

Higher-Order Type Theory and Toposes

(a) Type structure: we have a unit type 1

(possibly) primitive types S, T, \dots

product types $A \times B$ (A, B types)

power-types PA (A a type)

(possibly) a type N of natural nos.

(b) Terms: built up from variables x^A, y^A, \dots of each type by

(possibly) primitive function-symbols $f: A \rightarrow B$

(including constants $c: 1 \rightarrow B$)

a unique constant $*$ of type 1

pairing & unpairing operators $\langle x, y \rangle: A \times B$, $\text{fst}(z): A$, $\text{snd}(z): B$

where $x: A, y: B, z: A \times B$ are terms

comprehension operator $\{x: A \mid \varphi\}: PA$

where φ is a formula and x a variable (which is bound in $\{x: A \mid \varphi\}$)

a constant $0: N$ and function-symbol $s: N \rightarrow N$, plus a term-constructor rec (the 'recursor', which builds terms whose interpretations are recursively defined morphisms)

(c) Formulae: built up from atomic formulae

$(s =_A t)$ (s, t terms of type A)

$(s \in_A t)$ (s of type A , t of type PA)

(possibly) $R_A(s)$ (R_A a primitive predicate symbol)
(but not needed, since we can replace R_A by a constant $r: PA$)

by logical connectives $\top, \perp, \wedge, \vee, \Rightarrow$

and quantifiers $(\exists x^A), (\forall x^A)$

(N.B.: $\neg \varphi$ is defined to be $(\varphi \Rightarrow \perp)$, and $(\varphi \Leftrightarrow \psi)$ to be $((\varphi \Rightarrow \psi) \wedge (\psi \Rightarrow \varphi))$.)

(d) Axioms and rules of inference: standard axioms and rules for (constructive) first-order logic, plus the 'obvious' rules for product and power types (in particular, $((x \in \{y:A \mid \varphi\}) \Leftrightarrow \varphi[x/y])$ and $(\{x:A \mid x \in t\} = t)$ provided x not free in t) and (less obvious) rules for type N , if it exists.

These rules are all sound for interpretations in toposes, where we interpret the primitive types, function-symbols and relation symbols (if any) by appropriate objects, morphisms and subobjects in the topos. Moreover, interpretations are preserved by logical functors between toposes.

But they're also complete: given a theory \mathbb{T} in higher-order type theory, we can construct a topos $\mathcal{E}[\mathbb{T}]$ containing a model $G_{\mathbb{T}}$ of \mathbb{T} which is generic in the sense that, for any topos \mathcal{F} , models of \mathbb{T} in \mathcal{F} correspond to logical functors $\mathcal{E}[\mathbb{T}] \rightarrow \mathcal{F}$, and the sentences satisfied by $G_{\mathbb{T}}$ are exactly those deducible in \mathbb{T} .

Moreover, for every small topos \mathcal{E} , there is a higher-order theory \mathbb{T} such that $\mathcal{E} \simeq \mathcal{E}[\mathbb{T}]$.

[Boring details of the construction: objects of $\mathcal{E}[\mathbb{T}]$ are ~~expressions~~ closed terms $t:PA$ for some type A . Morphisms from $t:PA$ to $u:PB$ are named by formulae $\varphi(x,y)$ with free variables x^A, y^B , such that

$$(\forall x,y) (\varphi(x,y) \Rightarrow ((x \in t) \wedge (y \in u)))$$

$$(\forall x) ((x \in t) \Rightarrow (\exists y) \varphi(x,y))$$

$$(\forall x,y,y') ((\varphi(x,y) \wedge \varphi(x,y')) \Rightarrow (y=y'))$$

are deducible in \mathbb{T} ; two formulae φ, ψ name the same morphism iff $(\forall x,y) (\varphi(x,y) \Leftrightarrow \psi(x,y))$ is deducible. (In fact the one-way implication would be sufficient.)]

