CHAPTER 5

Type II Singularities

We suppose now to be in the *type II* singularity case, that is,

$$\limsup_{t\to T}\max_{p\in M}|\mathrm{A}(p,t)|\sqrt{T-t}=+\infty \tag{5.1}$$

for a mean curvature flow of a compact hypersurface $\varphi : M\times[0,T)\to\mathbb{R}^{n+1}$ in its maximal interval of existence.

A good question is if actually type II singularities can develop.
The simplest example is given by a closed self–intersecting curve with the shape of a "eight" figure in the plane, with zero rotation number. If $T>0$ is the singularity time, and we suppose to have a type I singularity, by the results of previous chapter there is a nonflat limit of rescaled curves, then such limit must be a circle or one of Abresch–Langer curves (with possible integer multiplicity larger than 1). In both cases, the limit would be a compact closed curve, and by the smooth convergence, the rotation number would still be zero. Hence, the circle has to be excluded and the contradiction with the hypotheses is given by the fact that there are no Abresch–Langer curves with rotation number zero. This shows that type I singularities do not describe all the possible ones.
Another example is given by a cardioid curve in the plane with a small loop: at some time the small loop has shrunk while the rest of the curve remained smooth, and a cusp has developed, it can be shown that such a singularity is of type II (see **[10, 12]** and in particular **[11]** where the blow up rate of the curvature is also discussed).

One could conjecture that all the singularities of *embedded* surfaces (at least in low dimension) are of Type I. Unfortunately, this is not true if the dimension is at least two, the following example excludes such a reasonable good behavior.

EXAMPLE 5.1 (The Degenerate Neckpinch). For a given $\lambda>0$, let us set

$$\phi_\lambda(x)=\sqrt{(1-x^2)(x^2+\lambda)},\qquad -1\leq x\leq 1.$$

For any $n\geq 2$, let M^λ be the n–dimensional hypersurface in $\mathbb{R}^{n+1}$ obtained by rotation of the graph of ϕ_λ in $\mathbb{R}^2$. The hypersurface M^λ looks like a dumbbell, where the parameter λ measures the width of the central part. Then, it is possible to prove the following properties (see **[5]**):

(1) if λ is large enough, the hypersurface M^λ_t eventually becomes convex and shrinks to a point in finite time;
(2) if λ is small enough, M^λ_t exhibits a neckpinch singularity as in the case of the *standard neckpinch* (see Section 4);

(3) there exists at least one intermediate value of $\lambda > 0$ such that M_t^λ shrinks to a point in finite time, has positive mean curvature up to the singular time, but never becomes convex. The maximum of the curvature is attained at the two points where the surface meets the axis of rotation;

(4) the singularity is of type II, otherwise the blow up at the singular time would give a sphere (for all $p \in M$ we would have $\widehat{p} = O \in \mathbb{R}^{n+1}$ hence, by estimate 4.5 in the previous chapter, any limit hypersurface is bounded). This is impossible as it would imply that the surface would have been convex at some time.

The flowing hypersurface at point (3) is called the *degenerate neckpinch* and was first conjectured by Hamilton for the Ricci flow [**49**, Section 3]. Intuitively speaking, it is a limiting case of the neckpinch where the cylinder in the middle and the balls on the sides shrink at the same time. One can also build the example in an asymmetric way, with only one of the two balls shrinking simultaneously with the neck, while the other one remains nonsingular.
A sharp analysis of the singular behavior for a class of rotationally symmetric surfaces exhibiting a degenerate neckpinch has been done by Angenent and Velázquez in [**15**].
Another interesting example of singularity formation (a family of evolving tori, proposed by De Giorgi) was carefully studied by Soner and Souganidis in [**88**, Prop. 3] (see also the numerical analysis performed by Paolini and Verdi in [**80**, Sect. 7.5]).

1. Hamilton's Blow Up

In order to deal with the blow up around type II singularities, we need a new set of estimates, which are actually independent of hypothesis (5.1) and are scaling invariant (see [**4**] and [**84**]).

PROPOSITION 5.2. *Let $\varphi : M \times [0,T) \to \mathbb{R}^{n+1}$ be a mean curvature flow of a compact hypersurface such that $\sup_{p\in M} |\mathrm{A}(p,0)| \leq \Lambda < +\infty$. Then, there exists a time $\tau = \tau(\Lambda) > 0$ and constants $C_m = C_m(\Lambda)$, for every $m \in \mathbb{N}$ such that $|\nabla^m \mathrm{A}(p,t)|^2 \leq C_m/t^m$ for every $p \in M$ and $t \in (0,\tau)$.*

PROOF. We prove the claim by induction. By the evolution equation for $|\mathrm{A}|^2$,

$$\frac{\partial}{\partial t}|\mathrm{A}|^2 = \Delta|\mathrm{A}|^2 - 2|\nabla\mathrm{A}|^2 + 2|\mathrm{A}|^4 \leq \Delta|\mathrm{A}|^2 + 2|\mathrm{A}|^4$$

we get

$$\frac{\partial}{\partial t}|\mathrm{A}|^2_{\max} \leq 2|\mathrm{A}|^4_{\max}$$

hence, there exists a time $\tau = \tau(\Lambda) > 0$ and a constant $C_0 = C_0(\Lambda)$ such that $|\mathrm{A}(p,t)|^2 \leq C_0$ for every $p \in M$ and $t \in [0,\tau)$. This is the case $m = 0$.
Recalling equation (3.4), setting $f = \sum_{k=0}^m |\nabla^k \mathrm{A}|^2 \lambda_k t^k$ and assuming the inductive hypothesis, $|\nabla^k \mathrm{A}(p,t)|^2 \leq C_k(\Lambda)/t^k$ for any $k \in \{0,\dots,m-1\}$, $p \in M$ and $t \in (0,\tau)$, we

compute,

$$
\begin{aligned}
\frac{\partial}{\partial t} f &= \frac{\partial}{\partial t} \sum_{k=0}^{m} |\nabla^k \mathrm{A}|^2 \lambda_k t^k \\
&= \sum_{k=1}^{m} |\nabla^k \mathrm{A}|^2 k \lambda_k t^{k-1} \\
&\quad + \sum_{k=0}^{m} \lambda_k t^k \Big(\Delta |\nabla^k \mathrm{A}|^2 - 2|\nabla^{k+1}\mathrm{A}|^2 + \sum_{p+q+r=k \,|\, p,q,r \in \mathbb{N}} \nabla^p \mathrm{A} * \nabla^q \mathrm{A} * \nabla^r \mathrm{A} * \nabla^k \mathrm{A} \Big) \\
&\leq \Delta f + \sum_{k=1}^{m} |\nabla^k \mathrm{A}|^2 (k\lambda_k - 2\lambda_{k-1}) t^{k-1} - 2 |\nabla^{m+1}\mathrm{A}|^2 \lambda_m t^m \\
&\quad + \sum_{k=0}^{m} \lambda_k t^k C(k) \sum_{p+q+r=k \,|\, p,q,r \in \mathbb{N}} |\nabla^p \mathrm{A}| |\nabla^q \mathrm{A}| |\nabla^r \mathrm{A}| |\nabla^k \mathrm{A}| \\
&\leq \Delta f + \sum_{k=1}^{m} |\nabla^k \mathrm{A}|^2 (k\lambda_k - 2\lambda_{k-1}) t^{k-1} + \sum_{k=0}^{m-1} \lambda_k C(k) \sum_{p+q+r=k \,|\, p,q,r \in \mathbb{N}} C_p C_q C_r C_k \\
&\quad + \lambda_m t^{m/2} C(m) \Big(\sum_{p+q+r=m \,|\, p,q,r<m} C_p C_q C_r \Big) |\nabla^m \mathrm{A}| + \lambda_m t^m C(m) |\mathrm{A}|^2 |\nabla^m \mathrm{A}|^2 \\
&\leq \Delta f + \sum_{k=1}^{m} |\nabla^k \mathrm{A}|^2 (k\lambda_k - 2\lambda_{k-1}) t^{k-1} + C \lambda_m t^m |\nabla^m \mathrm{A}|^2 + D
\end{aligned}
$$

where in the last passage we applied Peter–Paul inequality. If now we choose inductively positive constants $\lambda_1, \dots, \lambda_m$ such that $\lambda_k = 2\lambda_{k-1}/k$, starting with $\lambda_0 = 1$ (easily $\lambda_k = 2^k/k!$), we have,

$$
\frac{\partial}{\partial t} f \leq \Delta f + C \lambda_m t^m |\nabla^m \mathrm{A}|^2 + D \leq \Delta f + C f + D
$$

for every $p \in M$ and $t \in (0, \tau)$, and the constants C and D depends only on m and Λ, by the inductive hypothesis. Notice that the inequality holds also at $t = 0$ as the function f is smooth on $M \times [0, \tau)$.
This differential inequality, by maximum principle then implies that $f_{\max}(t)$ is bounded in the interval $[0, \tau)$ by some constant C depending only on m, Λ and $f_{\max}(0) = |\mathrm{A}|_{\max}(0) \leq \Lambda^2$, hence

$$
t^m |\nabla^m \mathrm{A}(p,t)|^2 \leq f(t)/\lambda_m \leq C/\lambda_m = C_m
$$

and we are done, $C_m = C_m(\Lambda)$. □

The following corollary is an easy consequence.

COROLLARY 5.3. *Let $\varphi : M \times [0, T) \to \mathbb{R}^{n+1}$ be a mean curvature flow of a compact hypersurface such that $\sup_{p \in M} |\mathrm{A}(p, 0)| \leq \Lambda < +\infty$. Then, there exists a value $\tau > 0$ and constants C_m, for every $m \in \mathbb{N}$, depending only on Λ such that $|\nabla^m \mathrm{A}(p,t)|^2 \leq C_m$ for every*

$p \in M$ and $t \in (\tau/2, \tau)$.
For instance, one can choose $\tau = 1/(4\Lambda^2)$.

PROOF. Only the last claim need an explanation, it follows by integrating the differential inequality

$$\frac{\partial}{\partial t}|\mathrm{A}|^2_{\max} \leq 2|\mathrm{A}|^4_{\max}\,.$$

□

We describe now Hamilton's procedure to get a blow up flow in the type II singularity case.

Let us choose a sequence of times $t_k \in [0, T - 1/k]$ and points $p_k \in M$ such that

$$|\mathrm{A}(p_k, t_k)|^2(T - 1/k - t_k) = \max_{\substack{t \in [0,T-1/k] \\ p \in M}} |\mathrm{A}(p, t)|^2(T - 1/k - t)\,. \tag{5.2}$$

This maximum goes to $+\infty$ as $k \to \infty$, indeed, if it is bounded on a subsequence $k_i \to \infty$, we would have

$$|\mathrm{A}(p, t)|^2(T - t) = \lim_{i\to\infty} |\mathrm{A}(p, t)|^2(T - 1/k_i - t) \leq C$$

for every $t \in [0, T - 1/k_i]$ and $p \in M$, hence for every $t \in [0, T)$ and $p \in M$. This is in contradiction with condition (5.1).
This fact forces the sequence t_k to converge to T as $k \to \infty$. If t_{k_i} is a subsequence not converging to T, we would have that the sequence $|\mathrm{A}(p_{k_i}, t_{k_i})|^2$ is bounded, hence also $\max_{\substack{t\in[0,T-1/k_i] \\ p\in M}} |\mathrm{A}(p, t)|^2(T - 1/k_i - t)$.
Hence, we can choose an increasing (not relabeled) subsequence t_k converging to T, such that $|\mathrm{A}(p_k, t_k)|$ goes monotonically to $+\infty$ and

$$|\mathrm{A}(p_k, t_k)|^2 t_k \to +\infty\,, \qquad |\mathrm{A}(p_k, t_k)|^2(T - 1/k - t_k) \to +\infty\,,$$

Moreover, we can also assume that $p_k \to p$ for some $p \in M$.
We rescale now *the flow* as follows: let $\varphi_k : M \times I_k \to \mathbb{R}^{n+1}$, where

$$I_k = [-|\mathrm{A}(p_k, t_k)|^2 t_k, |\mathrm{A}(p_k, t_k)|^2(T - 1/k - t_k)]\,,$$

be the evolution given by

$$\varphi_k(p, s) = |\mathrm{A}(p_k, t_k)|[\varphi(p, s/|\mathrm{A}(p_k, t_k)|^2 + t_k) - \varphi(p_k, t_k)]$$

and we set $M^k_s = \varphi_k(M, s)$ and A_k to be the second fundamental form of the flowing hypersurfaces φ_k.
It is easy to check that this is a parabolic rescaling hence, every φ_k is still a mean curvature flow, moreover, the following properties hold,

- $\varphi_k(p_k, 0) = 0 \in \mathbb{R}^{n+1}$ and $|\mathrm{A}_k(p_k, 0)| = 1$,
- for every $\varepsilon > 0$ and $\omega > 0$ there exists $\overline{k} \in \mathbb{N}$ such that

$$\max_{p\in M} |\mathrm{A}_k(p, s)| \leq 1 + \varepsilon \tag{5.3}$$

  for every $k \geq \overline{k}$ and $s \in [-|\mathrm{A}(p_k, t_k)|^2 t_k, \omega]$.

