$,

$$\widetilde{\varphi}_i(q, s) = \frac{\varphi(q, t) - \widehat{p}_i}{\sqrt{2(T-t)}} \qquad s = s(t) = -\frac{1}{2}\log(T-t)$$

with associated measures $\widetilde{\mu}_{i,s}$, and we set

$$\psi_i(q) = \widetilde{\varphi}_i(q, s_i) = \frac{\varphi(q, t_i) - \widehat{p}_i}{\sqrt{2(T-t_i)}} \qquad s_i = -\frac{1}{2}\log(T-t_i)\,,$$

with associated measures $\widetilde{\mu}_{i,s_i}$.
For every $\varepsilon > 0$, definitely

$$\begin{aligned}
\varepsilon \geq \theta(p_i, t_i) - 1 \geq\, & \theta(p_i, t_i) - \Theta(p_i)\\
&= \int_M \frac{e^{-\frac{|x-\widehat{p}_i|^2}{4(T-t_i)}}}{[4\pi(T-t_i)]^{n/2}}\,d\mu_{t_i} - \Theta(p_i)\\
&= \frac{1}{(2\pi)^{n/2}}\int_M e^{-\frac{|y|^2}{2}}\,d\widetilde{\mu}_{i,s_i} - \Theta(p_i)\\
&\geq \int_{s_i}^{+\infty}\int_M e^{-\frac{|y|^2}{2}}\left|\widetilde{\mathrm{H}} + \langle y \,|\, \widetilde{\nu}\rangle\right|^2 d\widetilde{\mu}_{i,s}\,ds\,.
\end{aligned}$$

Hence, since by the uniform curvature estimates of Proposition 4.14, see computation 4.15, we have,

$$\frac{d}{ds}\int_M e^{-\frac{|y|^2}{2}}\left|\widetilde{\mathrm{H}}+\langle y\,|\,\widetilde{\nu}\rangle\right|^2 d\widetilde{\mu}_s\leq C$$

where $C=C(n,\mathrm{Area}(\varphi_0),T)$ is a positive constant independent of s, we get

$$\begin{aligned}\varepsilon &\geq \int_{s_i}^{+\infty}\int_M e^{-\frac{|y|^2}{2}}\left|\widetilde{\mathrm{H}}+\langle y\,|\,\widetilde{\nu}\rangle\right|^2 d\widetilde{\mu}_{i,s}\,ds\\ &\geq \frac{1}{C^2}\left(\int_M e^{-\frac{|y|^2}{2}}\left|\widetilde{\mathrm{H}}+\langle y\,|\,\widetilde{\nu}\rangle\right|^2 d\widetilde{\mu}_{i,s_i}\right)^2.\end{aligned}$$

If now we proceed like in Proposition 4.15 and we extract from the sequence ψ_i a smoothly converging subsequence to some limit hypersurface $\widetilde{M}_\infty$, by Lemma 4.12, we have

$$\varepsilon\geq\int_{\widetilde{M}_\infty} e^{-\frac{|y|^2}{2}}\left|\widetilde{\mathrm{H}}+\langle y\,|\,\widetilde{\nu}\rangle\right|^2 d\widetilde{\mathcal{H}}^n\,,$$

for every $\varepsilon>0$, that is, $\widetilde{M}_\infty$ also satisfies $\widetilde{\mathrm{H}}+\langle y\,|\,\widetilde{\nu}\rangle=0$.
Finally, by Corollary 4.13,

$$\frac{1}{(2\pi)^{n/2}}\int_{\widetilde{M}_\infty} e^{-\frac{|y|^2}{2}}\,d\widetilde{\mathcal{H}}^n=\lim_{i\to\infty}\frac{1}{(2\pi)^{n/2}}\int_M e^{-\frac{|y|^2}{2}}\,d\widetilde{\mu}_{i,s_i}=\lim_{i\to\infty}\theta(p_i,t_i)=1$$

then, by Lemma 4.21, the hypersurface $\widetilde{M}_\infty$ has to be a hyperplane. But since the points $\psi_i(p_i)$ all belong to the ball of radius $C_0\sqrt{2n}\subset\mathbb{R}^{n+1}$ and the second fundamental form A^i of ψ_i satisfies $|\mathrm{A}^i(p_i)|\geq\delta>0$, for every $i\in\mathbb{N}$, by construction, passing to the limit, the second fundamental form of $\widetilde{M}_\infty$ is not zero at some point in the ball $B_{C_0\sqrt{2n}}(0)$.
Then, we get a contradiction and p cannot be a singular point of the flow. □

REMARK 4.31. This lemma is an immediate consequence of White's Theorem 4.26, but we wanted to emphasize the fact that the only needed "ingredient" by the line of analysis of Stone is the uniqueness of the hyperplanes as minimizers the integral $\frac{1}{(2\pi)^{n/2}}\int_M e^{-\frac{|y|^2}{2}}\,d\widetilde{\mathcal{H}}^n$.

The lower estimate on the blow up rate of the curvature

$$\max_{p\in M}|\mathrm{A}(p,t)|\geq\frac{1}{\sqrt{2(T-t)}}\,.$$

and the compactness of M clearly imply that there always exists at least one singular point but do not imply that there exists at least one *special* singular point.

