

Recall :

- every space X yields a topos $\text{Sh}(X)$
- every continuous map $X \xrightarrow{f} Y$ yields a geometric morphism $\text{Sh}(X) \xrightarrow{f^*} \text{Sh}(Y)$, i.e. a pair of functors with $(f^* \dashv f_*)$ and f^* preserving finite limits.

Topological properties of spaces / continuous maps can be translated into categorical properties of toposes / geometric morphisms.

Often, things that we think of as properties of a space X turn out to be properties of the geometric morphism $\text{Sh}(X) \rightarrow \text{Set}$, i.e. of the continuous map $X \rightarrow 1$. This leads to the possibility of ‘relativizing’ these properties.

Example 1 : open maps

- For a continuous map $X \xrightarrow{f} Y$, the following are equivalent:
 - (1) f is an open map.
 - (2) $f^{-1}: \mathcal{O}(Y) \rightarrow \mathcal{O}(X)$ has a left adjoint $f_!: \mathcal{O}(X) \rightarrow \mathcal{O}(Y)$, satisfying the ‘Frobenius reciprocity’ condition
$$f_!(V \cap f^*(U)) = f_!(V) \cap U.$$
 - (3) $f^{-1}: \mathcal{O}(Y) \rightarrow \mathcal{O}(X)$ is a homomorphism of complete Heyting algebras.
 - (4) $f^*: \text{Sh}(Y) \rightarrow \text{Sh}(X)$ preserves full first-order logic.

So we define a geometric morphism $\mathcal{E} \xrightarrow{f} \mathcal{E}'$ to be open iff f^* preserves first-order logic.

Given $F: \mathcal{E} \rightarrow \mathcal{F}$ preserving finite limits, we have for each $A \in \text{ob } \mathcal{E}$ a natural comparison map $F(\Omega_{\mathcal{E}}^A) \xrightarrow{\phi_A} \Omega_{\mathcal{F}}^{FA}$, obtained as follows:

$$\begin{array}{c} B \longrightarrow \Omega_{\mathcal{E}}^A \\ \hline ? \longrightarrow B \times A \\ \hline F? \longrightarrow FB \times FA \\ \hline FB \longrightarrow \Omega_{\mathcal{F}}^{FA} \end{array}$$

We say F is **logical** if ϕ_A is an isomorphism for all A , i.e. if F preserves full higher-order logic.

Theorem For a geometric morphism $\mathcal{F} \xrightarrow{f} \mathcal{E}$, the following are equivalent:

- (1) f is open, i.e., f^* preserves first-order logic.
- (2) f^* is 'sub-logical', i.e. ϕ_A is a monomorphism for all A .
- (3) For any $A \in \text{ob } \mathcal{E}$ and any $B \rightarrow f^*A$, the image of the composite $\mathcal{F}/B \rightarrow \mathcal{F}/f^*A \xrightarrow{f^*/A} \mathcal{E}/A$ is of the form $\mathcal{E}/f_!^A B$ for some $f_!^A B \rightarrow A$.

Site characterization: Suppose \mathcal{E} is a Grothendieck topos, and write \mathcal{S} for the topos of sets. Then

$\mathcal{E} \xrightarrow{P} \mathcal{S}$ is open $\Leftrightarrow \mathcal{E} \simeq \text{Sh}(\mathcal{C}, \mathcal{J})$ for some site $(\mathcal{C}, \mathcal{J})$ such that every \mathcal{J} -covering sieve is ~~empty~~ inhabited.

Classically, this result is vacuous. But it remains true over an arbitrary base topos \mathcal{S} !

Using it, we can show:

Theorem Given a pullback square

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{h} & \mathcal{F} \\ \downarrow k & & \downarrow f \\ \mathcal{G} & \xrightarrow{g} & \mathcal{E} \end{array}$$

of geometric morphisms,

- (1) f open implies k open;
- (2) f open & surjective implies k open & surjective;
- (3) f open implies the 'weak Beck-Chevalley condition' that the canonical natural transformation

$$f^* g_* \rightarrow h_* k^*$$

is monic.

(N.B.: if we stabilize (3) under pullback, it characterizes open maps.)

Example 2: (Locally) connected maps.

- For a space X , the following are equivalent:

- (1) X is connected.
- (2) $p^*: \text{Set} \rightarrow \text{Sh}(X)$ is full and faithful.
- (3) $p_*: \text{Sh}(X) \rightarrow \text{Set}$ preserves coproducts.