In particular, taking \mathbb{T} to be the 'empty' theory (no primitive types, function-symbols or relation-symbols, no non-logical axioms), we obtain the **free topos** $\mathcal{Z} = \mathcal{E}[\mathbb{T}]$ which is initial in the 2-category Log of toposes, logical functors and natural isomorphisms

If we include the type N , we get \mathcal{Z}^N which is initial in Log_N , the sub-2-category of toposes having natural number objects. (Remark: a logical functor preserves the natural number object if it exists, so Log_N is a cosieve in Log .)

Basic idea: although \mathcal{Z} and \mathcal{Z}^N are highly 'exotic' categories (understanding what goes on in them involves understanding the proof theory of higher-order intuitionistic logic), we can gain a lot of information about them simply from the fact that they are initial in Log and Log_N , since these 2-categories contain familiar (Grothendieck) toposes whose properties we understand.

The glueing construction (M. Artin, Tierney, Wraith)

Let $\mathcal{E} \xrightarrow{F} \mathcal{F}$ be a finite-limit-preserving functor between toposes. Consider the category $\text{Gl}(F)$ whose objects are triples (A, B, f) with $f: B \rightarrow FA$ in \mathcal{F} , and whose morphisms $(A, B, f) \rightarrow (A', B', f')$ are pairs (h, k) such that

$$\begin{array}{ccc} B & \xrightarrow{f} & FA \\ \downarrow k & & \downarrow Fh \\ B' & \xrightarrow{f'} & FA' \end{array}$$

commutes. Then $\text{Gl}(F)$ is a topos; the projections $\text{Gl}(F) \xrightarrow{P_1} \mathcal{E}$ and $\text{Gl}(F) \xrightarrow{P_2} \mathcal{F}$ are both inverse image functors, and the former (but not the latter) is logical.

(Remark: F can be recovered as the composite $\mathcal{E} \xrightarrow{R_1} \text{Gl}(F) \xrightarrow{P_2} \mathcal{F}$ where R_1 is the right adjoint of P_1 .)

Let $b: PB$ be an object of \mathcal{T} (or of \mathcal{T}^N), and write \mathcal{G}_b for the topos obtained by glueing along $\mathcal{T}(b, -): \mathcal{T} \rightarrow \text{Set}$.

We have a unique (up to isomorphism) logical functor $\mathcal{T} \xrightarrow{L_b} \mathcal{G}_b$, and $P_1 L_b$ must be the identity, so we can write

$L_b(a) = (a, \kappa_b(a), \kappa_a)$ where $\kappa_b = P_2 L_b$ and κ is a natural transformation $\kappa_b \rightarrow \mathcal{T}(b, -)$.

We say an object a is b -injective if κ_a is surjective; equivalently, if $L_b(a)$ is injective w.r.t. the particular monomorphism $(b, \phi, (\phi \rightarrow \mathcal{T}(b, b))) \rightarrow (b, \{*\}, (* \mapsto 1_b))$ in \mathcal{G}_b .

N.B.: injective objects of \mathcal{T} are b -injective for any b .

From this we deduce:

Theorem If \mathbf{A} is a connected type (i.e. not N , or a product having N as a factor) and $\mathbf{a}: PA$ is a negative term (i.e. satisfies $(\neg\neg(x \in a) \Rightarrow (x \in a))$) then a is b -injective for any $b \neq 0$.

The restriction to connected types is necessary: if $\bar{N} = \{x: N \mid T\}$, then $\kappa_b(\bar{N}) = N$ for any b , and $\kappa_{\bar{N}}$ sends each $n \in N$ to the morphism $b \rightarrow 1 \xrightarrow{s^n 0} \bar{N}$. But not every morphism $\bar{N} \rightarrow \bar{N}$ is of this form.

Corollary If $\mathbf{a}: PA$ is as in the Theorem, and in addition $\mathbf{a} \neq 0$, then

- (a) \mathbf{a} is an indecomposable projective: i.e., given $\coprod_{i=1}^n b_i \rightarrow \mathbf{a}$ in \mathcal{T} , there exists i s.t. $b_i \rightarrow \mathbf{a}$ is split epic.
- (b) \bar{N} is \mathbf{a} -injective, i.e. every morphism $\mathbf{a} \rightarrow \bar{N}$ is of the form $\mathbf{a} \rightarrow 1 \xrightarrow{s^n 0} \bar{N}$ for some $n \in \mathbb{N}$.