OPEN PROBLEM 4.32. To our knowledge, in the general case, even if we are dealing with the flow of embedded hypersurfaces, the existence of at least one special singular point is an open problem.
A related stronger statement would be *every singular point is a special singular point*.

This problem and Problem 4.29, in the embedded situation at least, are strongly connected. Indeed, if the initial hypersurface is embedded, any limit hypersurface $\widetilde{M}_\infty$ is also embedded, so the union of two or more hyperplanes cannot arise.
This means that if Problem 4.29 has a positive answer, for every nonspecial singular point $p \in M$ the limit hypersurfaces can be only single unit multiplicity hyperplanes through the origin, hence, by Proposition 4.30, the point is not singular.
If there are not special singular point then it would follows that there are no singular points at all, which is a contradiction.

In the general case, repeating this argument, unfortunately, one could obtain a union of hyperplanes, or even more disturbing, higher integer multiplicity hyperplanes. Hence, a possible flat limit.

COROLLARY 4.33. *At a singular point $p \in M$ a limit $\widetilde{M}_\infty$ of rescaled hypersurfaces is not empty, satisfies* $\mathrm{H} + \langle x \,|\, \nu \rangle = 0$ *and it is not a unit multiplicity hyperplane through the origin. In the embedded case $\widetilde{M}_\infty$ cannot be flat.*

REMARK 4.34. Another line to produce a homothetic blow up limit, is to apply, instead of Stone's argument, White's Theorem 4.26, excluding the presence of singularities in the case $\Sigma = 1$ (recall Definition 4.8). As the set of reachable points $\mathcal{S}$ is compact, if $\Sigma > 1$ there must exists a point $x_0 = \widehat{p}$ such that $\Theta(p) > 1$, otherwise, by a covering argument, White's Theorem would imply that the curvature is uniformly bounded as $t \nearrow T$ and Σ would be 1.

REMARK 4.35. Finally, we can also obtain a homothetic limit by rescaling the hypersurfaces around *moving* points as follows. Rescaling the *maximal* monotonicity formula (4.7) around the points x_t which are the maximum points realizing $\sigma(t)$ in Definition 4.8, that is,

$$\sigma(t) = \max_{x_0 \in \mathbb{R}^{n+1}} \int_M \frac{e^{-\frac{|x-x_0|^2}{4(T-t)}}}{[4\pi(T-t)]^{n/2}}\, d\mu_t = \int_M \frac{e^{-\frac{|x-x_t|^2}{4(T-t)}}}{[4\pi(T-t)]^{n/2}}\, d\mu_t\,,$$

where now the rescaled hypersurfaces with associated measures $\widetilde{\mu}_s$ are given by

$$\widetilde{\varphi}(q,s) = \frac{\varphi(q,t(s)) - x_{t(s)}}{\sqrt{2(T-t(s))}} \qquad s = s(t) = -\frac{1}{2}\log(T-t)\,,$$

we get

$$\frac{d}{ds}\int_M e^{-\frac{|y|^2}{2}}\, d\widetilde{\mu}_s = -\int_M e^{-\frac{|y|^2}{2}} \left|\widetilde{\mathrm{H}} + \langle y \,|\, \widetilde{\nu}\rangle\right|^2 d\widetilde{\mu}_s \le 0\,.$$

It follows that, integrating this formula as before, we get

$$\sigma(0) - \Sigma = \int_{-\frac{1}{2}\log T}^{+\infty} \int_M e^{-\frac{|y|^2}{2}} \left|\widetilde{\mathrm{H}} + \langle y \,|\, \widetilde{\nu}\rangle\right|^2 d\widetilde{\mu}_s\, ds < +\infty\,,$$

and with the same argument we can produce a homothetic limit hypersurface $\widetilde{M}_\infty$ such that

$$\int_{\widetilde{M}_\infty} e^{-\frac{|y|^2}{2}}\, d\widetilde{\mathcal{H}}^n = \Sigma \geq 1\,.$$

Since when $\Sigma = 1$ the curvature is bounded, the limit hypersurface $\widetilde{M}_\infty$ cannot be a single hyperplane for the origin. If the initial hypersurface was embedded, this limit also cannot be flat.

4. Embedded Hypersurfaces with Nonnegative Mean Curvature

If the compact initial hypersurface is embedded and has $\mathrm{H} \geq 0$ (this condition is often called *mean convexity*) or at some time the evolving hypersurface achieve it, then the analysis of the previous section can be pushed forward, since we have a new condition that all the possible limits of rescaled hypersurfaces have to satisfy. Under such hypothesis, Problem 4.29 and consequently Problem 4.32 have a satisfying solution.
Actually, in this class, *every* singular point is a special singular point and it is indeed possible to classify all the embedded hypersurfaces in $\mathbb{R}^{n+1}$ such that $\mathrm{H} + \langle x \,|\, \nu\rangle = 0$ and $\mathrm{H} \geq 0$, arising from a rescaling around a singularity point, see **[54, 66]** or **[92]**.