Indeed, (the first point is immediate), by the choice of the pair (p_k, t_k) we get

$$\begin{aligned}|\mathrm{A}_k(p,s)| &= |\mathrm{A}(p_k,t_k)|^{-1}|\mathrm{A}(p, s/|\mathrm{A}(p_k,t_k)|^2 + t_k)|\\ &\leq |\mathrm{A}(p_k,t_k)|^{-2}|\mathrm{A}(p_k,t_k)|^2 \frac{T-1/k-t_k}{T-1/k-t_k-s/|\mathrm{A}(p_k,t_k)|^2}\\ &= \frac{|\mathrm{A}(p_k,t_k)|^2(T-1/k-t_k)}{|\mathrm{A}(p_k,t_k)|^2(T-1/k-t_k)-s}\,,\end{aligned}$$

if $s/|\mathrm{A}(p_k,t_t)|^2 + t_k \in [0, T-1/k]$, that is, $s \in I_k$. Then, assuming that $s \leq \omega$, the claim follows as we know that $|\mathrm{A}(p_k,t_k)|^2(T-1/k-t_k) \to +\infty$.

This discussion implies that if we are able to take a (subsequential) limit of these *flows*, smoothly converging on every compact time interval, we would get a mean curvature flow such that the norm of the second fundamental form is uniformly bounded by one and the interval of existence becomes $(-\infty,+\infty) = \lim_{k\to\infty} I_k$, this is assured by the next proposition.

PROPOSITION 5.4. *The family of flows φ_k converges (up to a subsequence) in the C^∞_{loc} topology to a nonempty, smooth evolution of complete, hypersurfaces by mean curvature M^∞_s in the time interval $(-\infty,+\infty)$. Such a flow is called* eternal*.*
Moreover, the norm of the second fundamental form is uniformly bounded in space and time and it takes its absolute maximum, which is 1, at time $s=0$ at the origin of $\mathbb{R}^{n+1}$.
Finally, if the original initial hypersurface was embedded this limit flow consists of embedded hypersurfaces.

PROOF. By the previous discussion, on every bounded interval of time $[s_1, s_2]$, the evolutions φ_k have definitely uniformly bounded curvature, precisely $|\mathrm{A}_k| \leq (1+\varepsilon)$, then for $\varepsilon << 1$, by Corollary 5.3, in every interval $[s_1 + 1/16, s_1 + 1/8]$ we have uniform estimates $|\nabla^m \mathrm{A}| \leq C_m$ with C_m independent of s_1, for every $m \in \mathbb{N}$.
By means of monotonicity formula we can have an uniform estimate on $\mathcal{H}^n(\varphi_k(M,s) \cap B_R)$ as follows: recall that $\mathcal{H}^n$ is the n–dimensional Hausdorff measure, we set μ^k_s to be

the measure associated to the hypersurface φ_k at time s,

$$
\begin{aligned}
\mathcal{H}^n(\varphi_k(M,s)\cap B_R) &= \int_M \chi_{B_R}(y)\,d\mu^k_s(y)\\
&\leq \int_M \chi_{B_R}(y)e^{\frac{R^2-|y|^2}{4}}\,d\mu^k_s(y)\\
&\leq e^{R^2/4}\int_M e^{-\frac{|y|^2}{4}}\,d\mu^k_s(y)\\
&= (4\pi)^{n/2}e^{R^2/4}\int_M \frac{e^{-\frac{|y|^2}{4(s+1-s)}}}{[4\pi(s+1-s)]^{n/2}}\,d\mu^k_s(y)\\
&\leq C(R)\int_M \frac{e^{-\frac{|y|^2}{4(s+1+|\mathrm{A}(p_k,t_k)|^2t_k)}}}{[4\pi(s+1+|\mathrm{A}(p_k,t_k)|^2t_k)]^{n/2}}\,d\mu^k_{-|\mathrm{A}(p_k,t_k)|^2t_k}(y)\\
&= C(R)\int_M \frac{|\mathrm{A}(p_k,t_k)|^n e^{-\frac{|x-\varphi(p_k,t_k)|^2|\mathrm{A}(p_k,t_k)|^2}{4(s+1+|\mathrm{A}(p_k,t_k)|^2t_k)}}}{[4\pi(s+1+|\mathrm{A}(p_k,t_k)|^2t_k)]^{n/2}}\,d\mu_0(x)\\
&\leq C(R)\int_M \frac{|\mathrm{A}(p_k,t_k)|^n}{[4\pi(s+1+|\mathrm{A}(p_k,t_k)|^2t_k)]^{n/2}}\,d\mu_0(x)\\
&\leq C(R)\mathrm{Area}(\varphi_0)\frac{|\mathrm{A}(p_k,t_k)|^n}{[4\pi(s+1+|\mathrm{A}(p_k,t_k)|^2t_k)]^{n/2}}\,,
\end{aligned}
$$

hence,

$$
\limsup_{k\to\infty}\mathcal{H}^n(\varphi_k(M,s)\cap B_R)\leq C(R)\frac{\mathrm{Area}(\varphi_0)}{[4\pi T]^{n/2}} = C(R,\varphi_0)\,.
$$

This conclusion implies that if s stays in a compact interval $J\subset\mathbb{R}$, we have definitely,

$$
\mathcal{H}^n(\varphi_k(M,s)\cap B_R)\leq C(R,\varphi_0,J)
$$

uniformly for $s\in J$, where the constant is independent of $k\in\mathbb{N}$.
Then we use the same argument of Proposition 4.15, but applied to flows, that is, we consider the time–tracks of the flows φ_k as hypersurfaces $\widetilde{\varphi}_k : M\times I_k\to\mathbb{R}^{n+1}\times\mathbb{R} = \mathbb{R}^{n+2}$ defined by $\widetilde{\varphi}_k(p,s) = (\varphi_k(p,s),s)$ and we reparametrize them locally as graphs of smooth functions.
Reasoning like in the proof of Proposition 3.28, the estimates on the spatial covariant derivatives of A_k imply uniform locally estimates on space and also time derivatives (using the evolution equation) of the representing functions, so, up to a subsequence, we can get locally a limit smooth mean curvature flow. By a diagonal argument, we show the claim (follow the proof of Proposition 4.15).
The claims about the properties of the limit flow are immediate by the above discussion, only the embeddedness, if the initial hypersurface is embedded, requires a justification. In this case, by Proposition 3.14, all the hypersurfaces in the flows φ_k are embedded at every time, then the only possibility for M^∞_s not to be embedded is if two or more of its "inside" regions "touch" each other at some point $y\in\mathbb{R}^{n+1}$ with a common tangent

space.
We define the monotone nondecreasing function $G(t) = \max_{\substack{s\in[0,t]\\ p\in M}} |\mathrm{A}(p,s)|$ and we choose a smooth monotone nondecreasing function $K : [0,T) \to \mathbb{R}^+$ such that $G(t) \leq K(t) \leq 2G(t)$ for every $t \in [0.T)$.
Then, we consider the following set $\Omega \subset M \times M \times [0,T)$ given by $\{(p,q,t)\,|\, d_{g(t)}(p,q) \leq \varepsilon/K(t)\}$, where $d_{g(t)}$ is the geodesic distance in the Riemannian manifold $(M, g(t))$. Let

$$C = \inf_{\partial\Omega} |\varphi(p,t) - \varphi(q,t)|K(t)$$

and suppose that $C = 0$, whatever small $\varepsilon > 0$ we take. This means that there exists a sequence of times $t_i \nearrow T$ and points p_i, q_i with $d_{g(t_i)}(p_i, q_i) = \varepsilon/K(t_i)$ and $|\varphi(p_i,t_i) - \varphi(q_i,t_i)|K(t_i) \to 0$, that is, $|\widetilde{\varphi}(p_i,s_i) - \widetilde{\varphi}(q_i,s_i)| \to 0$ and $d_{\widetilde{g}(s_i)}(p_i,q_i) = \varepsilon$, where we rescaled the hypersurfaces at time t_i around $\varphi(p_i)$ by the dilation factor $K(t_i) \geq G(t_i)$.
As the curvatures of these rescaled hypersurfaces $\widetilde{\varphi}_i$ satisfies

$$|\mathrm{A}_i(p,t_i)| = |\mathrm{A}(p,t_i)|^2/K(t_i) \leq 1\,,$$

reasoning now like in the proof of the same statement in Proposition 4.15, we have a contradiction.
Now, fixed a $\varepsilon > 0$ such that the relative constant C is positive, if we look at the function

$$L(p,q,t) = |\varphi(p,t) - \varphi(q,t)|K(t)$$

on $\complement\Omega \subset M \times M \times [0,T)$, we have that if the minimum of L at time t is lower than ε, then such minimum is not taken on the boundary of the set but in its interior, say at the pair (p,q), then we compute at the point (p,q,t),

$$\frac{\partial L(p,q,t)}{\partial t} = K(t)\frac{\partial}{\partial t}|\varphi(p,t) - \varphi(q,t)| + |\varphi(p,t) - \varphi(q,t)|K'(t) \geq K(t)\frac{\partial}{\partial t}|\varphi(p,t) - \varphi(q,t)|$$

and a geometric argument analogous to the one in the proof of Proposition 3.14 shows that this last partial derivative is nonnegative (where it exists, almost everywhere). Then, by means of maximum principle (Hamilton's trick, Lemma 3.3) we conclude that when the minimum of L at time t is under the constant C, it is nondecreasing.
Hence, there is a positive lower bound on

$$\inf_{\complement\Omega} |\varphi(p,t) - \varphi(q,t)|K(t)\,,$$

consequently, also on

$$\inf_{\complement\Omega} |\varphi(p,t) - \varphi(q,t)|G(t)\,.$$

Now notice that, for the times t_k coming from definition (5.2), we have $|\mathrm{A}(p_k,t_k)| = G(t_k)$, otherwise there would exist a time $\widehat{t} < t_k$ with $\max_{p\in M} |\mathrm{A}(p,\widehat{p})| > |\mathrm{A}(p_k,t_k)|$ but this is in contradiction with the maximum in the right hand side of equation (5.2).
Moreover, $|\mathrm{A}(p_k,t_k)| \geq G(t)$ for every $t \leq t_k$ and, fixed ω, $\delta > 0$, definitely by (5.3), $\max_{p\in M} |\mathrm{A}_k(p,s)| \leq (1+\delta)|\mathrm{A}(p_k,t_k)|$ for every $s \leq \omega$, that is,

$$G(s/|\mathrm{A}(p_k,t_k)|^2 + t_k) = \max_{\substack{s\leq\omega\\ p\in M}} |\mathrm{A}(p, s/|\mathrm{A}(p_k,t_k)|^2 + t_k)| \leq (1+\delta)|\mathrm{A}(p_k,t_k)| = (1+\delta)G(t_k)\,.$$

