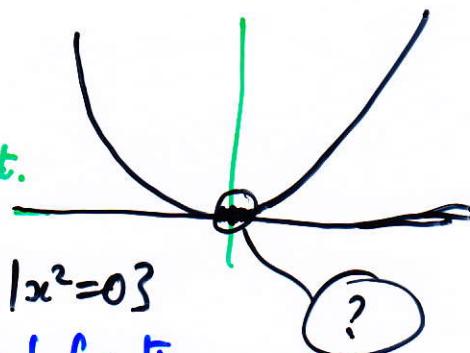


What do we know geometrically about the structure of the real line \mathbb{R} ?

If we fix two distinct points $0, 1$, then we can give geometric constructions of addition & multiplication, so \mathbb{R} is a (commutative) ring. But it's not geometrically clear that it should be a field.

In particular, the intersection of the parabola $y = x^2$ and the line $y = 0$ might be more than a single point.



Lawvere 1967: suppose the object $D = \{x : \mathbb{R} \mid x^2 = 0\}$

is 'just large enough' that every real function defined on it is uniquely linear, i.e.

$$(\forall f : \mathbb{R}^D) (\exists! a, b : \mathbb{R}) (\forall d : D) (f(d) = a + bd).$$

Clearly we have $a = f(0)$; and we can think of b as a 'synthetic derivative' $f'(0)$.

No such ring can exist in Set (or in any Boolean topos).

How about the smooth topos of Lecture II? Well, $\mathbb{R} = y \mathbb{R}$ is a ring in this topos, since \mathbb{R} is an internal ring in M_f . But the embedding y preserves all the limits which exist in M_f , in particular the pullback

$$\begin{array}{ccc} 1 & \xrightarrow{0} & \mathbb{R} \\ \downarrow 0 & & \downarrow (x \mapsto (x, x^2)) \\ \mathbb{R} & \xrightarrow{(x \mapsto (x, 0))} & \mathbb{R}^2 \end{array}$$

A. Kock 1977: the algebraic topos of Lecture II provides an example!

Before proving this, we need to develop the theory of Weil algebras.

Definition Let K be a ring. A Weil algebra over K is a K -algebra W having a K -module decomposition $W \cong K \cdot 1 \oplus M$ where M is a finitely-generated free K -module; $M \triangleleft W$; and all elements of M are nilpotent.

Example: $K[\varepsilon] = K[t]/(t^2)$, or more generally $K[t]/(t^{k+1})$.

Any Weil algebra has a presentation $K[t_1, t_2, \dots, t_n]/(E)$ where E is a set of ^{poly}nomials in the t_i , including some power $t_i^{d_i}$ for each i .

If W is a Weil algebra and R is an arbitrary K -algebra, then $R \otimes_K W$ is a Weil algebra over R .

Given W and R , let $R\text{-spec}(W)$ be the set of K -algebra homomorphisms $W \rightarrow R$. If W has presentation $K[t_1, \dots, t_n]/(E)$, then $R\text{-spec}(W) \cong \{(x_1, \dots, x_n) \in R^n \mid E[\vec{x}/\vec{E}] = 0\}$, an 'infinitesimal neighbourhood' of $\underline{0}$ in R^n .

These notions make sense when R is an internal K -algebra in a topos \mathcal{E} .

We have a mapping $R \times W \times R\text{-spec}(W) \rightarrow R$ sending (a, w, f) to $a \cdot f(w)$. This is bilinear in the first two variables, so induces a mapping $R \otimes_K W \times R\text{-spec}(W) \rightarrow R$, or equivalently $R \otimes_K W \rightarrow R^{R\text{-spec}(W)}$. We say R satisfies the Kock-Lawvere axiom (over K) if this mapping is bijective (and hence an isomorphism of R -algebras) for all W .

For $W = K[\varepsilon]$, this is Lawvere's original axiom for R .

Example Let $\mathcal{E} = \text{Sh}(\mathcal{C}, \mathcal{T})$ where $\mathcal{C} = (\text{K-Alg}_{\text{fp}})^{\text{op}}$ and \mathcal{T} is a topology (e.g. Zariski) for which the representable functors are sheaves. The composite $R: \mathcal{C}^{\text{op}} \hookrightarrow \text{K-Alg} \rightarrow \text{Set}$ is representable (by $\text{K}[t]$), and it's an internal K-algebra in \mathcal{E} . For any Weil algebra W , $R\text{-spec}(W)$ is simply the functor y_W ; and the exponential $R^{R\text{-spec}(W)}$ is the functor $A \mapsto R(A \otimes_K W)$. But this is isomorphic to $R \otimes_K W$.