We recall here that in this case, after a positive time $\varepsilon > 0$, there exist a constant $\alpha > 0$ such that $\alpha|\mathrm{A}| \leq \mathrm{H} \leq n|\mathrm{A}|$ everywhere on M for every time $t \geq \varepsilon$, Corollary 3.22.
Hence, for every $t \in [\varepsilon, T)$ we have

$$\frac{\alpha}{\sqrt{2(T-t)}} \leq \max_{p\in M} \mathrm{H}(p,t) \leq \frac{C}{\sqrt{2(T-t)}}\,.$$

PROPOSITION 4.36 (Huisken **[54, 66]**). *Let $M \subset \mathbb{R}^{n+1}$ be a mean convex, smooth, embedded hypersurface in $\mathbb{R}^{n+1}$ such that $\mathrm{H} + \langle x \,|\, \nu\rangle = 0$ at every $x \in M$ and there exists a constant C such that $|\mathrm{A}|, |\nabla \mathrm{A}| \leq C$ and $\mathcal{H}^n(M \cap B_R) \leq Ce^R$, for every ball of radius $R > 0$ in $\mathbb{R}^{n+1}$. Then, up to rotation in $\mathbb{R}^{n+1}$, M must be one of only $(n+1)$ possible hypersurfaces, namely, either a hyperplane for the origin, or the sphere $\mathbb{S}^n(\sqrt{n})$ or one of the cylinders $\mathbb{S}^m(\sqrt{m}) \times \mathbb{R}^{n-m}$.*

In the special one–dimensional case the only embedded smooth curves in $\mathbb{R}^2$ satisfying the structural equation $k + \langle x \,|\, \nu\rangle = 0$ are the lines through the origin and the unit circle.

PROOF. Let us assume that M is connected. If the theorem is true in this case, it it easy to see that it is not possible to have a nonconnected embedded hypersurface satisfying the hypotheses. Indeed, any connected component has to belong to the list of the statement and every two hypersurfaces in such list either coincide or have some intersections.

We deal separately with the case $n = 1$.
Fixing a reference point on a curve γ we have an arclength parameter s which gives a unit tangent vector field and a unit normal vector field ν which is the rotation of $\pi/2$ in $\mathbb{R}^2$ of the vector τ. Then, it follows that $k = \langle \partial_s \tau \,|\, \nu\rangle$.
The relation $k = -\langle \gamma \,|\, \nu\rangle$ implies $k_s = k\langle \gamma \,|\, \tau\rangle$. Suppose that at some point $k = 0$, then also $k_s = 0$ at the same point, hence, by the uniqueness theorem applied to this ODE for the curvature k we can conclude that k is identically zero and we are dealing with a line

L, which then, as $\langle x \,|\, \nu \rangle = 0$ for every $x \in L$, it must pass for the origin of $\mathbb{R}^2$.
So we suppose that k is always nonzero and possibly reversing the orientation of the curve we can also assume that $k > 0$ at every point, that is, the curve is strictly convex.
Computing the derivative of $|\gamma|^2$,

$$\partial_s |\gamma|^2 = 2\langle \gamma \,|\, \tau \rangle = 2k_s/k = 2\partial_s \log k$$

we get $k = Ce^{|\gamma|^2/2}$ for some constant $C > 0$, so if k is bounded above and below away from zero, the curve is also bounded in $\mathbb{R}^2$, hence, it is compact being embedded. As a consequence, it is closed.
We consider now a new coordinate $\theta = \arccos \langle e_1 \,|\, \nu \rangle$, this can be done globally as we know that the curve is convex.
Then, we have $\partial_s \theta = k$ and

$$k_\theta = k_s/k = \langle \gamma \,|\, \tau \rangle \qquad k_{\theta\theta} = \frac{\partial_s k_\theta}{k} = \frac{1 + k\langle \gamma \,|\, \nu \rangle}{k} = \frac{1}{k} - k\,.$$

Multiplying both sides for $2k_\theta$ we get $\partial_\theta[k_\theta^2 + k^2 - \log k^2] = 0$, that is, the quantity $k_\theta^2 + k^2 - \log k^2$ is equal to some constant E along all the curve. Notice that such quantity E cannot be less than 1, moreover, if $E = 1$ we have k constant equal to one and the curve must be the unit circle.
For other values of $E > 1$ it is easy to see, as the function, $x - \log x$ is convex, that k must be bounded above and below away from zero, hence, by what we said before the curve is a simple closed curve.
We look now at the critical points of the curvature k, they must be isolated (hence finite) and non degenerate ($k_{\theta\theta} \neq 0$), otherwise the ODE $k_{\theta\theta} = \frac{1}{k} - k$ implies that k_θ is identically zero, k is constant and again we are dealing with the unit circle.
Suppose now that k_- and k_+ are a pair of consecutive critical values of k, hence the two distinct positive zeroes of the function $E + \log k^2 - k^2$ when $E > 1$.
We have that the change $\Delta\theta$ in the angle θ along the piece of curve from the points corresponding to k_- and k_+ on γ is given by the integral

$$I(E) = \int_{k_-}^{k_+} \frac{dk}{\sqrt{E - k^2 + \log k^2}}\,.$$

As the four vertex theorem **[73, 79]** says that there are at least four critical points of k on the curve, there must be at least four pieces like the one above, hence, the total change in the angle θ along the curve must be at least $4I(E)$.
As the curve γ is simple, the total change must be 2π, so we have $4I(E) \leq 2\pi$, that is,

$$I(E) = \int_{k_-}^{k_+} \frac{dk}{\sqrt{E - k^2 + \log k^2}} \leq \pi/2\,.$$

The analysis of Abresch and Langer in **[1]** (see also the work of Epstein and Weinstein **[31]**) shows that $I(E)$ is always strictly larger than $\pi/2$ for every $E > 1$, which is a contradiction and γ must be a circle.