We take condition (2) as the definition of connectedness for a geometric morphism $\mathcal{F} \xrightarrow{f} \mathcal{E}$.

(It's still equivalent to (3), provided we 'relativize' the latter properly.)

- For a space X , the following are equivalent:

(1) X is locally connected.

(2) For every local homeomorphism $E \rightarrow X$, the components of E are open.

(3) $p^*: \text{Set} \rightarrow \text{Sh}(X)$ has a left adjoint $p_!$.

(4) $p^*: \text{Set} \rightarrow \text{Sh}(X)$ preserves exponentials.

We define a geometric morphism $\mathcal{F} \xrightarrow{f} \mathcal{E}$ to be locally connected if

f^* has an \mathcal{E} -indexed left adjoint $f_!$ (i.e., for each A ,

$(f/A)^*: \mathcal{E}/A \rightarrow \mathcal{F}/f^*A$ has a left adjoint, and the adjoints 'fit together nicely' as A varies). This is still equivalent to saying that $(f/A)^*$ preserves exponentials for all A .

N.B.: locally connected implies open.

Site characterization: suppose \mathcal{E} is a Grothendieck topos over \mathcal{S} .

Then $\mathcal{E} \xrightarrow{P} \mathcal{S}$ is (connected and) locally connected \Leftrightarrow

$\mathcal{E} \simeq \text{Sh}(\mathcal{C}, \mathcal{T})$ for some site $(\mathcal{C}, \mathcal{T})$ such that every

\mathcal{T} -covering sieve is connected (as a full subcategory of \mathcal{C}/U for some U) (and such that \mathcal{C} has a terminal object).

Using this, we can show:

Theorem Given a pullback square

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{h} & \mathcal{F} \\ \downarrow k & & \downarrow f \\ \mathcal{G} & \xrightarrow{g} & \mathcal{E} \end{array}$$

(1) f locally connected implies k locally connected;

(2) f connected & locally connected implies k connected & locally connected;

(3) f locally connected implies the 'strong Beck-Chevalley condition'

that $f^* g_* \rightarrow h_* k^*$ is an isomorphism.

(Again, the stabilization of (3) under pullback characterizes local connectedness.)

Example 3: atomic maps.

- For a continuous map $X \xrightarrow{f} Y$, the following are equivalent:
 - (1) f is a local homeomorphism.
 - (2) f and the diagonal $X \xrightarrow{\Delta} X \times_Y X$ are both open maps.

By definition, a local homeomorphism of toposes is a morphism of the form $\mathcal{E}/A \rightarrow \mathcal{E}$ for some $A \in \text{ob } \mathcal{E}$. Its inverse image is the functor A^* sending B to $(B \times A \xrightarrow{\pi_2} A)$, and this is a logical functor.

But this doesn't quite 'fit' with condition (2) above.

Theorem For a geometric morphism $\mathcal{F} \xrightarrow{f} \mathcal{E}$, the following are equivalent:

- (1) f^* is a logical functor.
- (2) f and the diagonal $\mathcal{F} \xrightarrow{\Delta} \mathcal{F} \times_{\mathcal{E}} \mathcal{F}$ are both open maps.

We call such morphisms atomic. Clearly, if $\mathcal{E} \rightarrow \text{Set}$ is atomic, then \mathcal{E} must be Boolean, and we have

Proposition For a complete Boolean algebra B , the following are equivalent:

- (1) $\text{Sh}(B) \rightarrow \text{Set}$ is atomic.
- (2) B is an atomic Boolean algebra.
- (3) $\text{Sh}(B) \simeq \text{Sh}(X)$ for a discrete space X .

But there are other toposes which are atomic over Set : for example, $[G, \text{Set}]$ for any group G , or more generally the category $\text{Cont}(G)$ of continuous G -sets for a topological group G .

Remark: atomic morphisms are (stable under pullback, & hence) locally connected.

For any locally connected $\mathcal{F} \xrightarrow{f} \mathcal{E}$, we have a factorization

$$\mathcal{F} \longrightarrow \mathcal{E}/f_! 1 \longrightarrow \mathcal{E}$$

where the first factor is connected & locally connected, and the second is a local homeomorphism.