Logical interpretation (M. Coste): let \mathbb{T} be a coherent extension of the theory of K-algebras, and let R be the generic model of \mathbb{T} . Then (R satisfies the Kock-Lawvere axiom for W and $R\text{-spec}(W)$ is 'tiny') iff \mathbb{T} is W-stable, i.e. $A \otimes_K W \models \mathbb{T}$ whenever $A \models \mathbb{T}$.

So we can develop synthetic differential geometry in $\text{Sh}(\mathcal{C}, \mathbb{Z})$.

Sample results: say M is infinitesimally smooth if

$M^{R\text{-spec}(-)}: \text{Weil}(K) \rightarrow \mathcal{E}$ preserves finite connected limits.
(N.B.: $\text{Weil}(K)$ is closed in K-Alg under finite connected limits.)
Kock-Lawvere $\Rightarrow R$ is infinitesimally smooth, and inf. smooth objects are closed under arbitrary limits. For an inf. smooth M , $M^D \rightarrow M$ has the structure of an R -module in \mathcal{E}/M (here $D = R\text{-spec}(K[\varepsilon])$, and $M^D \rightarrow M$ is induced by $1 \xrightarrow{0} D$) and we can think of it as the synthetic tangent bundle of M . If M is also a monoid in \mathcal{E} , we get an R -Lie algebra structure on the pullback

$$\begin{array}{ccc} T_1 M & \longrightarrow & M^D \\ \downarrow & & \downarrow \\ 1 & \xrightarrow{1} & M \end{array}$$

structure on vector fields on M (= sections of $M^D \rightarrow M$) by regarding them as tangents $D \rightarrow M^M$ at the identity of M^M .

But... the objects of the Zariski topos aren't really manifolds.

Even if we take $K = \mathbb{R}$, the only morphisms $\mathbb{R}^n \rightarrow \mathbb{R}$ in $\text{Sh}((\mathbb{R}\text{-Alg})_{\text{fp}}^{\text{op}}, \mathbb{Z})$ are polynomials in n variables.

Also, all \mathbb{Z} -covering sieves are finitely generated; so \mathbb{R} is compact in this category.

What we'd like is a well-adapted model of SDG : i.e. a model \mathcal{E} with a full embedding $i: \text{Mf} \rightarrow \mathcal{E}$ such that

(1) i preserves transversal pullbacks (where $M_1 \xrightarrow{f_1} N$ and $M_2 \xrightarrow{f_2} N$ are transversal if, whenever $f_1(x_1) = f_2(x_2) = y$, the images under df_i of the tangent spaces $T_{x_i} M_i$ ($i = 1, 2$) together span $T_y N$).

(2) i maps open covers to epimorphic families.

(3) $R = i(\mathbb{R})$ satisfies the Kock-Lawvere axiom (over \mathbb{R}) in \mathcal{E} .

The smooth topos of Lecture II satisfies (1) and (2) but not (3), since $R\text{-spec}(\mathbb{R}[\varepsilon]) = \{0\}$.

Idea (A. Kock/E. Dubuc): replace the theory of \mathbb{R} -algebras by a larger algebraic theory in which all smooth maps $\mathbb{R}^n \rightarrow \mathbb{R}$ become 'n-ary algebraic operations'.

Formally, a C^∞ -ring is a set A equipped with maps $f_A: A^n \rightarrow A$ for every smooth $f: \mathbb{R}^n \rightarrow \mathbb{R}$ (all $n \geq 0$), subject to the obvious compatibility conditions.

Examples: \mathbb{R} itself; $C^\infty(M) = \{\text{smooth functions } M \rightarrow \mathbb{R}\}$ for a manifold M ; $C(X) = \{\text{continuous functions } X \rightarrow \mathbb{R}\}$ for any space X .