Actually, Abresch and Langer (and also Epstein and Weinstein) classify *all* the closed curves in $\mathbb{R}^2$ satisfying the structural equation $k + \langle \gamma \,|\, \nu \rangle = 0$.
We remark that, like for other results, the one–dimensional case does not follows from the general one below. Moreover, even if the study of the integral $I(E)$ is done with elementary tools, the proof of the inequality $I(E) > \pi/2$ is quite involved making definitely nontrivial this classification result even for simple closed curves (we underline that the n–dimensional generalization, Problem 4.37, is open).

Suppose now that $n \geq 2$.
By covariant differentiation of the equation $\mathrm{H} + \langle x \,|\, \nu \rangle = 0$ in an orthonormal frame $\{e_1, \dots, e_n\}$ on M we get

$$\nabla_i \mathrm{H} = \langle x \,|\, e_k \rangle h_{ik}$$

$$\nabla_i \nabla_j \mathrm{H} = h_{ij} + \langle x \,|\, \nu \rangle h_{ik} h_{jk} + \langle x \,|\, e_k \rangle \nabla_i h_{jk} = h_{ij} - \mathrm{H} h_{ik} h_{jk} + \langle x \,|\, e_k \rangle \nabla_k h_{ij} \tag{4.18}$$

where we used Codazzi and Gauss–Weingarten equations.
Contracting now with g^{ij} and h^{ij} respectively we have

$$\Delta \mathrm{H} = \mathrm{H} - \mathrm{H}|\mathrm{A}|^2 + \langle x \,|\, e_k \rangle \nabla_k \mathrm{H} = \mathrm{H}(1 - |\mathrm{A}|^2) + \langle x \,|\, \nabla \mathrm{H} \rangle \tag{4.19}$$

$$h^{ij} \nabla_i \nabla_j \mathrm{H} = |\mathrm{A}|^2 - \mathrm{H}\mathrm{tr}(\mathrm{A}^3) + \langle x \,|\, e_k \rangle \nabla_k |\mathrm{A}|^2/2$$

which implies, by Simons' identity (2.3),

$$\Delta |\mathrm{A}|^2 = 2|\mathrm{A}|^2(1 - |\mathrm{A}|^2) + 2|\nabla \mathrm{A}|^2 + \langle x \,|\, \nabla |\mathrm{A}|^2 \rangle\,. \tag{4.20}$$

From equation (4.19) and the strong maximum principle for elliptic equations we see that, since M satisfies $\mathrm{H} \geq 0$ by assumption and $\Delta \mathrm{H} \leq \mathrm{H} + \langle x \,|\, \nabla \mathrm{H} \rangle$, we must either have that $\mathrm{H} \equiv 0$ or $\mathrm{H} > 0$ on all M.
Of these two possibilities the situation that $\mathrm{H} \equiv 0$ is easily handled: as x is tangent vector field on M, by the equation $\langle x \,|\, \nu \rangle = 0$, there is a solution of the ODE $\gamma'(t) = x(\gamma(t)) = \gamma(t)$ in M for $t \in \mathbb{R}$, but the solution is simply the line passing by x and the origin in $\mathbb{R}^{n+1}$, so M has to be a cone in $\mathbb{R}^{n+1}$. Being M smooth, the only possibility is that M is a hyperplane through the origin of $\mathbb{R}^{n+1}$.
Therefore we may assume henceforth, as we do, that the mean curvature satisfies the strict inequality $\mathrm{H} > 0$ everywhere (so that division by H and $|\mathrm{A}|$ is allowed).
Now let $R > 0$ and define η to be the inward unit conormal to $M \cap B_R(0)$ along $\partial\big(M \cap B_R(0)\big)$, which is a smooth boundary for almost every $R > 0$ (by Sard's theorem). Then, supposing that R is a *regular value* for the function $|x|$ on M, from equation (4.19) and

the divergence theorem, we obtain

$$\begin{aligned}\varepsilon_R &= -\int_{\partial(M\cap B_R(0))} |\mathrm{A}|\langle\nabla\mathrm{H}\,|\,\eta\rangle e^{-R^2/2}\,d\mathcal{H}^{n-1} \\ &= \int_{M\cap B_R(0)} |\mathrm{A}|\Delta\mathrm{H}e^{-|x|^2/2} + \langle\nabla(|\mathrm{A}|e^{-|x|^2/2})\,|\,\nabla\mathrm{H}\rangle\,d\mathcal{H}^n \\ &= \int_{M\cap B_R(0)} |\mathrm{A}|\mathrm{H}(1-|\mathrm{A}|^2)e^{-|x|^2/2} + |\mathrm{A}|\langle x\,|\,\nabla\mathrm{H}\rangle e^{-|x|^2/2}\,d\mathcal{H}^n \\ &\quad + \int_{M\cap B_R(0)} \frac{1}{2|\mathrm{A}|}\langle\nabla|\mathrm{A}|^2\,|\,\nabla\mathrm{H}\rangle e^{-|x|^2/2} - |\mathrm{A}|\langle x\,|\,\nabla\mathrm{H}\rangle e^{-|x|^2/2}\,d\mathcal{H}^n \\ &= \int_{M\cap B_R(0)} \Big(|\mathrm{A}|\mathrm{H}(1-|\mathrm{A}|^2) + \frac{1}{2|\mathrm{A}|}\langle\nabla|\mathrm{A}|^2\,|\,\nabla\mathrm{H}\rangle\Big)\,e^{-|x|^2/2}\,d\mathcal{H}^n\,.\end{aligned} \tag{4.21}$$