In particular, all morphisms $1 \rightarrow \bar{N}$ in \mathcal{T}^N are 'standard'.

Putting $a=1$ in the Corollary, and interpreting it in higher-order type theory, we get

Disjunction Property: If $\vdash (\varphi \vee \psi)$, then either $\vdash \varphi$ or $\vdash \psi$.

Existence Property: If $\vdash (\exists x:A)\chi$, then there's a closed term $a:A$ such that $\vdash \chi[a/x]$.

(Here φ, ψ and $(\exists x)\chi$ are understood to be closed formulae.)

Actually, (a) of the Corollary only reduces the Existence Property to the Unique Existence Property: given $\vdash (\exists x)\chi$, it produces a formula θ such that $\vdash (\exists! x)\theta$ and $\vdash (\forall x)(\theta \Rightarrow \chi)$. But fortunately HOTT has enough 'names' to witness theorems of the form $\vdash (\exists! x)\theta$: the only nontrivial case is when the type of x is N , and (b) takes care of this.

But we can say more, namely 'disjunction and existence with parameters':

If A is a connected type, then

$\vdash (\forall x:A)(\varphi \vee \psi)$ implies either $\vdash (\forall x:A)\varphi$ or $\vdash (\forall x:A)\psi$.

If $\vdash (\forall x:A)(\exists y:B)\chi$, then there's a term $b:B$ (possibly involving the variable x) such that $\vdash (\forall x:A)\chi[b/y]$.

When $B=N$ and A is a connected type, we get a still stronger result (Troelstra's Uniformity Principle): if $\vdash (\forall x:A)(\exists y:N)\chi$, then $\vdash (\exists y:N)(\forall x:A)\chi$.

We can extend still further, to any finitely-presented theory \mathbb{T} whose axioms are 'of negative type', i.e. are sentences φ such that $\vdash (\neg\neg\varphi \Rightarrow \varphi)$ in pure type theory. Any such theory has the Disjunction Property, and it has the Existence Property provided it has the Unique Existence Property.

Markov's Principle asserts that if $\vdash (\forall x: \mathbb{N}) (\varphi \vee \neg \varphi)$, then

$$\vdash (\neg \neg (\exists x: \mathbb{N}) \varphi \Rightarrow (\exists x: \mathbb{N}) \varphi).$$

It seems very likely that this is true in $\mathcal{T}^{\mathbb{N}}$. However, the best we can prove at present is 'Markov's Rule': if $\vdash (\forall x: \mathbb{N}) (\varphi \vee \neg \varphi)$ and $\vdash \neg \neg (\exists x: \mathbb{N}) \varphi$, then $\vdash (\exists x: \mathbb{N}) \varphi$.

Markov's Rule is easy: if $\vdash \neg \neg (\exists x) \varphi$, then there's a natural number n such that $\varphi [n/x]$ holds in Set. But in $\mathcal{T}^{\mathbb{N}}$ we have

$$\vdash (\varphi [\bar{n}/x] \vee \neg \varphi [\bar{n}/x]), \text{ and we can't have } \vdash \neg \varphi [\bar{n}/x],$$

so we must have $\vdash \varphi [\bar{n}/x]$.

Brouwer's Continuity Principle asserts that every $f: \mathbb{R} \rightarrow \mathbb{R}$ is continuous. In $\mathcal{T}^{\mathbb{N}}$ we can construct the objects \mathbb{Z} , \mathbb{Q} , \mathbb{R} of integers, rationals and reals (the latter via Dedekind sections of \mathbb{Q}), and they are preserved by logical functors.

Hence every Grothendieck topos contains an object R_d of Dedekind real numbers. To identify it, we use the fact that Dedekind sections of \mathbb{Q} are models of a geometric (propositional) theory, whose classifying topos is $\text{Sh}(\mathbb{R})$.