Then, if $d_{\widetilde{g}_k(s)}(p,q) > 3\varepsilon$ for $s \leq \omega$, definitely

$$\begin{aligned} d_{g(s/|\mathrm{A}(p_k,t_k)|^2+t_k)}(p,q) &= d_{\widetilde{g}_k(s)}(p,q)/|\mathrm{A}(p_k,t_k)| \\ &\geq d_{\widetilde{g}_k(s)}(p,q)/G(s/|\mathrm{A}(p_k,t_k)|^2+t_k) \\ &\geq \varepsilon/K(s/|\mathrm{A}(p_k,t_k)|^2+t_k) \end{aligned}$$

hence, $(p,q,s/|\mathrm{A}(p_k,t_k)|^2+t_k) \in \complement\Omega$ and

$$|\varphi(p,s/|\mathrm{A}(p_k,t_k)|^2+t_k) - \varphi(q,s/|\mathrm{A}(p_k,t_k)|^2+t_k)|G(s/|\mathrm{A}(p_k,t_k)|^2+t_k) \geq C > 0\,.$$

Then, if $d_{\widetilde{g}_k(s)}(p,q) > 3\varepsilon$, definitely

$$\begin{aligned} |\widetilde{\varphi}_k(p,s) - \widetilde{\varphi}_k(q,s)| &= |\varphi(p,s/|\mathrm{A}(p_k,t_k)|^2+t_k) - \varphi(q,s/|\mathrm{A}(p_k,t_k)|^2+t_k)|\,|\mathrm{A}(p_k,t_k)| \\ &\geq C|\mathrm{A}(p_k,t_k)|/G(s/|\mathrm{A}(p_k,t_k)|^2+t_k) \\ &\geq \frac{C}{1+\delta}\,. \end{aligned}$$

This conclusion, obviously passes to the limit hypersurface M_s^∞, that is, if a couple of points has intrinsic distance larger than 2ε, then their extrinsic distance is bounded below by some uniform positive constant. If ε is chosen smaller enough such that any hypersurface with $|\mathrm{A}| \leq 1$ (like M_s^∞) is an embedding on any intrinsic ball of radius smaller than 3ε, we are done, the hypersurface M_s^∞ cannot have self–intersections, hence it is embedded. □

EXERCISE 5.5. This blow up procedure can be applied also in the type I singularity case. The are some differences and the sequence t_k must be chosen in order that $t_k \to T$, it is not a consequence of the construction.
The limit mean curvature flow that one obtains is no more eternal but only *ancient*, that is, defined on some interval $(-\infty, \Omega)$ with $\Omega > 0$, and $|\mathrm{A}_\infty| \leq 1$ holds only on $(\infty, 0]$.
We let the analysis to the interested reader.

The analysis of singularities in the type II case is so reduced to classify all the eternal flows with bounded curvature (and its covariant derivatives) with the extra property that the norm of the second fundamental form takes its maximum, equal to one, at some point in space and time.

Examples of this class of flows are the *translating solutions* to mean curvature flow (with bounded second fundamental form and achieving its maximum), that is, hypersurfaces $M \subset \mathbb{R}^{n+1}$ such that during the motion do not change their shape but simply move in a fixed direction with constant velocity. One can see that this condition is equivalent to the existence of a vector $v \in \mathbb{R}^{n+1}$ such that $\mathrm{H}(p) + \lambda\langle v \,|\, \nu(p)\rangle = 0$ at every point $p \in M$.
In $\mathbb{R}^2$ all the possible translating solutions are given by rotations,translations and homotheties of the graph of the function $y = -\log\cos x$ in the interval $(-\pi/2, \pi/2)$, called the *grim reaper* **[41]**.
In higher dimension, looking for convex graph solutions over the hyperplane $\{x_{n+1} =$

$0\} = \mathbb{R}^n$, translating in the e_{n+1} direction with unit speed, one has to find a convex function $u : \mathbb{R}^n \to \mathbb{R}$ such that

$$\Delta u - \frac{\mathrm{H}u(\nabla u, \nabla u)}{1 + |\nabla u|^2} = 1$$

and $u(0) = \nabla u(0) = 0$,where $\mathrm{H}u$ is the Hessian of u.
Imposing rotational symmetry around the origin, that is, $u(x) = u(\rho)$ with $\rho = |x|$, this problem becomes the following ODE

$$u_{\rho\rho} + \frac{(n-1)u_\rho}{\rho} - \frac{u_{\rho\rho}u_\rho^2}{1 + u_\rho^2} = 1\,,$$

that is,

$$u_{\rho\rho} = (1 + u_\rho^2)\left(1 - \frac{(n-1)u_\rho}{\rho}\right)$$

with $\lim_{\rho\to 0} u(\rho) = \lim_{\rho\to 0} u'(\rho) = 0$ for a convex function $u : \mathbb{R}^+ \to \mathbb{R}$.
When $n = 1$ this ODE gives the *grim reaper*, when $n > 1$ there is only one solution, defined on all $\mathbb{R}^+$ and growing quadratically at infinity. This solution provides the only rotationally symmetric, convex, translating hypersurface moving by mean curvature, up to isometries and dilations.

EXERCISE 5.6. Show the claimed properties of the solution of such ODE.

It is a controversial open problem (to my knowledge) whether all the convex, translating hypersurfaces moving by mean curvature in $\mathbb{R}^{n+1}$ are given by the product of a rotationally symmetric one in $\mathbb{R}^m$ times $\mathbb{R}^{n-m}$ (see **[95]** and **[97**, p. 536, end of Section 6]).

REMARK 5.7. Recently, Nguyen **[76]** exhibited some new nonconvex, embedded examples of translating hypersurfaces, with a trident–like shape at "large scales".

OPEN PROBLEM 5.8. Classify all the eternal mean curvature flows M_s of complete, connected, hypersurfaces in $\mathbb{R}^{n+1}$ such that A and its derivatives are uniformly bounded in space and time and $|\mathrm{A}|$ takes its maximum at some point in space–time. Same problem assuming embeddedness. Same problem assuming the flow comes from Hamilton's blow up procedure.

Another problem is the analogous classification for ancient complete solutions with bounded curvature at every fixed time (see the discussion in **[97**, p. 536]). For closed convex curves, this problem has been solved by Daskalopoulos, Hamilton and Sesum **[24]**. Same problem for the *immortal* flows, that is, defined on $[0, +\infty)$.

Because of the results of the next section, we also state the following.

OPEN PROBLEM 5.9. All the eternal mean curvature flows M_s of complete hypersurfaces in $\mathbb{R}^{n+1}$ coming from Hamilton's blow up procedure are translating solutions? At least if they are embedded?

These problems are difficult in general, but like in the type I singularity case, if the evolving hypersurfaces are mean convex ($\mathrm{H} \geq 0$) or we are dealing with curves in the plane, they have a positive answer. This will be the subject of the next sections.

We point out that the rescaled hypersurfaces are unbounded, complete but not compact, since any compact hypersurface cannot be eternal by Corollary 3.12. Even if they satisfy uniform bounds on the curvature, the "bad blow up rate" is an obstacle to the use of Huisken's monotonicity formula in the contest of type II singularities analysis.

We conclude this section by giving Hamilton's line of proof of Theorems 4.41 and 4.42, which is different from the original ones.

PROOF OF 4.41, 4.42. Let T the maximal time of smooth existence of the flow. If $n = 1$, as the initial curve is embedded we will see, by a geometric argument in Section 5, that type II singularities cannot develops (Proposition 5.31).
If $n > 1$, as the initial hypersurface is convex, by the results of Section 5, in particular Proposition 3.38, we have that after any positive time, $\mathrm{H} > 0$ and there exists a positive constant α, independent of time, such that $\mathrm{A} \geq \alpha \mathrm{H} g$ as forms.
If at time T we have a type II singularity, we have an unbounded, eternal convex blow up limit with $\mathrm{H} \geq 0$. By strong maximum principle, actually $\mathrm{H} > 0$ for every time (otherwise $\mathrm{H} \equiv 0$ everywhere, but this with the convexity would imply that the limit hypersurface is a hyperplane) and the condition $\mathrm{A} \geq \alpha \mathrm{H} g$ passes to the limit. Then, by the following theorem of Hamilton **[48]**, the limit hypersurface is compact, in contradiction with the unboundedness, hence type II singularities are excluded.

THEOREM 5.10. *Let M be a smooth strictly convex n–dimensional complete hypersurface in Euclidean space, with $n \geq 2$. Suppose that for some $\alpha > 0$ its second fundamental form is α–pinched in the sense that $\mathrm{A} \geq \alpha \mathrm{H} g$, where g is the induced metric and H its mean curvature. Then M is compact.*

Since we have to deal only with type I singularities, in dimension one, the limit of a sequence of rescaled curves around a singular point must be the unit circle, in dimension higher than one, we have an embedded, strictly convex limit due to the fact that the smallest eigenvalue of h_{ij}/H is uniformly bounded below by a positive constant, around the singular time T, and $\mathrm{H} \sim \frac{1}{\sqrt{T-t}}$. By Corollary 4.38 such limit must be the unit sphere.
As the convergence of the sequence of rescaled hypersurfaces is in C^∞ to the unit sphere, a comparison argument with a slightly larger ball shows the shrinking to a point.
Rescaling then the flow with the procedure above (see Exercise 5.5), one can then prove that the full rescaled sequence converges in C^∞. □

2. Nonnegative Mean Curvature

We shall now consider the formation of type II singularities for hypersurfaces which are mean convex, that is, with nonnegative mean curvature everywhere.
An important result for the analysis of singularities of mean convex hypersurfaces is

the following estimate on the elementary symmetric polynomials of the curvatures S_k, proved in [**59**], which holds in general for any mean curvature flow.

THEOREM 5.11 (Huisken–Sinestrari [**59**]). *Let $\varphi : M \times [0,T) \to \mathbb{R}^{n+1}$ be the mean curvature flow of a compact mean convex immersed hypersurface. Then, for any $\eta > 0$ there exists a constant $C = C(\eta, \varphi_0)$ such that $\mathrm{S}_k \geq -\eta \mathrm{H}^k - C$ for any $k = 2, \dots, n$ at every point of M and $t \in [0,T)$.*

Such an estimate easily implies the following one, which has a more immediate interpretation.

COROLLARY 5.12. *Under the same hypotheses of the previous theorem, for any $\eta > 0$ there exists a constant $C = C(\eta, \varphi_0)$ such that $\lambda^{\min} \geq -\eta \mathrm{H} - C$ at every point of M and $t \in [0,T)$, where $\lambda^{\min}$ is the smallest eigenvalue of the second fundamental form.*

The interest of the above estimates lies in the fact that η can be chosen arbitrarily small and C is a constant not depending on the curvatures and on time. Thus we see that, roughly speaking, the negative curvatures become negligible with respect to the others when the singular time is approached. This implies that the hypersurface becomes asymptotically convex near a singularity.
Let us observe that these results cannot be valid for general hypersurfaces, even in low dimension. In fact, Angenent's homothetically shrinking torus in [**13**] has a behavior which is incompatible with the validity of these convexity estimates.