Similarly,

$$\begin{aligned}\delta_R &= -\int_{\partial(M\cap B_R(0))} \frac{\mathrm{H}}{|\mathrm{A}|}\langle\nabla|\mathrm{A}|^2\,|\,\eta\rangle e^{-R^2/2}\,d\mathcal{H}^{n-1} \\ &= \int_{M\cap B_R(0)} \frac{\mathrm{H}}{|\mathrm{A}|}\Delta|\mathrm{A}|^2 e^{-|x|^2/2} + \Big\langle\nabla\Big(\frac{\mathrm{H}}{|\mathrm{A}|}|e^{-|x|^2/2}\Big)\,\Big|\,\nabla|\mathrm{A}|^2\Big\rangle\,d\mathcal{H}^n \\ &= \int_{M\cap B_R(0)} 2|\mathrm{A}|\mathrm{H}(1-|\mathrm{A}|^2)e^{-|x|^2/2} + \frac{2\mathrm{H}|\nabla\mathrm{A}|^2}{|\mathrm{A}|}e^{-|x|^2/2} + \frac{\mathrm{H}}{|\mathrm{A}|}\langle x\,|\,\nabla|\mathrm{A}|^2\rangle e^{-|x|^2/2}\,d\mathcal{H}^n \\ &\quad + \int_{M\cap B_R(0)} \frac{\langle\nabla\mathrm{H}\,|\,\nabla|\mathrm{A}|^2\rangle}{|\mathrm{A}|}e^{-|x|^2/2} - \frac{\mathrm{H}|\nabla|\mathrm{A}|^2|^2}{2|\mathrm{A}|^3}e^{-|x|^2/2} - \frac{\mathrm{H}}{|\mathrm{A}|}\langle x\,|\,\nabla|\mathrm{A}|^2\rangle e^{-|x|^2/2}\,d\mathcal{H}^n \\ &= \int_{M\cap B_R(0)} \Big(2|\mathrm{A}|\mathrm{H}(1-|\mathrm{A}|^2) + \frac{2\mathrm{H}|\nabla\mathrm{A}|^2}{|\mathrm{A}|} + \frac{\langle\nabla\mathrm{H}\,|\,\nabla|\mathrm{A}|^2\rangle}{|\mathrm{A}|} - \frac{\mathrm{H}|\nabla|\mathrm{A}|^2|^2}{2|\mathrm{A}|^3}\Big)\,e^{-|x|^2/2}\,d\mathcal{H}^n\,.\end{aligned} \tag{4.22}$$

Hence,

$$\begin{aligned}\sigma_R = 2\delta_R - 4\varepsilon_R &= \int_{M\cap B_R(0)} \Big(\frac{4\mathrm{H}|\nabla\mathrm{A}|^2}{|\mathrm{A}|} - \frac{\mathrm{H}|\nabla|\mathrm{A}|^2|^2}{|\mathrm{A}|^3}\Big)\,e^{-|x|^2/2}\,d\mathcal{H}^n \\ &= \int_{M\cap B_R(0)} (4|\mathrm{A}|^2|\nabla\mathrm{A}|^2 - |\nabla|\mathrm{A}|^2|^2)\frac{\mathrm{H}}{|\mathrm{A}|^3}e^{-|x|^2/2}\,d\mathcal{H}^n\,.\end{aligned} \tag{4.23}$$

As we have $4|\mathrm{A}|^2|\nabla\mathrm{A}|^2 \geq |\nabla|\mathrm{A}|^2|^2$, this quantity σ_R is nonnegative and nondecreasing in R.
If now we show that $\liminf_{R\to+\infty}\sigma_R = 0$ (on the set of regular values) we can conclude that at every point of M,

$$4|\mathrm{A}|^2|\nabla\mathrm{A}|^2 = |\nabla|\mathrm{A}|^2|^2\,. \tag{4.24}$$

We have,

$$\begin{aligned}
|\sigma_R| &= \left| -2\int_{\partial(M\cap B_R(0))} \frac{\mathrm{H}}{|\mathrm{A}|}\langle \nabla|\mathrm{A}|^2\,|\,\eta\rangle e^{-R^2/2}\,d\mathcal{H}^{n-1} + 4\int_{\partial(M\cap B_R(0))} |\mathrm{A}|\langle \nabla \mathrm{H}\,|\,\eta\rangle e^{-R^2/2}\,d\mathcal{H}^{n-1}\right| \\
&\leq 4e^{-R^2/2}\int_{\partial(M\cap B_R(0))} \frac{\mathrm{H}}{|\mathrm{A}|}|\nabla|\mathrm{A}|^2| + |\mathrm{A}||\nabla \mathrm{H}|\,d\mathcal{H}^{n-1} \\
&\leq 8e^{-R^2/2}\int_{\partial(M\cap B_R(0))} \mathrm{H}|\nabla \mathrm{A}| + |\mathrm{A}||\nabla \mathrm{H}|\,d\mathcal{H}^{n-1} \\
&\leq Ce^{-R^2/2}\mathcal{H}^{n-1}(\partial(M\cap B_R(0)))\,,
\end{aligned}$$