If $\mathcal{F} \xrightarrow{f} \mathcal{E}$ is connected and atomic, then the image of f^* is closed under arbitrary subobjects in \mathcal{F} .

We call connected morphisms with this property hyperconnected:

they are exactly the morphisms for which f_* preserves Ω , or equivalently, such that $f^*: \text{Sub}_{\mathcal{E}}(A) \longrightarrow \text{Sub}_{\mathcal{F}}(f^*A)$ is bijective for all $A \in \text{ob } \mathcal{E}$.

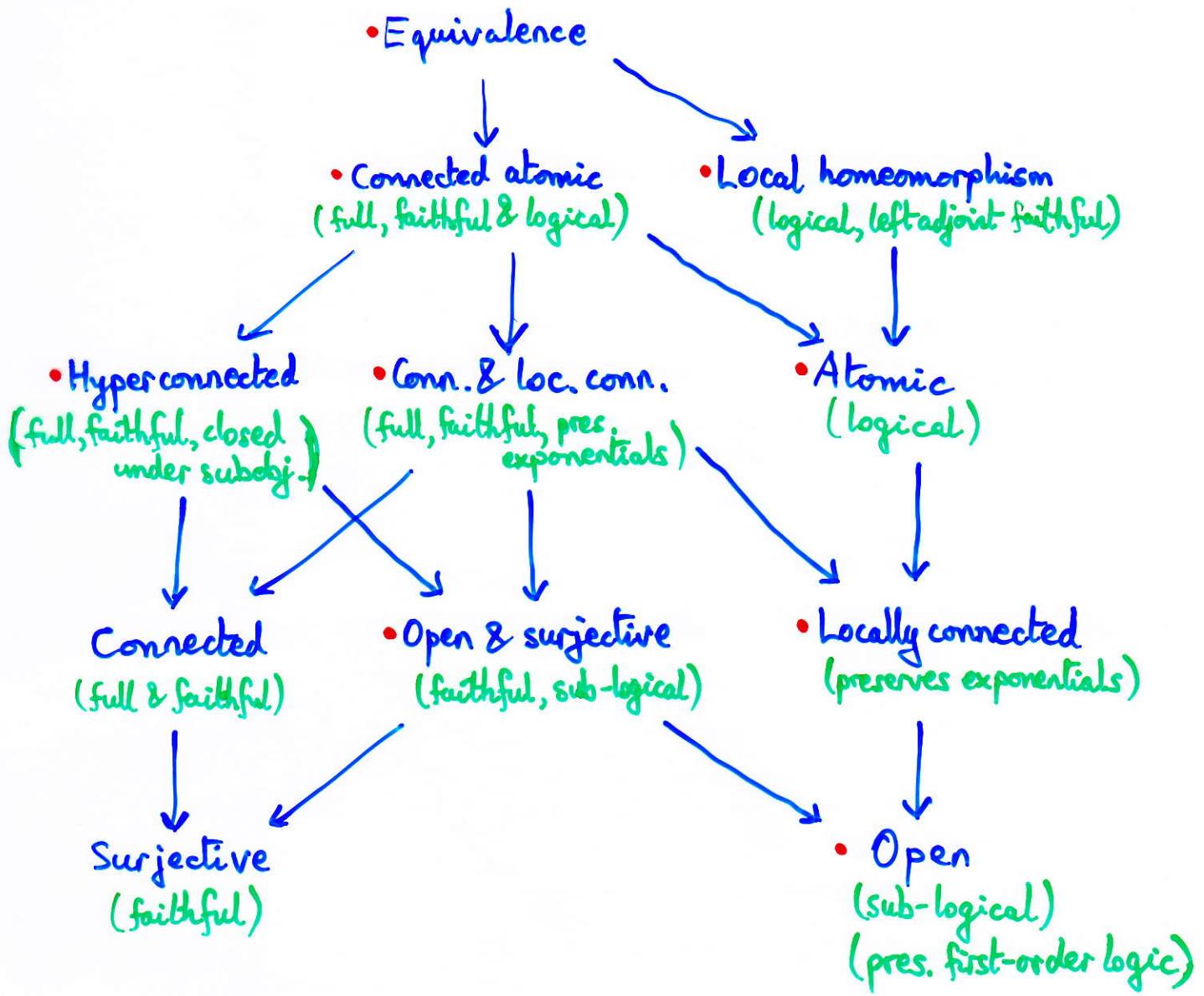
Any hyperconnected morphism between toposes of the form $\text{Sh}(X)$ is an equivalence. So the only atomic morphisms between such toposes are local homeomorphisms.