PROPOSITION 5.13. *The hypersurfaces of the limit flow M^∞_s obtained by the Hamilton's procedure described above are all convex.*

PROOF. First, since we are taking the limit of hypersurfaces with $\mathrm{H} \geq 0$, also the limit is mean convex. By strong maximum principle applied to the equation $\partial_t \mathrm{H} = \Delta \mathrm{H} + \mathrm{H}|\mathrm{A}|^2$ actually $\mathrm{H}^\infty(p,t) > 0$ for the limit flow, for every point in space and time.
Fixing a pair (p,s), if $Q_k \to +\infty$ is the rescaling factor for the flow φ_k, we have $\mathrm{H}_k(p,s) = \mathrm{H}(p, s/Q_k + t_k)/Q^k \to \mathrm{H}^\infty(p,t) > 0$, hence, $\mathrm{H}(p, s/Q_k + t_k) \to +\infty$. Now, since we have $\lambda^{\min} \geq -\eta \mathrm{H} - C$ for the original flow φ and $\mathrm{H} > \varepsilon$ at least for $t > \delta > 0$, we have $\lambda^{\min}/\mathrm{H} \geq -\eta - C/\mathrm{H}$ everywhere. When we rescale the hypersurfaces,

$$\frac{\lambda^{\min}_k(p,s)}{\mathrm{H}_k(p,s)} = \frac{\lambda^{\min}(p, s/Q_k + t_k)}{\mathrm{H}(p, s/Q_k + t_k)} \geq -\eta - \frac{C}{\mathrm{H}(p, s/Q_k + t_k)}$$

and sending $k \to \infty$ we conclude $\lambda^{\min}_\infty(p,s)/\mathrm{H}_\infty(p,s) \geq -\eta$.
Since $\eta > 0$ was arbitrary and the argument holds for every pair (p,s), the second fundamental is nonnegative definite on the whole limit flow, hence the hypersurfaces are all convex. □

REMARK 5.14. Instead of using Corollary 5.12, one can apply the same argument to the estimates of Theorem 5.11 obtaining that all the elementary symmetric functions of the eigenvalues of the second fundamental form are nonnegative at every point in space and time of the limit flow. By relation 3.5 it follows that the hypersurfaces are convex at every time.

REMARK 5.15. This conclusion holds also if the procedure is applied in the case of type I singularities, see Exercise 5.5. Actually, it is a consequence of the classification result in Chapter 4.

REMARK 5.16. This proposition (in a slightly stronger form) has been also obtained by White [**98**] by completely different techniques. His approach also works for the singularities of weak solutions which are defined after the first singular time.

The limiting hypersurfaces are convex, but in general not strictly convex. However, if they are not strictly convex then they necessarily split as the product of a flat factor and of a strictly convex one, as shown by the following result.

PROPOSITION 5.17 (Theorem 4.1 in [**59**]). *Let M^∞_s be as in the previous proposition. If the hypersurfaces are not strictly convex, then (up to a rigid motion) they can be written as $M^\infty_s = N^k_s \times \mathbb{R}^{n-k}$, where $1 \leq k \leq n$ and N^k_s is a family of strictly convex k–dimensional hypersurfaces moving by mean curvature in $\mathbb{R}^{k+1}$.*

PROOF. The proof is based on Hamilton's strong maximum principle for tensors in [**44**, Section 8] (see Appendix C, Theorem C.3).
Arguing as in Remark 3.36, as the conclusion of Hamilton's strong maximum principle for tensors in not affected if the manifold M is not compact (as it can happen in our case), we have that M (as a subset of $\mathbb{R}^{n+1}$) contains an $(n-k)$–dimensional affine subspace of $\mathbb{R}^{n+1}$ which is invariant in time.
Thus, the hypersurface splits at a product of an $(n-k)$–dimensional flat part and a strictly convex k–dimensional submanifold of $\mathbb{R}^{n+1}$ evolving by mean curvature. □

REMARK 5.18. If the singularity is of type I (see Exercise 5.5), then M^∞_s is a family of homothetically shrinking hypersurfaces. More precisely, the hypersurfaces M^∞_s are either of the form $\mathbb{S}^{n-k}_s \times \mathbb{R}^k$, for some $0 \leq k \leq (n-1)$ where $\mathbb{S}^{n-k}_s$ is an $(n-k)$–dimensional shrinking sphere, or of the form $\gamma(s) \times \mathbb{R}^{n-1}$, where $\gamma(s)$ is an Abresch–Langer curve, homothetically shrinking in the plane.

In the case of the evolution of mean convex hypersurfaces in a time interval $[0, T)$, by Proposition 3.21 and Corollary 3.22, the mean curvature H and $|\mathrm{A}|$ are comparable quantities, that is, there exists a constant α, independent of time such that $\alpha|\mathrm{A}| \leq \mathrm{H} \leq \sqrt{n}|\mathrm{A}|$ for $t \in [\delta, T)$. This implies that we can modify Hamilton's blow up procedure, substituting H^2 in place of $|\mathrm{A}|^2$ in equation (5.2), obtaining the same estimates on the second fundamental form and its derivatives. We still get an eternal smooth limit flow, complete with bounded curvature with the only difference that this time it is the mean curvature H which gets a global maximum (equal to one) at some point in space and time. This will be crucial to continue the analysis in the next sections.
Analogously, it is easy to see that the conclusions of Propositions 5.13 and 5.17 are not affected by this modification so also in this case the limit flow consists of convex hypersurfaces.
We call this limit flow "modified" Hamilton's blow up.

REMARK 5.19. Notice that for curves in $\mathbb{R}^2$ the two procedures coincide as $|\mathrm{A}| = |\mathrm{H}| = |k|$, where k is the usual curvature of a curve in the plane.

3. Curves in the Plane

Again, the case of a closed curve in $\mathbb{R}^2$ is special.
We suppose to deal with a generic, non convex, initial closed curve γ_0 in the plane $\mathbb{R}^2$ moving by mean curvature $\gamma : \mathbb{S}^1 \times [0, T) \to \mathbb{R}^2$ where at time $t = T$ we have a type II singularity. Setting ξ and k to be respectively the arclength and the curvature of γ_t, we have the evolution equation $k_t = k_{\xi\xi} + k^3$, then we define the function $z(t) = \#\{p \in \gamma_t \,|\, k(p) = 0\}$, counting the number of points on γ_t such that $k = 0$.
We need the following result of Angenent in [**10**, Proposition 1.2] (see [**9**] for the proof).

PROPOSITION 5.20. *If we have a mean curvature flow of a (possibly unbounded) curve in $\mathbb{R}^2$ in a open interval of time (a, b), at every fixed time, the points where k is zero are isolated in space. In particular, this implies that for a closed curve, the function z is finite at every positive time.*
The function z is nonincreasing during the flow, hence if at some time it is finite, it remains finite.
Finally, if at some point $p \in \gamma_t$ we have $k(p) = 0$ and $k_\xi(p) = 0$ then the zero point p for k immediately vanishes. To be precise, this means that there exists a small space interval I around p and a small $r > t$ such that k is never zero in $I \times (t, r)$.

We only mention that the proof is based on the application of maximum principle to the above evolution equation for the curvature.
By this proposition, we can define $\mathcal{I}_t$ to be the finite family of open intervals on γ_t where $k \neq 0$ and the following computation is justified,

$$\begin{aligned}
\frac{d}{dt}\int_{\gamma_t} |k|\, d\xi &= \sum_{I\in\mathcal{I}_i}\int_I [(\mathrm{sign}k)(k_{\xi\xi} + k^3) - |k|^3]\, d\xi \\
&= \sum_{I\in\mathcal{I}_i}\int_I [(\mathrm{sign}k)(k_{\xi\xi} + k^3) - |k|^3]\, d\xi \\
&= \sum_{I\in\mathcal{I}_t}\int_I (\mathrm{sgn}k) k_{\xi\xi}\, d\xi \\
&= -2 \sum_{p\in\gamma_t \,|\, k(p)=0} |k_\xi(p)|\,.
\end{aligned}$$

Hence, the integral $\int_{\gamma_t} |k|\, d\xi$, which is positive and finite (by compactness), is not increasing during the flow, so it converges to some value as $t \to T$, moreover, it is scaling invariant. We have, for every $t_1 < t_2$,

$$\int_{\gamma_{t_1}} |k|\, d\xi - \int_{\gamma_{t_2}} |k|\, d\xi = 2\int_{t_1}^{t_2} \sum_{p\in\gamma_t \,|\, k(p)=0} |k_\xi(p)|\, dt\,.$$

If now we rescale the curves following the procedure above for type II singularities, calling γ_s^n the rescaled flow at step n, converging to γ_s^∞ and denoting with $K_n \to +\infty$ the rescaling factor, we have

$$\begin{aligned} 2\int_a^b \sum_{p\in\gamma_s^n\,|\,k^n(p)=0} |k^n_\xi|\,ds &= \int_{\gamma_a^n} |k^n|\,d\xi - \int_{\gamma_b^n} |k^n|\,d\xi \\ &= \int_{\gamma_{a/K_n+t_n}} |k^n|\,d\xi - \int_{\gamma_{b/K_n+t_n}} |k^n|\,d\xi\,. \end{aligned}$$

Then, passing to the limit in $n \to \infty$ we conclude that at almost every $s \in \mathbb{R}$, by the arbitrariness of a and b, that

$$\sum_{p\in\gamma_s^\infty\,|\,k^\infty(p)=0} |k^\infty_\xi(p)| = 0$$

that is $k^\infty_\xi(p,s)$ is zero at every point p in space and s in time, where $k^\infty(p,s)$ is zero.
Again by means of Proposition 5.20, we conclude that for every $s \in \mathbb{R}$ as above, choosing any small $r > s$, the zero points of the curvature vanish for the curve γ_r^∞, hence k is positive and γ_r^∞ is strictly convex for every $r > s$ (strict convexity is preserved). Since we can draw this conclusion for almost every $s \in \mathbb{R}$, at every time the flow γ^∞ consists of strictly convex curves and k^∞ is never zero.

4. Hamilton's Harnack Estimate for Mean Curvature Flow

We have seen in the previous two sections that if a closed curve or a compact hypersurface with $\mathrm{H} \geq 0$ develops a type II singularity then the limit of the rescaled flows by the "modified" Hamilton's procedure is an eternal mean curvature flow of a convex, complete, hypersurface such that H takes its maximum in space and time at some point. We want now to see that this implies that this limit flow is a translating solution of motion by mean curvature. This can be obtained by means of the following two deep results of Hamilton [**50**].

THEOREM 5.21 (Harnack Estimate for Mean Curvature Flow). *Let $\varphi : M \times (A,T)$ be the mean curvature flow of complete convex hypersurfaces with uniformly bounded second fundamental forms.*
Let $C \in [A,T)$ and X a time dependent smooth tangent vector field on M. Then the following inequality holds,

$$\frac{\partial \mathrm{H}}{\partial t} + \frac{\mathrm{H}}{2(t-C)} + 2\langle \nabla \mathrm{H} \,|\, X\rangle + h_{ij}X^iX^j \geq 0$$

for every $t \in (C,T)$.

THEOREM 5.22. *Let $\varphi : M \times (A,T)$ be a complete strictly convex eternal solution to mean curvature flow with bounded curvature and be such that* H *takes its maximum in space and time at some point. Then, φ is a translating solution in $\mathbb{R}^{n+1}$.*

The proofs of these two theorems involve some smart and heavy computations with a strong use of maximum principle, we show the proof only in the one–dimensional and compact case, referring the interested reader to the original paper **[50]** (see also **[45]**).

