

We begin with four examples.

(A) Continuous Let Sp be the category of topological spaces & continuous maps. We have a Grothendieck topology \mathcal{T} on Sp s.t. a sieve $R \subseteq \text{Sp}(-, X)$ is covering iff there exists an open cover $\{U_i \mid i \in I\}$ of X for which all the inclusions $U_i \rightarrow X$ are in R .

Unfortunately Sp is 'too big' for $\text{Sh}(\text{Sp}, \mathcal{T})$ to be a topos.

But we can take any small full subcategory $\mathcal{C} \subseteq \text{Sp}$, closed under passage to open subspaces, and consider $\text{Sh}(\mathcal{C}, \mathcal{T})$.

For any space X (not just those in \mathcal{C}), the functor $yX = \text{Sp}(-, X)|_{\mathcal{C}}$ is a sheaf; and $X \mapsto yX$ is a functor $\text{Sp} \rightarrow \text{Sh}(\mathcal{C}, \mathcal{T})$ (in general, not full & faithful; but it can be so on a larger category than \mathcal{C}).

For any $X \in \text{ob } \mathcal{C}$, the full embedding $\mathcal{O}(X) \rightarrow \mathcal{C}/X$ preserves and reflects covers; so it induces a local geometric morphism $\text{Sh}(\mathcal{C}, \mathcal{T})/yX \rightarrow \text{Sh}(X)$.

(Again, this can work for some spaces not in \mathcal{C} , but not for all of them.)

(B) Smooth Let M_f be the category of (paracompact) smooth manifolds & smooth maps. (N.B.: M_f doesn't have all pullbacks; but it does have pullbacks of open inclusions along arbitrary maps.) We define the Grothendieck topology \mathcal{T} just as we did on S_p ; then $Sh(M_f, \mathcal{T})$ is a topos. For any manifold M , the functor $yM = M_f(-, M)$ is a sheaf, and $M \mapsto yM$ is a full embedding $M_f \rightarrow Sh(M_f, \mathcal{T})$. And, for any M , the inclusion $O(M) \rightarrow M_f/M$ induces a local geometric morphism $Sh(M_f, \mathcal{T})/yM \rightarrow Sh(M)$.

(C) Algebraic Let K be a field, and let \mathcal{G} be the category of affine schemes of finite type over K (i.e. \mathcal{G}^{op} is the category of finitely-presented K -algebras). For a K -algebra A , an open subscheme of A is the dual of $A \xrightarrow{\quad} A[a^{-1}]$ for some $a \in A$; note that these are epimorphisms of K -algebras and stable under pushout.

The Zariski topology \mathcal{Z} on \mathcal{G} is generated by families $(A \xrightarrow{\quad} A[a_i^{-1}] \mid i \in I)$ s.t. $\{a_i \mid i \in I\}$ generates A as an ideal. We have a topos $Sh(\mathcal{G}, \mathcal{Z})$; for any K -scheme X , the functor $Sch_K(-, X)|_{\mathcal{G}} = yX$ is a sheaf, and $X \mapsto yX$ is a full embedding $Sch_K \rightarrow Sh(\mathcal{G}, \mathcal{Z})$.

And for $X = \text{spec } A$ in \mathcal{G} , we have a local morphism $Sh(\mathcal{G}, \mathcal{Z})/yX \rightarrow Sh(X)$.

Remark: $Sh(\mathcal{G}, \mathcal{Z})$ is the classifying topos for the theory of local K -algebras.

(D) Combinatorial Let Δ be the category of nonempty finite totally ordered sets & order-preserving maps, and let $\mathcal{E} = [\Delta^{\text{op}}, \text{Set}]$ be the topos of simplicial sets.

This doesn't look much like the other three examples, but ...

In \mathcal{E} , we have a non-full subcategory \mathcal{D} of 'degeneracy-reflecting' maps: $A_{\cdot} \xrightarrow{f} B_{\cdot}$ is in \mathcal{D} if, given $\sigma \in A_n$ st. $f(\sigma) = s_i^{n-1}(\tau)$ for some $\tau \in B_{n-1}$, there exists $\tilde{\tau} \in A_{n-1}$ st. $\sigma = s_i^{n-1}(\tilde{\tau})$ (and hence also $f(\tilde{\tau}) = \tau$).