Using descent theory, one can show that if $\mathcal{F} \xrightarrow{f} \mathcal{E}$ is connected atomic and has a section $\mathcal{E} \xrightarrow{\Delta} \mathcal{F}$, then there exists an internal localic group G in \mathcal{E} (like a topological group) such that $\mathcal{F} \cong \text{Cont}_{\mathcal{E}}(G)$, the topos of objects of \mathcal{E} equipped with a continuous G -action.

However, there exist connected atomic morphisms (even when $\mathcal{E} = \text{Set}$) which have no sections.

Summary Table of Implications

(words in green denote properties of inverse image functors)



• = stable under pullback.

Example 4 : local maps

- For a space X , the following are equivalent:

- (1) X has a focal point, i.e. a point whose only neighbourhood is X .
- (2) $p_* : \text{Sh}(X) \rightarrow \text{Set}$ has a right adjoint.

(Example: $X = \text{spec } R$ for a local ring R .)

We define a geometric morphism $\mathcal{F} \xrightarrow{f} \mathcal{E}$ to be local if it satisfies the following equivalent conditions:

- (1) $f_* : \mathcal{F} \rightarrow \mathcal{E}$ has an \mathcal{E} -indexed right adjoint f^* .
- (2) f is connected and f_* has an (ordinary) right adjoint.
- (3) f has a section $c : \mathcal{E} \rightarrow \mathcal{F}$ which is left adjoint to f in the 2-category Top of toposes and geometric morphisms.

Examples: (1) a continuous map $X \xrightarrow{f} Y$ induces a local map $\text{Sh}(X) \rightarrow \text{Sh}(Y)$ iff each fibre $f^{-1}(y)$ has a focal point, and the mapping $y \mapsto (\text{focal point of } f^{-1}(y))$ is continuous.

(2) a functor $\mathcal{C} \xrightarrow{F} \mathcal{D}$ between small categories induces a local map $[\mathcal{C}^{\text{op}}, \text{Set}] \rightarrow [\mathcal{D}^{\text{op}}, \text{Set}]$ iff F has a full and faithful right adjoint.

Theorem Given a pullback square

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{h} & \mathcal{F} \\ \downarrow k & & \downarrow f \\ \mathcal{G} & \xrightarrow{g} & \mathcal{E} \end{array}$$

with f local,

- (1) k is local, and its left adjoint is obtained by factoring c_g through h (where c is the left adjoint of f);
- (2) the 'dual Beck-Chevalley condition' holds, i.e. $g^* f_* \rightarrow k_* h^*$ is an isomorphism.

(But (2) doesn't characterize local maps, even if we stabilize it under pullback.)

There's also a site characterization: given a Grothendieck topos \mathcal{E} , $\mathcal{E} \xrightarrow{\text{P}} \text{Set}$ is local iff $\mathcal{E} \simeq \text{Sh}(\mathcal{C}, \mathcal{J})$ where \mathcal{C} has a terminal object which is \mathcal{J} -irreducible, i.e. has no covering sieves except the maximal one.

By 'relativizing' this, we get

Theorem Let $(\mathcal{C}, \mathcal{J})$ and $(\mathcal{D}, \mathcal{K})$ be sites such that

\mathcal{C} and \mathcal{D} have finite limits, and let $F: \mathcal{C} \rightarrow \mathcal{D}$ be a full & faithful functor which preserves finite limits and both preserves and reflects covers, i.e.

$R \in \mathcal{J}(U)$ implies $\{F\alpha \mid \alpha \in R\}$ generates a \mathcal{K} -covering sieve on FU , and

$S \in \mathcal{K}(FU)$ implies $\{\alpha: V \rightarrow U \mid F\alpha \in S\} \in \mathcal{J}(U)$.

Then there is a local geometric morphism

$$\text{Sh}(\mathcal{D}, \mathcal{K}) \longrightarrow \text{Sh}(\mathcal{C}, \mathcal{J})$$

whose direct image is given by composing with F .

Moreover, all local morphisms between Grothendieck toposes arise in this way (and we can even choose the sites in such a way that F has a left adjoint).