PROOF OF THEOREM 5.21 – ONE–DIMENSIONAL COMPACT CASE. We suppose that the curves are compact and $C > A$, then k and all its derivatives are bounded in $[C, T-\varepsilon]$, for a small $\varepsilon > 0$. Moreover, by Proposition 3.20, in the same interval, $k > k_0 > 0$ for some positive constant k_0.
We fix the following notations: given the flow of convex curves $\gamma : \mathbb{S}^1 \times (\tau, T)$ we denote with θ the parameter on $\mathbb{S}^1$ and with s the arclength, τ is the tangent unit vector and $\nu = \mathrm{R}\tau$ is the unit normal, where $\mathrm{R} : \mathbb{R}^2 \to \mathbb{R}^2$ is the counterclockwise rotation of $\pi/2$, finally $k = \langle \partial_s \tau \,|\, \nu \rangle$ is the curvature.
Notice that $\partial_s = |\gamma_\theta|^{-1}\partial_\theta$.
We have the following commutation rule,

$$\partial_t \partial_s = \partial_s \partial_t + k^2 \partial_s \tag{5.4}$$

which implies easily the evolution equations

$$\begin{aligned}
\partial_t \tau &= \partial_t \partial_s \gamma = \partial_s \partial_t \gamma + k^2 \partial_s \gamma = \partial_s(k\nu) + k^2 \tau = k_s \nu \\
\partial_t \nu &= \partial_t(\mathrm{R}\tau) = \mathrm{R}\,\partial_t \tau = -k_s \tau \\
\partial_t k &= \partial_t \langle \partial_s \tau \,|\, \nu \rangle = \langle \partial_t \partial_s \tau \,|\, \nu \rangle = \langle \partial_s \partial_t \tau \,|\, \nu \rangle + k^2 \langle \partial_s \tau \,|\, \nu \rangle = \partial_s \langle \partial_t \tau \,|\, \nu \rangle + k^3 = k_{ss} + k^3 \,.
\end{aligned}$$

We define the Hamilton's quadratic

$$Z(\lambda) = \partial_t k + \frac{k}{2(t-C)} + 2\lambda k_s + k\lambda^2 = k_{ss} + k^3 + \frac{k}{2(t-C)} + 2\lambda k_s + k\lambda^2$$

which clearly is bounded below by

$$Z = k_{ss} + k^3 - k_s^2/k + \frac{k}{2(t-C)} \,.$$

We also define

$$W = k_{ss} + k^3 - k_s^2/k$$

and we start computing the evolution equation for this latter quantity,

$$
\begin{aligned}
(\partial_t - \partial_{ss})W =& \partial_t k_{ss} - \frac{2k_s\partial_t k_s}{k} + \frac{k_s^2 k_t}{k^2} + 3k^2 k_t - k_{ssss} + \frac{2k_s k_{sss}}{k} + \frac{2k_{ss}^2}{k} \\
& - \frac{5k_s^2 k_{ss}}{k^2} + \frac{2k_s^4}{k^3} - 6kk_s^2 - 3k^2 k_{ss} \\
=& \partial_s\partial_t k_s + k^2 k_{ss} - \frac{2k_s\partial_s k_t}{k} - 2kk_s^2 + \frac{k_s^2 k_{ss}}{k^2} + kk_s^2 + 3k^2 k_{ss} + 3k^5 \\
& - k_{ssss} + \frac{2k_s k_{sss}}{k} + \frac{2k_{ss}^2}{k} - \frac{5k_s^2 k_{ss}}{k^2} + \frac{2k_s^4}{k^3} - 6kk_s^2 - 3k^2 k_{ss} \\
=& \partial_{ss}(k_{ss} + k^3) + 2k^2 k_{ss} - 5kk_s^2 - \frac{2k_s\partial_s(k_{ss} + k^3)}{k} \\
& + \frac{k_s^2 k_{ss}}{k^2} + 3k^5 - k_{ssss} + \frac{2k_s k_{sss}}{k} + \frac{2k_{ss}^2}{k} - \frac{5k_s^2 k_{ss}}{k^2} + \frac{2k_s^4}{k^3} \\
=& k_{ssss} + 5k^2 k_{ss} - 5kk_s^2 - \frac{2k_s k_{sss}}{k} \\
& - \frac{4k_s^2 k_{ss}}{k^2} + 3k^5 - k_{ssss} + \frac{2k_s k_{sss}}{k} + \frac{2k_{ss}^2}{k} + \frac{2k_s^4}{k^3} \\
=& -5kk_s^2 + 3k^5 + \frac{2k_s^4}{k^3} + 5k^2 k_{ss} + \frac{2k_{ss}^2}{k} - \frac{4k_s^2 k_{ss}}{k^2}
\end{aligned}
$$

as $k_{ss} = (W + k_s^2/k - k^3)$, substituting, we get

$$
\begin{aligned}
(\partial_t - \partial_{ss})W =& -5kk_s^2 + 3k^5 + \frac{2k_s^4}{k^3} \\
& + 5k^2(W + k_s^2/k - k^3) \\
& + \frac{2(W + k_s^2/k - k^3)^2}{k} - \frac{4k_s^2(W + k_s^2/k - k^3)}{k^2} \\
=& -5kk_s^2 + 3k^5 + \frac{2k_s^4}{k^3} \\
& + 5k^2 W + 5kk_s^2 - 5k^5 \\
& + \frac{2W^2}{k} + \frac{2k_s^4}{k^3} + 2k^5 + \frac{4Wk_s^2}{k^2} - 4Wk^2 - 4kk_s^2 \\
=& \frac{2W^2}{k} + Wk^2\,.
\end{aligned}
\tag{5.5}
$$

Notice that, by maximum principle, if W is positive at some time, it stays positive. Unfortunately it can happen that the minimum of W can be negative at every time, so we add the strongly positive term $k/(2(t-C))$, that is $Z = W + k/(2(t-C))$.

$$
\begin{aligned}
(\partial_t - \partial_{ss})Z &= (\partial_t - \partial_{ss})W + \frac{k^3}{2(t-C)} - \frac{k}{2(t-C)^2}\\
&= \frac{2W^2}{k} + Wk^2 + \frac{k^3}{2(t-C)} - \frac{k}{2(t-C)^2}\\
&= \frac{2(Z - k/(2(t-C))^2) + k^3(Z - k/(2(t-C)))}{k} + \frac{k^3}{2(t-C)} - \frac{k}{2(t-C)^2}\\
&= \frac{2Z^2 + k^2/(2(t-C)^2) - 2Zk/(t-C)}{k} + \frac{k^3 Z - k^4/(2(t-C))}{k}\\
&\quad + \frac{k^3}{2(t-C)} - \frac{k}{2(t-C)^2}\\
&= \frac{2Z^2}{k} - \frac{2Z}{t-C} + k^2 Z\,.
\end{aligned}
$$

Letting $\underline{Z}(t) = \min_{\gamma_t} Z(\cdot, t)$ we have $\lim_{t\to C^+} \underline{Z}(t) = +\infty$, so $Z \geq \underline{Z}(t)$ is positive in some interval $(C, C+\delta)$, by maximum principle, Z cannot be zero on γ_t for $t \in (C, T-\varepsilon)$. Clearly, as we have that $Z > 0$, also $Z(\lambda) > 0$ for every function λ. Sending $\varepsilon \to 0$ and C to A (if necessary) we have the claim of the theorem. □

REMARK 5.23. When the curves γ_t are not compact there are two nontrivial technical points to take care of: the possible non existence of the minimum of $Z(t)$ and the fact that it is not granted that $\lim_{t\to C^+} \inf_{\gamma_t} Z(\cdot, t) = +\infty$, as k can go to zero at infinity. This requires some ε–perturbation in space of Z by means of a function growing enough at infinity and the addition of another function assuring that the resulting term diverges as $t \to C^+$ (see **[45]** for these technical details).

REMARK 5.24. The higher complexity in dealing with the case of general dimension is essentially due to the fact that the minimum of the quadratic

$$Z(X) = \frac{\partial \mathrm{H}}{\partial t} + \frac{\mathrm{H}}{2(t-C)} + 2\langle \nabla \mathrm{H} \,|\, X\rangle + h_{ij}X^i X^j$$

is not so explicit like for curves. Indeed, it is preferable to keep generic the vector field X in doing the computations.

PROOF OF THEOREM 5.22 – ONE–DIMENSIONAL COMPACT CASE. Suppose we have an eternal mean curvature flow γ_t of strictly convex curves in the plane. By Theorem 5.21 we have

$$Z = \partial_t k - k_s^2/k + k/(t-C) \geq 0$$

at every point and for every $t, C \in \mathbb{R}$ with $t > C$. Sending $C \to -\infty$ we get

$$W = \partial_t k - k_s^2/k \geq 0\,.$$

As we computed in equation (5.5) that

$$(\partial_t - \partial_{ss})W = \frac{2W^2}{k} + Wk^2\,,$$

if W is zero at some point in space and time, it must be zero everywhere. But, by hypotheses k takes a maximum at some point, hence, at such a point $k_t = k_s = 0$ which implies $W = 0$.
Hence, $k_t = k_s^2/k$ for all the curves of the evolution or equivalently, $k_{ss} + k^3 - k_s^2/k = 0$.
We set $v = -(\kappa_s/k)\tau + k\nu$ as a vector field in $\mathbb{R}^2$ along γ_t, obviously $\langle v \,|\, \nu\rangle_{\mathbb{R}^2} = k$. Then,

$$\begin{aligned}\partial_s v &= -\,(k_{ss}/k - k_s^2/k^2)\tau - (\kappa_s/k)k\nu + k_s\nu - k^2\tau\\ &= -\,(-k^2 + k_s^2/k^2 - k_s^2/k^2)\tau - k^2\tau = 0\end{aligned}$$

and

$$\begin{aligned}\partial_t v &= (-k_{ts}/k + k_s k_t/k^2 - kk_s)\tau + (-k_s^2/k + k_t)\nu\\ &= (-k_{st}/k - kk_s + k_s^3/k^3 - kk_s)\tau\\ &= (-[\partial_s(k_s^2/k)]/k + k_s^3/k^3 - 2kk_s)\tau\\ &= (-2k_sk_{ss}/k^2 + 2k_s^2/k^3 - 2kk_s)\tau\\ &= -\,2\frac{k_s}{k}(k_{ss} - k_s^2/k + k^3)\tau = 0\,.\end{aligned}$$

Hence, v is a vector field along γ_t, constant in space and time.
As $k = \langle v \,|\, \nu\rangle_{\mathbb{R}^2}$, we have that γ_t moves by translation under mean curvature flow. □

Then, we have the following theorem.

THEOREM 5.25. *The blow up limit by the Hamilton's* modified *procedure at a type II singularity of a closed curve in the plane or of a hypersurface with* $\mathrm{H} \geq 0$ *is a convex translating solution of mean curvature flow (possibly with multiplicities if the initial hypersurface is not embedded).*

REMARK 5.26. For curves in the plane, possibly with self–intersections, such that the initial curvature is never zero, this result was obtained via a different method by Angenent [**11**] (see also [**4**]), studying directly the parabolic equation satisfied by the curvature function.

In view of these results and the discussion about the classification of translating solution in Section 1, the strongest conjecture in this context is that every blow up limit via the Hamilton's *modified* procedure at a type II singularity of the evolution of an embedded hypersurface with $\mathrm{H} \geq 0$ is the only rotationally symmetric, strictly convex, translating solution.
In [**98**], White was able to exclude the possibility to get as a blow up limit the product of a grim reaper with $\mathbb{R}^{n-1}$.

More in general, also without assuming the condition $\mathrm{H} \geq 0$, one can conjecture that blow up limits like the minimal catenoid surface M in $\mathbb{R}^3$ given by

$$\Omega = \left\{(x, y) \in \mathbb{R}^2 \times \mathbb{R} \mid \cosh |y| = |x|\right\} .$$

See Ecker **[27]** for more details and check the recent paper by Sheng and Wang **[83]**.

5. The Special Case of Embedded Closed Curves in the Plane

In the special case of the evolution of an embedded closed curve in the plane, it is possible to exclude at all type II singularities. This, together with the case of convex, compact, hypersurfaces (as we have seen in the proof of Theorem 4.41 and 4.42) are the only cases in which this can be done.

By the previous section and embeddedness, any blow up limit must be translating and with unit multiplicity, that is, a grim reaper. We apply now a very geometric argument by Huisken in **[57]** in order to exclude also such possibility (see also **[51]** for another similar quantity).