by the estimates on A and $\nabla\mathrm{A}$ in the hypotheses.
Now, suppose that definitely on the set of regular values in $\mathbb{R}^+$ we have

$$\mathcal{H}^{n-1}(\partial(M\cap B_R(0))) \geq \delta R e^{R^2/4}$$

for some constant $\delta > 0$, for every $R > r_1$. Setting x^M to be the projection of the vector x on the tangent space to M, as the function $R \mapsto \mathcal{H}^n(M\cap B_R(0))$ is monotone and continuous from the left and actually continuous at every regular value of $|x|$ on M, we can differentiate it almost everywhere in $\mathbb{R}^+$ and we have (by the coarea formula, see **[32]** or **[84]**),

$$\begin{aligned}
\mathcal{H}^n(M\cap B_R(0)) - \mathcal{H}^n(M\cap B_r(0)) &\geq \int_r^R \frac{d}{d\xi}\mathcal{H}^n(M\cap B_\xi(0))\,d\xi \\
&\geq \int_r^R \int_{\partial(M\cap B_\xi(0))} |\nabla^M|x||^{-1}\,d\mathcal{H}^{n-1}\,d\xi \\
&= \int_r^R \int_{\partial(M\cap B_\xi(0))} |x|/|x^M|\,d\mathcal{H}^{n-1}\,d\xi \\
&\geq \int_r^R \int_{\partial(M\cap B_\xi(0))} d\mathcal{H}^{n-1}\,d\xi\,,
\end{aligned}$$

where the derivative in the integral is taken only at the points where it exists and $\nabla^M|x|$ denotes the projection of the gradient of the function $|x|$ on the tangent space to M.
Hence, if r is larger than r_1,

$$\begin{aligned}
\mathcal{H}^n(M\cap B_R(0)) - \mathcal{H}^n(M\cap B_r(0)) &\geq \int_r^R \int_{\partial(M\cap B_\xi(0))} d\mathcal{H}^{n-1}\,d\xi \\
&\geq \delta\int_r^R \xi e^{\xi^2/4}\,d\xi \\
&= 2\delta(e^{R^2/4} - e^{r^2/4})
\end{aligned}$$

so if R goes to $+\infty$, the quantity $\mathcal{H}^n(M\cap B_R(0))e^{-R}$ diverges, contradicting the hypotheses in the statement. Hence, the $\liminf$ on the set of regular values as R goes to $+\infty$ of the quantity $e^{-R^2/4}\mathcal{H}^{n-1}(\partial(M\cap B_R(0)))$ has to be zero. It follows the same for σ_R and equation (4.24) holds.

Making explicit such equation, by the equality condition in the Cauchy–Schwartz inequality, it immediately follows that, fixing k, at every point there exists a constant c_k such that

$$\nabla_k h_{ij} = c_k h_{ij}$$

for every i, j. Tracing with the metric and with h^{ij}, we get $\nabla_k \mathrm{H} = c_k\mathrm{H}$ and $\nabla_k|\mathrm{A}|^2 = 2c_k|\mathrm{A}|^2$, hence $c_k = \nabla_k \log \mathrm{H}$ and $\nabla_k \log|\mathrm{A}|^2 = 2c_k = 2\nabla_k \log \mathrm{H}$.
This implies that locally $|\mathrm{A}| = \alpha\mathrm{H}$ for some constant $\alpha > 0$, by connectedness, this relation has to hold globally on M.
Suppose now that at a point $|\nabla\mathrm{H}| \neq 0$, then, $\nabla_k h_{ij} = c_k h_{ij} = \frac{\nabla_k \mathrm{H}}{\mathrm{H}} h_{ij}$ which is a symmetric 3–tensor by Codazzi equations, hence, $\nabla_k \mathrm{H} h_{ij} = \nabla_j \mathrm{H} h_{ik}$. Computing then in normal coordinates, with an orthonormal basis $\{e_1 \dots, e_n\}$ such that $e_1 = \nabla\mathrm{H}/|\nabla\mathrm{H}|$ we have

$$0 = |\nabla_k \mathrm{H} h_{ij} - \nabla_j \mathrm{H} h_{ik}|^2 = 2|\nabla\mathrm{H}|^2 \left(|\mathrm{A}|^2 - \sum_{i=1}^n h_{1i}^2\right).$$

Hence, $|\mathrm{A}|^2 = \sum_i^n h_{1i}^2$ then

$$|\mathrm{A}|^2 = h_{11}^2 + 2\sum_{i=2}^n h_{1i}^2 + \sum_{i,j\neq 1}^n h_{ij} = |\mathrm{A}|^2 \sum_{i=2}^n h_{1i}^2 + \sum_{i,j\neq 1}^n h_{ij}$$

so $h_{ij} = 0$ unless $i = j = 1$. This means that A has rank one.
Thus, we have two possible (non mutually excluding) situations at every point of M: either A has rank one or $\nabla\mathrm{H} = 0$.