Thus, for any space X , the real number object in $\text{Sh}(X)$ is the sheaf of continuous real-valued functions on open subsets of X .

Now consider the 'continuous' topos $\mathcal{E} = \text{Sh}(\mathcal{C}, \mathcal{I})$ from Lecture II, where we now assume that our subcategory \mathcal{C} of Sp contains both \mathbb{R} and the space $[\mathbb{R}, \mathbb{R}]$ of continuous maps $\mathbb{R} \rightarrow \mathbb{R}$ (with compact-open topology), and that it's closed under finite products.

Recall that we had a local geometric morphism $\mathcal{E}/_y X \rightarrow \text{Sh}(X)$ for any object X of \mathcal{C} ; using the fact that \mathbb{R} is Hausdorff, this induces a bijection

$$\begin{array}{ccc} \text{Top}(\text{Sh}(X), \text{Sh}(\mathbb{R})) & \longrightarrow & \text{Top}(\mathcal{E}/_y X, \text{Sh}(\mathbb{R})) \\ \parallel & & \parallel \\ \text{Sp}(X, \mathbb{R}) & & \mathcal{E}(yX, \mathbb{R}_d) \end{array}$$

Thus the Dedekind real number object in \mathcal{E} is simply $y\mathbb{R}$!

Moreover, since $[\mathbb{R}, \mathbb{R}]$ is the exponential $\mathbb{R}^{\mathbb{R}}$ in Sp , and hence in \mathcal{E} , it follows that $y[\mathbb{R}, \mathbb{R}]$ is the exponential $\mathbb{R}_d^{\mathbb{R}_d}$ in \mathcal{E} .

It's now easy to prove that Brouwer's Theorem holds internally in \mathcal{E} , and hence (by a gluing argument due to A. Joyal) that it holds in \mathcal{T}^N .

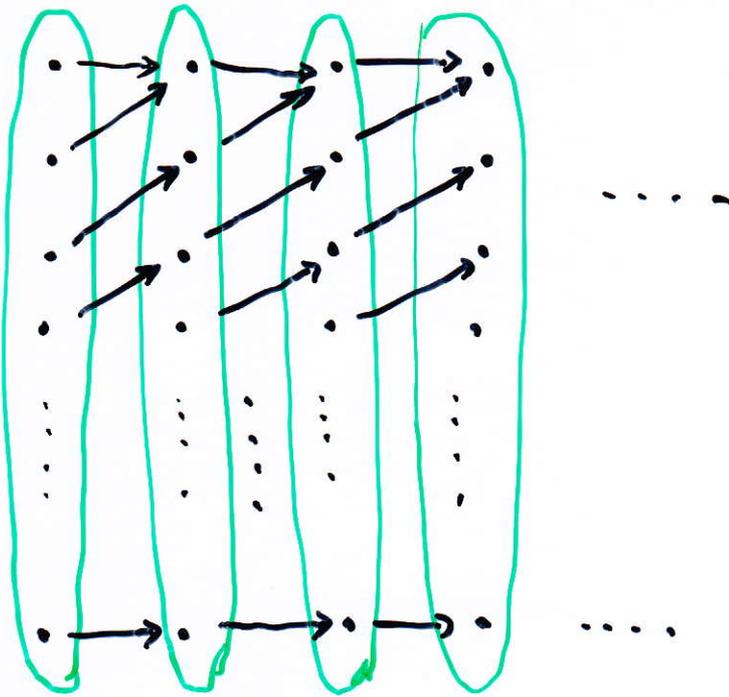
Remark: Brouwer's Theorem implies that \mathbb{R} is indecomposable.

Indeed, in \mathcal{E} one has $2^{\mathbb{R}_d} \cong 2$ (since $2^{\mathbb{R}_d}$ can be identified with $y[\mathbb{R}, 2]$); but this can't be true in \mathcal{T}^N .