For any object A_{\cdot} of \mathcal{E} , \mathcal{D}/A_{\cdot} is a full, coreflective subcategory of \mathcal{E}/A_{\cdot} ; the coreflection sends $(B_{\cdot} \rightarrow A_{\cdot})$ to $(B'_{\cdot} \rightarrow A_{\cdot})$ where B'_{\cdot} is the subobject of B_{\cdot} generated by simplices which map to non-degenerate simplices in A_{\cdot} .

Moreover, \mathcal{D}/A_{\cdot} is a topos, and the coreflection is the direct image of a connected (in fact local) geometric morphism $\mathcal{E}/A_{\cdot} \rightarrow \mathcal{D}/A_{\cdot}$.

We can think of \mathcal{D}/A_{\cdot} as the 'little topos' of the 'combinatorial space' A_{\cdot} : in general it's not spatial, but note that $\mathcal{D}/1_{\cdot} \simeq \text{Set}$, since $B_{\cdot} \rightarrow 1_{\cdot}$ is degeneracy-reflecting iff B_{\cdot} is generated by its 0-simplices.

How do we recognize when a given Grothendieck topos is 'big'?

(1) Lawvere's approach: 'axiomatic cohesion'.

Start from the assumption that there's a connected and locally connected morphism $p: \mathcal{E} \rightarrow \mathcal{S}$ where \mathcal{S} is a topos of abstract 'non-cohesive' sets.

N.B.: Example (A) isn't locally connected over Set if we allow non-locally-connected spaces into the site \mathcal{B} ; but it is if all spaces in \mathcal{B} are locally connected.

Thus \mathcal{S} sits inside \mathcal{E} (via p^*) as a full subcategory; and it is both coreflective (via p_* , sending an object A to the discrete set of its points $1 \rightarrow A$) and reflective (via $p_!$, sending A to its set of connected components).

There's a canonical natural transformation $\theta: p_* \rightarrow p_!$:

θ_A is the unique morphism such that

$$\begin{array}{ccc} p^* p_* A & \xrightarrow{p^* \theta_A} & p^* p_! A \\ \text{counit:} \swarrow & & \searrow \text{unit} \\ A & & \end{array}$$

commutes.

Lawvere's axioms for a 'category of cohesion':

- (a) $p_!$ preserves finite products (i.e., finite products of connected objects are connected).
- (b) The canonical map $p_! (A^{p^* I}) \longrightarrow p_! A^{p! p^* I} \cong p_! A^I$ is an isomorphism (i.e., \mathcal{S} -indexed products of connected objects are connected).
- (c) θ_A is epic for all A (i.e., every connected object has a point).

Comments on the axioms :

1. (a) is needed to ensure that the intrinsic notion of homotopy, where $A \xrightarrow{f} B$ are homotopic iff $1 \xrightarrow{\frac{f}{g}} B^A$ have the same image under $p_!$, is a congruence on \mathcal{E} .
2. (b) says that we can 'chain together' \mathcal{S} -indexed families of homotopies. It holds in the topological example (A), but not in the combinatorial example (D).
3. (c) holds (with $\mathcal{S} = \text{Set}$) in the algebraic example (C) iff the ground field K is algebraically closed. For a non-algebraically closed K , perhaps we should take \mathcal{S} to be $\text{Cont}(G)$ where G is the Galois group of \bar{K}/K ?
4. For $\mathcal{E} = \text{Sh}(X)$, (a) holds iff $p_!$ preserves all finite limits (so that the adjoint pair $(p_! \dashv p^*)$ defines a right adjoint for p in Top), iff X is totally connected (i.e. every open set in X is connected), iff X has a dense point.
(c) holds for $\text{Sh}(X)$ iff p is an equivalence.
5. (c) holds iff the unit $A \rightarrow p^* p_! A$ is epic for all A and p_* preserves epimorphisms.

The first condition is equivalent to saying that the image of p^* is closed under subobjects, i.e. that p is hyperconnected.

The second is equivalent to saying that p is local !

Thus we have $p^*: \mathcal{S} \rightarrow \mathcal{E}$ embedding non-cohesive sets in \mathcal{E} as 'indiscrete spaces'; corresponding to Θ we have $\phi: p^* \rightarrow p^*$, and it's pointwise monic.