Given the smooth flow γ_t of an initial embedded closed curve γ_0 on some interval $[0, T)$, we know that the curve stay embedded during the flow, so we can refer to every curve γ_t as a subset of $\mathbb{R}^2$. At every time $t \in [0, T)$, for every pair of points p and q in γ_t we define $d_t(p, q)$ to be the *geodesic* distance in γ_t of p and q, $|p - q|$ the standard distance in $\mathbb{R}^2$ and L_t the length of γ_t.
We consider the function $\Phi_t : \gamma_t \times \gamma_t \to \mathbb{R}$ defined as

$$\Phi_t(p, q) = \begin{cases} \frac{\pi|p-q|}{L_t} / \sin \frac{\pi d_t(p,q)}{L_t} & \text{if } p \neq q, \\ 1 & \text{if } p = q\,, \end{cases}$$

which is a perturbation of the quotient between the extrinsic and the intrinsic distance of a pair of points on γ_t.
Since γ_t is smooth and embedded for every time, the function Φ_t is well defined and positive. Moreover, it is easy to check that even if d_t is not C^1 at the pairs of points such that $d_t(p, q) = L_t/2$, the function Φ_t is C^1 in the open set $\{p \neq q\} \subset \gamma_t \times \gamma_t$ and continuous on $\gamma_t \times \gamma_t$.
By compactness, for every $t \in [0, T)$, the following infimum is actually a minimum in this case,

$$E(t) = \inf_{p,q \in \gamma_t} \Phi_t(p, q)\,. \tag{5.6}$$

As the curve γ_t has no self–intersections we have $0 < E(t) \leq 1$, the converse is clearly also true. Finally, since the evolution is smooth it is easy to see that the function $E : [0, T) \to \mathbb{R}$ is continuous.

LEMMA 5.27 (Huisken **[57]**). *The function $E(t)$ is monotone increasing in every interval where $E(t) < 1$.*

PROOF. We start differentiating in time $\Phi_t(p,q)$,

$$\begin{aligned}
\frac{d}{dt}\Phi_t(p,q) = & \frac{\pi}{L_t}\frac{\langle p-q\,|\,k(p)\nu(p)-k(q)\nu(q)\rangle}{|p-q|}\Big/\sin\frac{\pi d_t(p,q)}{L_t}\\
&+\left(\frac{\pi|p-q|}{L_t^2}\int_{\gamma_t}k^2\,ds\right)\Big/\sin\frac{\pi d_t(p,q)}{L_t}\\
&-\frac{\pi^2|p-q|}{L_t^2}\cos\frac{\pi d_t(p,q)}{L_t}\left(\frac{d_t(p,q)}{L_t}\int_{\gamma_t}k^2\,ds-\int_q^p k^2\,ds\right)\Big/\sin^2\frac{\pi d_t(p,q)}{L_t}\\
=&\left[\frac{\langle p-q\,|\,k(p)\nu(p)-k(q)\nu(q)\rangle}{|p-q|^2}+\frac{1}{L_t}\int_{\gamma_t}k^2\,ds\right.\\
&\left.-\frac{\pi}{L_t}\cot\frac{\pi d_t(p,q)}{L_t}\left(\frac{d_t(p,q)}{L_t}\int_{\gamma_t}k^2\,ds-\int_q^p k^2\,ds\right)\right]\Phi_t(p,q)\\
=&\left[\frac{\langle p-q\,|\,k(p)\nu(p)-k(q)\nu(q)\rangle}{|p-q|^2}+\frac{1}{L_t}\left(1-\frac{\pi d_t(p,q)}{L_t}\cot\frac{\pi d_t(p,q)}{L_t}\right)\int_{\gamma_t}k^2\,ds\right.\\
&\left.+\frac{\pi}{L_t}\cot\frac{\pi d_t(p,q)}{L_t}\int_q^p k^2\,ds\right]\Phi_t(p,q)
\end{aligned}$$

where s is the arclength and k the curvature of γ_t. It is then easy to see that being the function E the infimum of a family of locally uniformly Lipschitz functions, it is also locally Lipschitz, hence differentiable almost everywhere. Then, to prove the statement it is enough to show that $\frac{dE(t)}{dt} > 0$ for every time t such that this derivative exists. We will do that as usual, by Hamilton's trick, Lemma 3.3.
Let (p,q) a minimizing pair at a differentiability time t and suppose that $E(t) < 1$. By the very definition of Φ_t, it must be $p \neq q$.
We set $\alpha = \pi d_t(p,q)/L_t$ and notice that $\alpha\cot\alpha < 1$ as $\alpha \in (0,\pi/2]$. Moreover, $\int_{\gamma_t} k^2\,ds \geq \left(\int_{\gamma_t} k\,ds\right)^2/L_t \geq 4\pi^2/L_t$. Then, we have

$$\frac{d}{dt}E(t) \geq \left[\frac{\langle p-q\,|\,k(p)\nu(p)-k(q)\nu(q)\rangle}{|p-q|^2}+\frac{4\pi^2}{L_t^2}(1-\alpha\cot\alpha)+\frac{\pi}{L_t}\cot\alpha\int_q^p k^2\,ds\right]E(t)$$

that is,

$$\frac{d}{dt}\log E(t) \geq \frac{\langle p-q\,|\,k(p)\nu(p)-k(q)\nu(q)\rangle}{|p-q|^2}+\frac{4\pi^2}{L_t^2}(1-\alpha\cot\alpha)+\frac{\pi}{L_t}\cot\alpha\int_q^p k^2\,ds\,, \qquad (5.7)$$

at any minimizing pair (p,q).
Assume that the curve is parametrized counterclockwise in arclength, that $d_t(p,q) < L_t/2$ and that the geodesic connecting p and q is the counterclockwise oriented part of the curve from q to p, like in the figure.

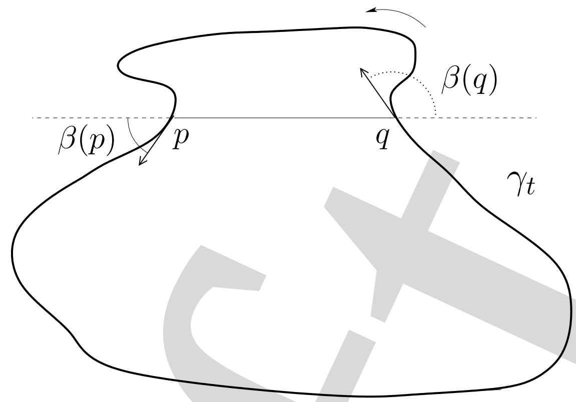

FIGURE 1.

We set $p(s) = \gamma_t(s_0 + s)$ with $p = \gamma_t(s_0)$, then, by minimality we have

$$0 = \frac{d}{ds}\Phi_t(p(s), q)\bigg|_{s=0} = \frac{\pi}{L_t}\frac{\langle p - q \,|\, \tau(p)\rangle}{|p-q|}\Big/\sin\frac{\pi d_t(p,q)}{L_t} - \frac{\pi|p-q|}{L_t \sin^2\frac{\pi d_t(p,q)}{L_t}}\frac{\pi\cos\frac{\pi d_t(p,q)}{L_t}}{L_t}$$

where we denoted with $\tau(p)$ the oriented unit tangent vector to γ_t at p.
By this last equality we get

$$\cos\beta(p) = \frac{\langle p - q \,|\, \tau(p)\rangle}{|p-q|} = \frac{\pi|p-q|}{L_t\sin\frac{\pi d_t(p,q)}{L_t}}\cos\frac{\pi d_t(p,q)}{L_t} = E(t)\cos\alpha$$

where $\beta(p)$ is the angle between the vectors $p - q$ and $\tau(p)$.
Repeating this argument for the other point q we get

$$\cos\beta(q) = -E(t)\cos\alpha$$

where, as before, $\beta(q)$ is the angle between $q - p$ and $\tau(q)$, see Figure 1. Clearly, $\beta(q) = \pi - \beta(p)$.
Notice that if one of these intersection is tangential, we would have $E(t)\cos\alpha = 1$ which is impossible as we assumed that $E(t) < 1$. Moreover, by the relation $\cos\beta(p) = E(t)\cos\alpha < \cos\alpha$ it follows that $\beta > \alpha$.

We look now at the second variation of Φ_t, at the same minimizing pair of points (p, q). With the same notation, if $p = \gamma_t(s_1)$ and $q = \gamma_t(s_2)$ we set $p(s) = \gamma_t(s_1 + s)$ and $q(s) = \gamma_s(s_2 - s)$. After a straightforward computation one gets,

$$\begin{aligned}0 \le \frac{d^2}{ds^2}\Phi_t(p(s), q(s))\bigg|_{s=0} &= \frac{\pi}{L_t}\left(\frac{\langle p - q \,|\, k(p)\nu(p) - k(q)\nu(q)\rangle}{|p-q|} + \frac{4\pi^2|p-q|}{L_t^2}\right)\Big/\sin\frac{\pi d_t(p,q)}{L_t}\\ &= \left[\frac{\langle p - q \,|\, k(p)\nu(p) - k(q)\nu(q)\rangle}{|p-q|^2} + \frac{4\pi^2}{L_t^2}\right]E(t)\,.\end{aligned}$$

Hence, getting back to inequality (5.7), we have

$$\begin{aligned}\frac{d}{dt}\log E(t) &\geq \frac{\langle p-q\,|\,k(p)\nu(p)-k(q)\nu(q)\rangle}{|p-q|^2}+\frac{4\pi^2}{L_t^2}(1-\alpha\cot\alpha)+\frac{\pi}{L_t}\cot\alpha\int_q^p k^2\,ds\\ &\geq -\frac{4\pi^2}{L_t^2}\alpha\cot\alpha+\frac{\pi}{L_t}\cot\alpha\int_q^p k^2\,ds\\ &=\frac{\pi\cot\alpha}{L_t}\left(\int_q^p k^2\,ds-\frac{4\pi}{L_t}\alpha\right),\end{aligned}$$

so it remains to show that this last expression is positive. As

$$\int_p^q k^2\,ds \geq \left(\int_p^q k\,ds\right)^2 / d_t(p,q)$$

and noticing that $\int_p^q k\,ds$ is the angle between the tangent vectors $\tau(p)$ and $\tau(q)$, we have $\left(\int_p^q k\,ds\right)^2 = 4\beta(p)^2 > 4\alpha^2$, as we concluded above.
Thus,

$$\begin{aligned}\frac{d}{dt}\log E(t) \geq &\frac{\pi\cot\alpha}{L_t}\left(\int_q^p k^2\,ds-\frac{4\pi}{L_t}\alpha\right)\\ > &\frac{\pi\cot\alpha}{L_t}\left(\frac{4\alpha^2}{d_t(p,q)}-\frac{4\pi}{L_t}\alpha\right)\\ = &\,0\end{aligned}$$

recalling that $\alpha = \pi d_t(p,q)/L_t$. ☐

REMARK 5.28. Clearly, by its definition and this lemma, the function E is always nondecreasing. Actually, to be more precise, by means of a simple geometric argument it can be proved that if $E(t) = 1$ the curve must be a circle. Hence, in any other case E is strictly increasing in time.

REMARK 5.29. This lemma clearly implies that an initial embedded closed curve cannot develop a self–intersection during mean curvature flow, otherwise E would get zero, which is impossible as $E(0) > 0$ and E is nondecreasing.

An immediate consequence of this lemma is that for every initial embedded, closed curve in $\mathbb{R}^2$, there exists a positive constant C depending on the initial curve such that on all $[0,T)$ we have $E(t) \geq C$. The same conclusion holds for any rescaling of such curves as the function E is scaling invariant by construction.

REMARK 5.30. This lemma also provide an alternative proof of the fact that an initial embedded, closed curve stays embedded, that is, it cannot develop a self–intersection during mean curvature flow, otherwise E would get zero.

We can then exclude *Type II* singularities, indeed, as γ^∞ is a grim reaper and it is the limit of rescalings of curves of the family γ_t, the function E for such grim reaper (which

is constant in time, since it moves by translation) is not smaller, at any time, than the infimum of the corresponding functions for the approximating curves, hence, by the discussion above, following the lemma, it is bounded below by some positive constant C.
But, if we consider a pair of points p, q on any grim reaper Γ_t such that the segment $[p, q]$ is orthogonal to the velocity vector $w \in \mathbb{R}^2$ and we send such segment infinity, we can see that $\Phi_t(p, q) \to 0$, hence $E(\Gamma_t) = 0$, indeed, the distance $|p - q|$ is bounded by a constant (the width of the strip where the grim reaper lives) and the intrinsic distance $d_t(p, q)$ diverges.
This is in contradiction with the above conclusion.