If the kernel of A is empty everywhere, A must have rank at least two, as we assumed $n \geq 2$, then we have $\nabla\mathrm{H} = 0$ which implies $\nabla\mathrm{A} = 0$ and, by equation (4.18) $h_{ij} = \mathrm{H} h_{ik} h_{kj}$. This means that all the eigenvalues of A are 0 or $1/\mathrm{H}$. As the kernel is empty, $\mathrm{A} = \mathrm{H}g/n$, precisely $\mathrm{H} = \sqrt{n}$ and $\mathrm{A} = g/\sqrt{n}$. Then, the hypersurface M has to be the sphere $\mathbb{S}^n(\sqrt{n})$.
Indeed, computing

$$\Delta|x|^2 = 2n + 2\langle x \,|\, \Delta x\rangle = 2n + 2\mathrm{H}\langle x \,|\, \nu\rangle = 2n - 2\mathrm{H}^2 = 0\,,$$

by the structural equation $\mathrm{H} + \langle x \,|\, \nu\rangle = 0$, being $|x|^2$ a harmonic function on M, looking at the point of M of minimum distance from the origin, by the strong maximum principle for elliptic equations, it must be constant on M.

We suppose now that the kernel of A is not empty at some point $p \in M$, then let $v_1(p), \dots, v_{n-m}(p) \in \mathbb{R}^{n+1}$ be a family of unit orthonormal tangent vectors spanning such $(n-m)$–dimensional kernel, that is $h_{ij}v_k^j = 0$. Then the geodesic $\gamma(s)$ in M from p with initial velocity $v_k(p)$ satisfies

$$\nabla_s(h_{ij}\gamma_s^j) = \mathrm{H}^{-1}\langle\nabla\mathrm{H} \,|\, \gamma_s\rangle h_{ij}\gamma_s^j$$

hence, by Gronwall's lemma, it holds $h_{ij}(\gamma(s))\gamma_s^j(s) = 0$ for every s.
Being γ a geodesic in M, the normal to the curve in $\mathbb{R}^{n+1}$ is the normal to M, then setting

k to be the curvature of γ in $\mathbb{R}^{n+1}$, we have

$$k = \left\langle \nu \,\Big|\, \frac{d}{ds}\gamma_s \right\rangle = h_{ij}\gamma_s^i\gamma_s^j = 0\,,$$

thus γ is a line in $\mathbb{R}^{n+1}$.
Hence, all the $(n-m)$–dimensional affine subspace $p + S(p) \subset \mathbb{R}^{n+1}$ is contained in M, where we set $S(p) = \langle v_1(p), \dots, v_m(p)\rangle \subset \mathbb{R}^{n+1}$.
Let now $\sigma(s)$ a geodesic from p to another point q, parametrized in arclength, and extend by parallel transport the vectors v_k along σ,

$$\nabla_s(h_{ij}v_k^j) = \langle \nabla \mathrm{H} \,|\, \sigma_s\rangle h_{ij}v_k^j$$

and again by Gronwall's lemma, $h_{ij}v_k^j(s) =$ for every s, in particular $v_k(q)$ is contained in the kernel of A at $q \in M$. This argument clearly shows that the kernel of $S(p)$ of A at p has constant dimension $(n-m)$ with $0 < m < n$ (as A is never zero) and all the affine $(n-m)$–dimensional subspaces $p + S(p) \subset \mathbb{R}^{n+1}$ are all contained in M.
Moreover, as $h_{ij}v_k^j = 0$ along such geodesic, looking at things in $\mathbb{R}^{n+1}$, denoting with D the covariant derivative in $\mathbb{R}^{n+1}$, we have

$$D_s v_k = \nabla_s v_k + h_{ij}v_k^j\sigma_s^i\nu = 0$$

so the subspaces $S(p)$ are all a common $(n-m)$–dimensional vector subspace that we denote with S and $M = M + S \subset \mathbb{R}^{n+1}$.
By Sard's theorem, there exist a vector $y \in S$ such that $N = M \cap (y + S^\perp)$ is a smooth m–dimensional submanifold of $\mathbb{R}^{n+1}$, then as $M = M + S$, it is easy to see that $M = N \times S$, but this means that $L = S^\perp \cap M$ is a smooth m–dimensional submanifold of $S^\perp = \mathbb{R}^{m+1}$ with $M = L \times S$.
Moreover, as S is in the tangent space to every point of L, the normal ν to M at a point of L stays in $S^\perp = \mathbb{R}^{m+1}$ so it coincides with the normal to L in $S^\perp = \mathbb{R}^{m+1}$, then a simple computation shows that the mean curvature of M at the points of L is equal to the mean curvature of L as a hypersurface in $S^\perp = \mathbb{R}^{m+1}$. This shows that L is a hypersurface in $\mathbb{R}^{m+1}$ satisfying the relative structural equation. Finally, as, by construction, the second fundamental form of L has empty kernel, by the previous discussion, $L = \mathbb{S}^m(\sqrt{m})$ and $M = \mathbb{S}^m(\sqrt{m}) \times \mathbb{R}^{n-m}$ and we are done.

□

OPEN PROBLEM 4.37. Without the assumption $\mathrm{H} > 0$ this result is not true, an example is the Angenent torus **[13]**. It is an open question if there exists a smooth embedding of $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$ such that $\mathrm{H} + \langle x \,|\, \nu\rangle = 0$, different by the unit sphere.