Moreover, (c) implies (a).

A 'degenerate' case (pure quality): suppose θ is an isomorphism, so that p_* is both left and right adjoint to p^* . Then the homotopy relation collapses ($f \simeq g$ iff $f = g$).

Given a connected & locally connected $\mathcal{E} \xrightarrow{p} \mathcal{S}$, we could consider the full subcategory \mathcal{F} on objects A s.t. θ_A is an isomorphism. p^* factors through $\mathcal{F} \subseteq \mathcal{E}$, and the induced morphism $\mathcal{F} \rightarrow \mathcal{S}$ is pure quality; moreover, \mathcal{F} is both reflective & coreflective in \mathcal{E} , so we have a connected & locally connected $\mathcal{E} \xrightarrow{s} \mathcal{F}$. s is hyperconnected if p is; but it needn't be local if p is.

Lawvere's axiom (d): For every object A of \mathcal{E} , there's a monomorphism $A \rightarrow B$ with B contractible (i.e. all powers B^c connected). This holds iff Ω is contractible, iff there's a contractible object I with two disjoint points $0, 1$. (In particular, it holds in each of our four basic examples.)

On the other hand, it's incompatible with pure quality.

If \mathcal{S} is Boolean (or even satisfies De Morgan's law), then axiom (d) holds iff $p^*(2)$ is connected.

If $\mathcal{S} = \text{Set}$, it holds iff $p_! p^*(I) \cong 1$ for all $I \neq \emptyset$.

Conjecture: for general \mathcal{S} , it holds iff $p_! p^*$ is the 'support' functor $\mathcal{S} \rightarrow \mathcal{S}$, i.e. $I \rightarrowtail p_! p^*(I) \rightarrowtail 1$ for all I .

(2) Calibrations (Joyal-Moerdijk, Duncan, PTJ)

Definition A pre-calibration (or class of étale maps) in a topos \mathcal{E} is a class of maps \mathcal{D} such that

- (a) \mathcal{D} contains all isomorphisms, & is closed under composition;
- (b) \mathcal{D} is stable under pullback;
- (c) \mathcal{D} descends along epimorphisms;
- (d) $B_i \xrightarrow{f_i} A_i \in \mathcal{D}$ for all $i \in I$ implies $\coprod_{i \in I} B_i \longrightarrow \coprod_{i \in I} A_i \in \mathcal{D}$;
- (e) $\coprod_{i \in I} 1 \longrightarrow 1 \in \mathcal{D}$ for all I ;
- (f) Given $C \xrightarrow{g} B \xrightarrow{f} A$ with $g, fg \in \mathcal{D}$, then $f \in \mathcal{D}$;
- (g) If $B \xrightarrow{f} A \in \mathcal{D}$, then $B \xrightarrow{\Delta} B \times_A B \in \mathcal{D}$.

These axioms imply that, for any A , \mathcal{D}/A is a full subcategory of \mathcal{E}/A , closed under finite limits and arbitrary colimits.

Say \mathcal{D} is a calibration if \mathcal{D}/A is coreflective in \mathcal{E}/A for all A .

In this case it's a topos, and the coreflection is a connected geometric morphism $\mathcal{E}/A \xrightarrow{p_A} \mathcal{D}/A$. We say a calibration is local/hyperconnected/... if all the p_A have the corresponding property.

Remark: if a pre-calibration contains all monomorphisms in \mathcal{E} , then it's a (hyperconnected) calibration.

If \mathcal{D} is a calibration, then by (b) we have a commutative square

$$\begin{array}{ccc} \mathcal{E}/A & \xrightarrow{f} & \mathcal{E}/B \\ \downarrow p_A & & \downarrow p_B \\ \mathcal{D}/A & \longrightarrow & \mathcal{D}/B \end{array}$$

for any $A \xrightarrow{f} B$ in \mathcal{E} .

If $f \in \mathcal{D}$ then this square is a pullback;

if f is epic then (c) implies that it's a pushout.