PROPOSITION 5.31. Type II *singularities cannot develop during the mean curvature flow of an embedded, closed curve in* $\mathbb{R}^2$.

Collecting together the results of Chapter 4 about type I singularities and this last proposition, we obtain the following Theorem due to Grayson **[41]**, whose original proof is more geometric and direct, showing that the intervals of negative curvature vanish in finite time, before any singularity.

THEOREM 5.32. *Let γ_0 be a closed, smooth embedded curve in the plane and let γ_t, for $t \in [0, T)$ be its maximal evolution by mean curvature. There exists a time $\widehat{t} < T$ such that $\gamma_{\widehat{t}}$ is convex.*
As a consequence, the result of Gage and Hamilton 4.41 applies and subsequently the curve shrinks smoothly to a point $t \to T$.

PROOF. As we said no type II singularities are possible and the only type I singularities have a circle as limit of rescalings, see Section 5.
Hence, at some point the curve must have become convex. □

We add a final remark in this case of embedded curves.
Letting $A(t)$ to be the area enclosed by γ_t which moves by mean curvature, we have

$$\frac{d}{dt}A(t) = -\int_{\gamma_t} k\, ds = -2\pi\,,$$

hence, as the evolution is smooth till the curve shrinks to a point at time $T > 0$ and clearly $A(t)$ goes to zero, we have $A(0) = 2\pi T$. That is, the existence time is exactly equal to the initial enclosed area divided by 2π.

5.1. An Alternative Proof of Grayson's Theorem. Ideas and techniques are related to the work of Ilmanen **[65]**.

In the very special case of curves in the plane, one can avoid the use of Hamilton's Harnack inequality in order to deal with type II singularities, but still work with Huisken's monotonicity formula and produce a homothetic blow up.

As underlined in Remarks 4.25 and 4.27, in general $\Sigma > 1$, otherwise the curvature is uniformly bounded as $t \nearrow T$. Moreover, the estimates in Lemma 4.12 also are independent of the type I hypothesis. Then, rescaling the curves around the moving points

x_t like in Remark 4.35, we have

$$\sigma(0) - \Sigma = \int_{-\frac{1}{2}\log T}^{+\infty} \int_{\gamma_r} e^{-\frac{|y|^2}{2}} \left| \widetilde{k} + \langle y \,|\, \widetilde{\nu} \rangle \right|^2 \, ds \, dr < +\infty \,.$$

Clearly, since we are not assuming the type I hypothesis, the curvatures of the rescaled curves $\widetilde{k}$ are not bounded, but by this formula, it follows that for every family of disjoint intervals $(a_i, b_i) \subset [-\frac{1}{2}\log T, +\infty)$ such that $\sum_{i\in\mathbb{N}}(b_i - a_i) = +\infty$ we can find a sequence $r_i \in (a_i, b_i)$ such that

$$\lim_{i\to\infty} \frac{1}{\sqrt{2\pi}} \int_{\widetilde{\gamma}_{r_i}} e^{-\frac{|y|^2}{2}} \left| \widetilde{k} + \langle y \,|\, \widetilde{\nu} \rangle \right|^2 \, ds = 0 \tag{5.8}$$

and

$$\lim_{i\to\infty} \frac{1}{\sqrt{2\pi}} \int_{\widetilde{\gamma}_{r_i}} e^{-\frac{|y|^2}{2}} \, ds = \lim_{i\to\infty} \sigma(t(r_i)) = \Sigma \,. \tag{5.9}$$

Clearly, the sequence r_i converges monotonically increasing to $+\infty$. From the estimate (4.9) on the local length, it follows that the sequence of curves $\widetilde{\gamma}_{r_i}$ has curvatures *locally* equibounded in L^2. Hence, we can extract a subsequence (not relabeled) that, after a possible reparametrization, converges in C^1_{loc} to a limit curve $\widetilde{\gamma}_\infty$. Such curve satisfies $\widetilde{k} + \langle x \,|\, \widetilde{\nu} \rangle = 0$, as the integral $\int_{\widetilde{\gamma}} e^{-\frac{|y|^2}{2}} \left| \widetilde{k} + \langle y \,|\, \widetilde{\nu} \rangle \right|^2 \, ds$ is lower semicontinuous under C^1_{loc}–convergence. Moreover, by a bootstrap argument, $\widetilde{\gamma}_\infty$ is smooth, then, it must be an Abresch–Langer curve.
If the initial curve was embedded, as the Huisken's quantity E is scaling invariant and upper semicontinuous under the C^1_{loc}–convergence of curves, E is bounded below also for the limit curve by a positive constant, hence, $\widetilde{\gamma}_\infty$ is embedded, then it must be a line for the origin or the unit circle by the classification theorem 4.36.
Since the second point of Lemma 4.12 implies that

$$\lim_{i\to\infty} \frac{1}{\sqrt{2\pi}} \int_{\widetilde{\gamma}_{q_i}} e^{-\frac{|y|^2}{2}} \, ds = \frac{1}{\sqrt{2\pi}} \int_{\widetilde{\gamma}_\infty} e^{-\frac{|y|^2}{2}} \, ds \,,$$

and the first limit is equal to $\Sigma > 1$ by equation (5.9), we conclude that $\widetilde{\gamma}_\infty$ is the unit circle.

By what we said above we can find $r_i \nearrow +\infty$ such that the curves γ_{r_i} converge in C^1_{loc} to the unit circle. Moreover, being the unit circle compact, the convergence is actually C^1 with equibounded curvatures in L^2 (not only locally).

Fixing $i \in \mathbb{N}$ and letting $\rho = r - r_i$, we look at the evolution of the following quantity,

$$
\begin{aligned}
\frac{d}{dr}\int_{\widetilde{\gamma}_r}(\widetilde{k}^2+\rho\widetilde{k}_s^2)\,ds =\, & 2(T-t)\frac{d}{dt}\int_{\gamma_t}\sqrt{2(T-t)}\,k^2\,ds+\int_{\widetilde{\gamma}_r}\widetilde{k}_s^2\,ds\\
& + 2(T-t)\rho\frac{d}{dt}\int_{\gamma_t}(\sqrt{2(T-t)})^3\,k_s^2\,ds\\
=& -\sqrt{2(T-t)}\int_{\gamma_t}k^2\,ds+(\sqrt{2(T-t)})^3\int_{\gamma_t}(2kk_{ss}+k^4)\,ds+\int_{\widetilde{\gamma}_r}\widetilde{k}_s^2\,ds\\
& -3(\sqrt{2(T-t)})^3\rho\int_{\gamma_t}k_s^2\,ds+(\sqrt{2(T-t)})^5\rho\int_{\gamma_t}(2k_sk_{sss}+7k^2k_s^2)\,ds\\
=& \int_{\widetilde{\gamma}_r}[-\widetilde{k}^2+2\widetilde{k}\widetilde{k}_{ss}+\widetilde{k}^4+\widetilde{k}_s^2-3\rho\widetilde{k}_s^2+2\rho\widetilde{k}_s\widetilde{k}_{sss}+7\rho\widetilde{k}^2\widetilde{k}_s^2]\,ds\\
\leq& \int_{\widetilde{\gamma}_r}[-\widetilde{k}_s^2+\widetilde{k}^4-2\rho\widetilde{k}_{ss}^2+7\rho\widetilde{k}^2\widetilde{k}_s^2]\,ds\\
=& \int_{\widetilde{\gamma}_r}[-\widetilde{k}_s^2+\widetilde{k}^4+\rho(-2\widetilde{k}_{ss}^2+7\widetilde{k}^3\widetilde{k}_{ss}/3)]\,ds\\
\leq& \int_{\widetilde{\gamma}_r}[-\widetilde{k}_s^2+\widetilde{k}^4+\rho(-2\widetilde{k}_{ss}^2+C\widetilde{k}^6+\widetilde{k}_{ss}^2)]\,ds\\
=& \int_{\widetilde{\gamma}_r}[-\widetilde{k}_s^2+\widetilde{k}^4+C\rho\widetilde{k}^6]\,ds\,.
\end{aligned}
$$

Using the following interpolation inequalities for any closed curve in the plane of length L (see Aubin [**16**, p. 93]),

$$
\|\widetilde{k}\|_{L^4}^4\leq C\|\widetilde{k}_s\|_{L^2}\|\widetilde{k}\|_{L^2}^3+\frac{C}{L}\|\widetilde{k}\|_{L^2}^4\quad\text{and}\quad\|\widetilde{k}\|_{L^6}^6\leq C\|\widetilde{k}_s\|_{L^2}^2\|\widetilde{k}\|_{L^2}^4+\frac{C}{L^2}\|\widetilde{k}\|_{L^2}^6
$$

which imply, by means of Young inequality,

$$
\begin{aligned}
\int_{\widetilde{\gamma}_r}\widetilde{k}^4\,ds\,&\leq 1/2\int_{\widetilde{\gamma}_r}\widetilde{k}_s^2\,ds+C\left(\int_{\widetilde{\gamma}_r}\widetilde{k}^2\,ds\right)^3+\left(\int_{\widetilde{\gamma}_r}\widetilde{k}^2\,ds\right)^3+\frac{C}{L^3(\widetilde{\gamma}_r)}\\
C\rho\int_{\widetilde{\gamma}_r}\widetilde{k}^6\,ds\,&\leq\left(\rho\int_{\widetilde{\gamma}_r}\widetilde{k}_s^2\,ds\right)^3+C\left(\int_{\widetilde{\gamma}_r}\widetilde{k}^2\,ds\right)^3+\frac{C}{L^2(\widetilde{\gamma}_r)}\left(\int_{\widetilde{\gamma}_r}\widetilde{k}^2\,ds\right)^3,
\end{aligned}
$$

we can conclude, as we know that $L(\widetilde{\gamma}_r)\geq\int_{\widetilde{\gamma}_r}e^{-\frac{|y|^2}{2}}\,ds\geq\sqrt{2\pi}$,

$$
\frac{d}{dr}\int_{\widetilde{\gamma}_r}(\widetilde{k}^2+\rho\widetilde{k}_s^2)\,ds\leq C\left(\int_{\widetilde{\gamma}_r}\widetilde{k}^2\,ds\right)^3+\left(\rho\int_{\widetilde{\gamma}_r}\widetilde{k}_s^2\,ds\right)^3+C\leq C\left(\int_{\widetilde{\gamma}_r}(\widetilde{k}^2+\rho\widetilde{k}_s^2)\,ds\right)^3+C\,,
$$

for a constant C independent of $r\geq r_i$ and $i\in\mathbb{N}$.
Integrating this differential inequality for the quantity $Q_i(r)=\int_{\widetilde{\gamma}_r}(\widetilde{k}^2+(r-r_i)\widetilde{k}_s^2)\,ds$ in the interval $[r_i, r_i+\delta]$ it is easy to see that if $\delta>0$ is small enough, we have $Q_i(r)\leq$

$C(\delta, Q_i(r_i)) = C\left(\delta, \int_{\widetilde{\gamma}_{r_i}} \widetilde{k}^2\, ds\right) = C(\delta)$, for every $r \in [r_i, r_i + 2\delta]$, as the curves $\widetilde{\gamma}_{r_i}$ have uniformly bounded curvature in L^2. Hence, if $r \in [r_i + \delta, r_i + 2\delta]$ we have the estimate

$$\int_{\widetilde{\gamma}_r} (\widetilde{k}^2 + \delta \widetilde{k}_s^2/2)\, ds \leq \int_{\widetilde{\gamma}_r} (\widetilde{k}^2 + (r - r_i)\widetilde{k}_s^2)\, ds \leq C(\delta)$$

which implies

$$\int_{\widetilde{\gamma}_r} \widetilde{k}^2\, ds \leq C(\delta) \quad \text{and} \quad \int_{\widetilde{\gamma}_r} \widetilde{k}_s^2\, ds \leq \frac{2C(\delta)}{\delta}\,.$$

We can now, as before, find a sequence of values $q_i \in [r_i + \delta/2, r_i + \delta]$ such that

$$\lim_{i\to\infty} \frac{1}{\sqrt{2\pi}} \int_{\widetilde{\gamma}_{q_i}} e^{-\frac{|y|^2}{2}} \left|\widetilde{k} + \langle y \,|\, \widetilde{\nu}\rangle\right|^2 ds = 0\,.$$

As this new sequence of rescaled curves $\widetilde{\gamma}_{q_i}$ also satisfies the length estimate (4.9) and has $\widetilde{k}$ and $\widetilde{k}_s$ uniformly bounded in L^2, we can extract another subsequence (not relabeled) that, after a possible reparametrization, converges in C^2 to a limit curve which is still the unit circle.