Conservativity of the Axiom of Infinity (P. Freyd)

Write Ω for P1. Consider the morphism $c: \Omega \rightarrow \Omega$ in \mathcal{T} defined by the term $(\forall q: \Omega) (q \vee (q \Rightarrow p))$ where p is a free variable of type Ω . c is called the Cantor coderivative: in a topos of the form $\text{Sh}(X)$, its effect on points $1 \rightarrow \Omega$ (that is, open subsets of X) is dual to the Cantor derivative on closed sets — i.e. $c(U)$ is the largest open $V \supseteq U$ such that $V \setminus U$ is discrete.

Now consider the topos $[\mathbb{N}, \text{Set}]$ where \mathbb{N} denotes the ordered set of natural numbers: so objects are sequences $A_0 \rightarrow A_1 \rightarrow A_2 \rightarrow \dots$. In this topos we can picture Ω as follows:



$$\Omega_0 \rightarrow \Omega_1 \rightarrow \Omega_2 \rightarrow \Omega_3 \dots$$

The effect of c in this case is to fix the top and bottom elements, and to move everything else up one place. In particular $c^n \neq c^m$ for any $n \neq m$, and in fact the mapping $n \mapsto c^n$ is a monomorphism $\mathbb{N} \rightarrow \Omega^\Omega = \text{PP1}$ in $[\mathbb{N}, \text{Set}]$.

(This is best possible; in any topos, we have

$$(\forall p, q, r: \Omega) \neg (\neg(p=q) \wedge \neg(p=r) \wedge \neg(q=r))$$

so we can't have a monomorphism $\mathbb{N} \rightarrow \text{P1}$.)

In \mathcal{T} , we can form the smallest subobject $W \rightarrow \Omega^\Omega$ such that $1 \xrightarrow{\bar{1}_\Omega} \Omega^\Omega$ factors through W and $\Omega^\Omega \xrightarrow{c \circ (-)} \Omega^\Omega$ ~~factors~~ restricts to a map $W \rightarrow W$. Moreover, the construction of W is preserved by logical functors. Let $U \rightarrow 1$ be the interpretation of the sentence $(\exists w: W)(c \circ w = w)$ and let $V \rightarrow 1$ be the negation (= Heyting pseudocomplement) of U .

Theorem For a logical functor $L: \mathcal{C} \rightarrow \mathcal{E}$, we have $L(V) \cong 0$ iff $L(W)$ is a natural number object in \mathcal{E} .

Corollary $V \neq 0$ in \mathcal{C} (since it maps to 1 in $[\mathbb{N}, \text{Set}]$), and the category \mathcal{C}/V has a natural number object. (In fact $\mathcal{C}/V \cong \mathcal{C}^{\mathbb{N}}/V$.)

But that's not all. Since V is defined by a negative formula, it's an indecomposable projective in \mathcal{C} , and $\mathcal{C}(V, -): \mathcal{C} \rightarrow \text{Set}$ preserves the natural number object. Hence it also preserves free constructions of all kinds.

Since we have a natural number object in \mathcal{C}/V , we can carry out the construction of the free topos $\mathcal{C}^{\mathbb{N}}$ in the internal logic of \mathcal{C}/V , to get an internal category \mathcal{C} in \mathcal{C}/V which is mapped by $\mathcal{C}/V(1, -)$ to the 'classical' $\mathcal{C}^{\mathbb{N}}$ in Set .

And if we apply the right adjoint $\Pi_V: \mathcal{C}/V \rightarrow \mathcal{C}$ of V^* to \mathcal{C} , we get an internal category in \mathcal{C} mapped by $\mathcal{C}(1, -)$ to $\mathcal{C}^{\mathbb{N}}$.

In summary: there exists an algorithm translating sentences in pure type theory with NNO to sentences in pure type theory without NNO, in such a way that provability is preserved and reflected.

Contrast with the Boolean case: if we write $B \rightarrow 1$ for the interpretation of $(\forall p: \Omega) (p \vee \neg p)$, then \mathcal{C}/B (the free Boolean topos) is just Set_f , whereas $\mathcal{C}^{\mathbb{N}}/B$ is highly complicated (it has no faithful logical functor to any Grothendieck topos).