Examples of (pre) calibrations:

- (1) If $\mathcal{E} \xrightarrow{p} \mathcal{S}$ is a connected geometric morphism, then there's a pre-calibration \mathcal{D} consisting of those $A \xrightarrow{f} B$ in \mathcal{E} such that

$$\begin{array}{ccc} p^* p_* A & \longrightarrow & A \\ \downarrow p^* p_* f & & \downarrow f \\ p^* p_* B & \longrightarrow & B \end{array}$$

is a pullback. (If p is hyperconnected, then this is a hyperconnected calibration.) Note that $\mathcal{D}/p^* I \simeq \mathcal{S}/I$ for any $I \in \text{ob } \mathcal{S}$, and in particular $\mathcal{D}/I \simeq \mathcal{S}$.

- (2) For any \mathcal{E} , there is a 'canonical' calibration in $[\mathcal{Z}, \mathcal{E}]$ consisting of those morphisms which are pullback squares in \mathcal{E} . This calibration is locally connected, but not local or hyperconnected.

- (3) If $\mathcal{F} \xrightarrow{f} \mathcal{E}$ is a geometric morphism and \mathcal{D} is a ~~ext~~(pre)calibration in \mathcal{F} , then $f_* \mathcal{D} = \{g \mid f^*(g) \in \mathcal{D}\}$ is a (pre)calibration in \mathcal{E} .

Note that in the calibration case, the square

$$\begin{array}{ccc} \mathcal{F}/f^* A & \xrightarrow{f/A} & \mathcal{E}/A \\ \downarrow & & \downarrow \\ \mathcal{D}/f^* A & \longrightarrow & f_* \mathcal{D}/A \end{array}$$

is a pushout for any A .

- (4) For any small category \mathcal{C} and any set S of morphisms of \mathcal{C} , we have a calibration on $[\mathcal{C}^{\text{op}}, \text{Set}]$ consisting of morphisms s.t. the naturality squares for all members of S are pullbacks. (If S consists of split epimorphisms, this calibration is hyperconnected.) In particular, the degeneracy-reflecting maps are a calibration in $[\Delta^{\text{op}}, \text{Set}]$, which is locally connected, hyperconnected and local.

(5) Let $\mathcal{E} = \text{Sh}(\mathcal{B}, \mathcal{J})$ where \mathcal{J} is such that all representables are sheaves.

Suppose given, for each $U \in \text{ob } \mathcal{B}$, a connected morphism $\mathcal{E}/yU \xrightarrow{q_U} \mathbb{F}_U$, and for each $V \xrightarrow{\alpha} U$ in \mathcal{B} , a geometric morphism $\hat{\alpha}$ making

$$\begin{array}{ccc} \mathcal{E}/yV & \xrightarrow{y\alpha} & \mathcal{E}/yU \\ \downarrow q_V & \hat{\alpha} & \downarrow q_U \\ \mathbb{F}_V & \xrightarrow{\quad} & \mathbb{F}_U \end{array}$$

commute. Suppose also that the class \mathcal{D}_0 of morphisms $A \rightarrow_{yU}$ isomorphic to objects in the image of q_U^* (for some U) satisfies

- Given $B \xrightarrow{g} A \xrightarrow{f} yU$ with $f \in \mathcal{D}_0$, we have $fg \in \mathcal{D}_0$ iff the pullback of g along any $yV \rightarrow A$ belongs to \mathcal{D}_0 .
- Given $A \xrightarrow{f} yU$, if there's a \mathcal{J} -covering sieve R on U s.t. $(y\alpha)^* f \in \mathcal{D}_0$ for all $\alpha \in R$, then $f \in \mathcal{D}_0$.
- Any complemented monomorphism $A \rightarrow_{yU}$ is in \mathcal{D}_0 .

Then we get a calibration \mathcal{D} on \mathcal{E} by saying $B \xrightarrow{f} A \in \mathcal{D}$ iff all pullbacks of f along morphisms $yU \rightarrow A$ are in \mathcal{D}_0 .

Moreover, $\mathcal{D}/yU \cong \mathbb{F}_U$ for all $U \in \text{ob } \mathcal{B}$.

This yields calibrations on the toposes in Examples (A), (B) and (C).

However, in general they're not local. To get a local calibration by this method, we need not only that each q_U is local, but also the Beck-Chevalley condition $\hat{\alpha}^* q_{U*} \xrightarrow{\sim} q_{V*} (y\alpha)^*$. (Recall from Lecture I that this would be automatic if the squares were pullbacks, but they aren't in general.)