Then, the curves $\widetilde{\gamma}_{q_i}$ definitely have positive curvature, hence, they are convex. This means that the same hold for γ_t for some time t, which is Grayson's result.

REMARK 5.33. Pushing a little forward this analysis, one can actually prove along the same lines also the asymptotic convergence of the *full* sequence of rescaled curves to the unit circle in C^∞, as proved by Gage and Hamilton in [**36, 37, 38**].

REMARK 5.34. Actually, the C^1_{loc}–convergence to a line in the case $\Sigma = 1$ also simplifies the application of White's Theorem in this special case of curves. Indeed, the boundedness of the curvature around every $x_0 \in \mathcal{S}$ also follows by the interior estimates of Ecker and Huisken. We give a sketch of the proof.

As $\Sigma = 1$, by the C^1_{loc}–convergence of the rescaled curves, for every $R > 2$ there is a sequence of times $t_i \nearrow T$ and a line L passing for x_0 such that every curve γ_{t_i} is a graph over L in the ball $B_{2R\sqrt{2(T-t_i)}}(x_0)$, indeed, the distance of $\gamma_{t_i} \cap B_{2R\sqrt{2(T-t_i)}}(x_0)$ from $\mathrm{L} \cap B_{2R\sqrt{2(T-t_i)}}(x_0)$ in the C^1–norm goes to zero.

Then, supposing that $x_0 = 0$ and that L is $\langle e_1\rangle$ in $\mathbb{R}^2$, the pieces of curves $\gamma_t \cap B_{2R\sqrt{2(T-t_i)}}$ can be represented as a graph of a function at least for a small time. Moreover, the quantity $v(x,t) = \langle \nu(x,t) \,|\, e_2\rangle^{-1}$ is small at time $t = t_i$ and $x \in \gamma_{t_i} \cap B_{2R\sqrt{2(T-t_i)}}$. As the sphere $\partial B_{\sqrt{2(T+\varepsilon-t)}}$ is moving by curvature and, choosing $\varepsilon > 0$ small enough, at time $t = t_i$ it is contained in the ball $B_{2R\sqrt{2(T-t_i)}}$, by a geometric comparison argument it is not possible that other parts of the moving curve "get into" the ball $B_{\sqrt{2(T+\varepsilon-t)}}$ at time $t > t_i$. Hence, the only way that $\gamma_t \cap B_{\sqrt{2(T+\varepsilon-t)}}$ can possibly stop to be a graph is that the tangent vector to such graph becomes vertical at some time, equivalently, the function v is not bounded.

The interior estimates of Ecker and Huisken (B.1) and (B.2) exclude this fact if we start

with v small enough. Hence, with a suitable choice of one of the times t_i, the curvature of γ_t for $t \in [t_i, T)$ is bounded in the ball $B_{\sqrt{2(T+\varepsilon-t)}}$, in particular it is bounded in $B_{\sqrt{2\varepsilon}}(x_0) \subset B_{\sqrt{2(T+\varepsilon-t)}}$ for every $t \in [t_i, T)$.
By a compactness argument, the curvature is then uniformly bounded as $t \to T$, which is impossible as T is the maximal time of existence of the flow.

Notice the key point in getting a bound on the curvature by means of this argument which is the C^1_{loc}–convergence of the rescaled curves to a line, implying that locally they are graphs.

We underline that the interesting point of this line in proving Grayson's Theorem (or equivalently, in analysing the possible singularities) is the fact that we did not distinguish between *type I* and *type II* singularities. Indeed, the curvature of the rescaled curves can be unbounded, but the control in L^2_{loc} is enough to imply the C^1_{loc}–convergence which is sufficient to have the smoothness of the limit curve. The fact that the control of the mean curvature in L^2_{loc} is not strong enough to imply give the C^1_{loc}–convergence of a subsequence, is the main reason why this unitary line of analysis is difficult to be pursued in higher dimensions, in order to obtain homothetic blow up limits even for type II singularities.
Look anyway at the very interesting results of Ilmanen in dimension two [**65**] (which is, in some sense, the critical case).
All this discussion underline the variational nature of the arguments (in particular, monotonicity formula) in the analysis of type I singularities, against the non–variational point of view (substantially maximum principle) in dealing with type II ones. Indeed, one is always able to produce an "homothetic" blow up limit, even dealing with "nonsmooth hypersurfaces" (or simply sets) moving according to some generalized (variational) notion of mean curvature, if monotonicity formula holds (see [**65**, Lemma 7]). The difficulty is to show the regularity of such limit. In some special cases this is possible, even if considering evolutions of singular sets, see [**62**] for networks of curves and [**65**, Lemma 7], for surfaces in $\mathbb{R}^3$.

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Interior Estimates of Ecker and Huisken

Let $M_t = \operatorname{graph} u(\,\cdot\,,t)$ be a mean curvature flow such that it is locally the graph of a function $u(y,t)$ over the hyperplane $e_{n+1}^\perp \approx \mathbb{R}^n$, for $t\in[0,T]$.
Let ν be the normal to the moving graph of u and define the function $v = \langle \nu \,|\, e_{n+1}\rangle^{-1}$ which we assume being positive.

The estimates contained in the following series of theorems have been obtained by Ecker and Huisken in the paper **[29]**, see also **[55]** and **[26]**.

THEOREM B.1. *Let $R>0$ and $x_0\in\mathbb{R}^{n+1}$ be arbitrary and define*

$$\varphi(x,t) = R^2 - |x-x_0|^2 - 2nt\,.$$

If φ_+ denotes the positive part of φ, we have the estimate

$$v(x,t)\varphi_+(x,t) \leq \sup_{M_0} v\varphi_+ \tag{B.1}$$

as long as $v(x,t)$ is defined everywhere on the support of φ_+.

THEOREM B.2. *The gradient of the height function u satisfies the estimate*

$$\begin{aligned}\sqrt{1+|\nabla u(y_0,t)|^2} \leq C_1(n) &\sup_{B_R(y_0)} \sqrt{1+|\nabla u_0|^2}\\ &\times \exp\left[C_2(n)R^{-2}\sup_{s\in[0,T]}\Big(\sup_{r\in[0,T],y\in B_R(y_0)} u(y,r)-u(y_0,s)\Big)^2\right]\end{aligned}$$

where $t\in[0,T]$, $B_R(y_0)$ is a ball in $e_{n+1}^\perp$ and $u_0 = u(\,\cdot\,,0)$.

THEOREM B.3. *Let $R>0$ and $\theta\in[0,1)$, then for $x_0\in\mathbb{R}^{n+1}$ we have the estimate*

$$\sup_{K(x_0,t,\theta R^2)} |\mathrm{A}|^2 \leq C(n)(1-\theta)^{-2}t^{-1}\sup_{0\leq s\leq t}\sup_{K(x_0,s,R^2)} v^4 \tag{B.2}$$

for all $t\in[0,T]$, where $K(x_0,t,\theta R^2) = \{x\in M_t \,|\, |x-x_0|^2+2nt\leq\theta R^2\}$.

Working by induction, one gets the following general result.

THEOREM B.4. (1) *For arbitrary $R>0$, $\theta\in[0,1)$ and $m\in\mathbb{N}$, we have the estimates*

$$\sup_{K(x_0,t,\theta R^2)} |\nabla^m \mathrm{A}|^2 \leq C_m(n)t^{-(m+1)}\,,$$

where $C_m(n)$ depends on n, m, θ and $\sup_{0\leq s\leq t}\sup_{K(x_0,s,R^2)} v^2$.

(2) *In case of additional smoothness, the constants above can be replaced by constants* $D_{m,k}(n)$*, depending on* n, m, k, θ *and* $\sup_{0\leq s\leq t}\sup_{K(x_0,s,R^2)}\sum_{i=0}^{m}|\nabla^i \mathrm{A}|^2$ *obtaining the improved estimates*

$$\sup_{K(x_0,t,\theta R^2)} |\nabla^{m+k}\mathrm{A}|^2 \leq D_{m,k}(n)t^{-k}\,.$$

(3) *In particular, choosing* $m = 0$ *and* $k = 1$*,*

$$\sup_{K(x_0,t,\theta R^2)} |\nabla \mathrm{A}|^2 \leq E(n)/t\,,$$

where $E(n)$ *depends on* n, θ *and* $\sup_{0\leq s\leq t}\sup_{K(x_0,s,R^2)}|\mathrm{A}|^2$.

Analogous estimates were obtained by Angenent **[12]** for evolving curves in the plane, and by Altschuler **[4]** and Altschuler and Grayson **[3]** for curves in space.

REMARK B.5. Compare these interior estimates with the ones of Colding and Minicozzi **[23]**.

APPENDIX C

Hamilton's Maximum Principle for Tensors

Let V a vector bundle over a compact manifold M. Let h a fixed metric on V, g a Riemannian metric on M and L a connection on V compatible with h. Both g and $L = \{L^{\beta}_{i\alpha}\}$ may depend on time t. We can form the Laplacian of a section f of V as the trace of the second covariant derivative with respect to g, using the connection L on V and the Levi–Civita connection on TM.

Let U an open subset of V and $\Psi(f)$ a vector field on V tangent to the fibers. We consider the nonlinear PDE

$$\partial_t f = \Delta f + \Psi(f) \tag{PDE}$$

and the ODE

$$\partial_t f = \Delta f\,. \tag{ODE}$$

THEOREM C.1 (Hamilton [**44**, Section 4]). *Let X be a closed subset of $U \subset V$ invariant under parallel transport by the connection L and every fiber of X is convex.*
If the solutions of the ODE starting in a fiber of X remain in X, then also any solution of the PDE remains in X.

THEOREM C.2 (Hamilton [**44**, Section 8]). *Let f a smooth section of V satisfying $\partial_t f = \Delta f + \Psi(f)$. Let $Z(f)$ be a convex function on the bundle, invariant under parallel translation whose level curves $Z(f) \leq \lambda$ are preserved by the ODE. Then, the inequality $Z(f) \leq \lambda$ is preserved by the PDE for any constant λ.*
Moreover, if at time $t = 0$ at some point we have $Z(f) < \lambda$, then $Z(f) < \lambda$ everywhere on M at time $t > 0$.

THEOREM C.3 (Hamilton [**44**, Section 8]). *Let B be a symmetric bilinear form on V. Suppose that B satisfies a heat equation $\partial_t \mathrm{B} \geq \Delta \mathrm{B} + \Psi(\mathrm{B})$ where the matrix $\Psi(\mathrm{B}) \geq 0$ for all $\mathrm{B} \geq 0$.*
Then, if $\mathrm{B} \geq 0$ at time $t = 0$ it remains nonnegative for $t \geq 0$. Moreover, there exists an interval $0 < t < \delta$ on which the rank of B is constant and the null space of B is invariant under parallel translation and invariant in time, finally it also lies in the null space of $\Psi(\mathrm{B})$.