COROLLARY 4.38. *Every limit hypersurface obtained by rescaling around a type I singularity point of the motion by mean curvature of a compact, embedded initial hypersurface with* $\mathrm{H} \geq 0$, *up to rotation in* $\mathbb{R}^{n+1}$, M *must be either a hyperplane for the origin, or the sphere* $\mathbb{S}^n(\sqrt{n})$ *or one of the cylinders* $\mathbb{S}^m(\sqrt{m}) \times \mathbb{R}^{n-m}$.

We discuss now what are the possible values of the limit heat density function, following Stone **[91]**. As the value of $\Theta(p)$ is the Huisken's functional on any limit of rescaled hypersurfaces, and these latter are "finite", we have that the possible values are 1 in the case of a hyperplane and

$$\Theta^{n,m} = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{S}^m(\sqrt{m})\times\mathbb{R}^{n-m}} e^{-\frac{|x|^2}{2}}\, d\mathcal{H}^n$$

for $m \in \{1, \dots, n\}$.
A straightforward computation gives for $m > 0$

$$\Theta^{n,m} = \left(\frac{m}{2\pi e}\right)^{m/2} \omega_m$$

where ω_m denotes the volume of the unit m–sphere. Notice that $\Theta^{n,m}$ does not depend on n, so we can simply write $\Theta^m = \Theta^{n,m}$.

LEMMA 4.39 (Stone [**91**]). *The values of Θ^m are all distinct and larger than 1 for $m > 0$. Indeed the numbers $\{\Theta^m : m = 1, 2, \dots\}$ form a strictly decreasing sequence in m, with $\Theta^m \searrow \sqrt{2}$ as $m \to \infty$.*

This lemma implies that the "shape" of the limit hypersurfaces arising from a blow up at a type I singularity of mean curvature flow of a mean convex, compact, embedded hypersurfaces are classified by the value of the limit heat density function at the blow up points.

This discussion gives a positive answer to problems 4.17 and 4.29, in the subclass with $\mathrm{H} \geq 0$ of the possible limit embedded hypersurfaces. Indeed, the limit of rescalings around a non special singular point is an embedded hypersurface with at least one point with $\mathrm{A} = 0$, the only possibility is then a single hyperplane, by the classification result.
Finally, as we noticed also in the general case, combining such conclusion with Proposition 4.30, also Problem 4.32 has a full answer in this class.

PROPOSITION 4.40. Every *singular point of a type I singularity of the motion by mean curvature of a compact, embedded initial hypersurface with* $\mathrm{H} \geq 0$ *is a special singular point.*

In the special situation that the singularity is asymptotically spherical, we can infer (as the convergence is in C^∞) that the hypersurface moving by mean curvature has become convex at some time. Then the following pair of theorems describes the last part of the evolution.

THEOREM 4.41 (Gage and Hamilton [**36, 37, 38**]). *Under the mean curvature flow a convex closed curve of $\mathbb{R}^2$ smoothly shrinks to a point in finite time. Rescaling in order to keep the length constant, it converges to a circle in C^∞.*

THEOREM 4.42 (Huisken [**53**]). *Under the mean curvature flow a compact and convex hypersurface in $\mathbb{R}^{n+1}$ with $n \geq 2$ smoothly shrinks to a point in finite time. Rescaling in order to keep the* Area *constant, it converges to a sphere in C^∞.*

REMARK 4.43. The theorem for curves is not merely a consequence of the general result. The proof in dimension $n \geq 2$ does not work in the one–dimensional case.

At the end of the first section of next chapter, we will show a line of proof by Hamilton in [**48**], different by the original ones.

Another proof was also given by Andrews in [**8**], analyzing the behavior of the eigenvalues of the second fundamental form, close to the singular time.

As a consequence, if the flow develops a type I singularities and some blow up is a sphere, the flow is smooth till the hypersurface shrinks to a point becoming asymptotically spherical.

REMARK 4.44. What is missing in all this story, even in the mean convex case, is a full solution to Problem 4.16. The limit heat density function tell us that any limit gives the same value of the Huisken's integral, hence its "shape" is fixed: plane, sphere or cylinder. If the limit is a sphere, the limit is unique and there is full convergence, if it is a plane we already had such conclusion by White's Theorem 4.26. But, if the limit is a cylinder, its axis could possibly change, depending on the choice of the converging sequence.

5. Embedded Curves in the Plane

The case of an embedded, closed curve γ in $\mathbb{R}^2$ is special, indeed, the classification theorem 4.36 holds without *a priori* assumptions on the curvature. So there are only two possible limit of rescaled curves without self–intersections: a line through the origin or the circle $\mathbb{S}^1$. This gives immediately a general positive answer to problems 4.17 and 4.29 and implies as before that every singular point is a special singular point.
In this very special case also Problem 4.16 is solved affirmatively, as there are no "cylinders". The limit is always unique.

COROLLARY 4.45. *Let $\gamma \subset \mathbb{R}^2$ be a simple closed curve, then every curve obtained by limit of rescalings around a type I singularity point of the motion by mean curvature, is a line through the origin or the circle $\mathbb{S}^1$.*

By the same reasoning of the previous section, at a type I singularity any simple closed curve vanishes becoming asymptotically spherical.

REMARK 4.46. We mention here that an extensive and deep analysis of the behavior of general curves moving by curvature (even when the ambient is a generic surface different from $\mathbb{R}^2$) is provided by the pair of papers by Angenent [**10, 12**] (see also the discussion in [**13**]).