A C^∞ -ring is in particular a ring (since addition & multiplication are smooth maps $\mathbb{R}^2 \rightarrow \mathbb{R}$) and in fact an \mathbb{R} -algebra (since each $1 \xrightarrow{x} \mathbb{R}$ is a nullary operation). It's also formally real (since we have an n -ary operation corresponding to $(x_1, x_2, \dots, x_n) \mapsto (1+x_1^2 + \dots + x_n^2)^{-1}$). In fact any local C^∞ -ring is a separably real-closed local ring, i.e. a Henselian local ring with real-closed residue field.

In any well-adapted model, $R = i(\mathbb{R})$ is an internal C^∞ -ring (since \mathbb{R} is an internal C^∞ -ring in M_f), and in fact it's local: its object of invertible elements is $i(\mathbb{R} \setminus \{0\})$, since

$$\begin{array}{ccc} \mathbb{R} \setminus \{0\} & \longrightarrow & \mathbb{R}^2 \\ \downarrow & & \downarrow ((x,y) \mapsto xy) \\ 1 & \xrightarrow{1} & \mathbb{R} \end{array}$$

is a transversal pullback, and the subobjects $i(\mathbb{R} \setminus \{0\})$ and $i(\mathbb{R} \setminus \{1\})$ cover $i(\mathbb{R})$, by (2).

Can also show that iM is infinitesimally smooth for any manifold M , and that i maps the 'analytic' tangent bundle $TM \rightarrow M$ to the synthetic one $iM^D \rightarrow iM$.

Lemma For any ring ideal I of a C^∞ -ring A , congruence modulo I is a C^∞ -congruence; i.e., we can give A/I a unique C^∞ -ring structure such that the quotient map $A \rightarrow A/I$ is a C^∞ -homomorphism.

Note that $C^\infty(\mathbb{R}^n)$ is by definition the free C^∞ -ring on n generators. Hence finitely-presented C^∞ -rings are those of the form $C^\infty(\mathbb{R}^n) / (f_1, f_2, \dots, f_k)$.

Further examples :

- We have a surjective ring homomorphism $C^\infty(\mathbb{R}^n) \rightarrow \mathbb{R}[[t_1, \dots, t_n]]$ (Taylor series), so $\mathbb{R}[[t_1, \dots, t_n]]$ is a C^∞ -ring.
- We can regard any Weil algebra W over \mathbb{R} as a quotient of $\mathbb{R}[[t_1, \dots, t_n]]$ rather than $\mathbb{R}[t_1, \dots, t_n]$, so they are (finitely-presented) C^∞ -rings.
- If $U \subseteq \mathbb{R}^n$ is open, we can find a smooth characteristic function $\chi_U: \mathbb{R}^n \rightarrow \mathbb{R}$ for U , and then let \widehat{U} denote the closed submanifold $\{(x_1, \dots, x_n, y) \mid y \cdot \chi_U(x_1, \dots, x_n) = 1\}$ of \mathbb{R}^{n+1} . Then $C^\infty(U) \cong C^\infty(\widehat{U}) \cong C^\infty(\mathbb{R}^{n+1}) / (1 - y \cdot \chi_U(x_1, \dots, x_n))$ so it's a finitely-presented C^∞ -ring.
- Any (connected) manifold M can be embedded as a smooth retract of an open $U \subseteq \mathbb{R}^n$; so $C^\infty(M)$ is a retract of $C^\infty(U)$, & hence finitely-presented.

Lemma Any C^∞ -homomorphism (in fact, any ring homomorphism) $C^\infty(M) \rightarrow \mathbb{R}$ is of the form $C^\infty(x)$ for a unique $x: 1 \rightarrow M$.
Hence $C^\infty(-)$ is a full & faithful functor $\text{Mf}^{\text{op}} \rightarrow C^\infty\text{-Rng}$.

By a point of a C^∞ -ring A , we mean a homomorphism $A \rightarrow \mathbb{R}$.

We say A is point-determined if its points are jointly injective (e.g. if $A = C^\infty(M)$ for some M).

A Weil algebra has a unique point, so it's not point-determined. But it is germ-determined: define the 'germ ring' of A at a point p to be the C^∞ -ring of fractions $A[\Sigma_p^{-1}]$ where $\Sigma_p = \{a \in A \mid p(a) \neq 0\}$, and say A is germ-determined if the homomorphisms $A \rightarrow A[\Sigma_p^{-1}]$ are jointly injective.

Lemma Any finitely-presented C^∞ -ring is germ-determined.

Not every finitely-generated C^∞ -ring is germ-determined; e.g. $C^\infty(\mathbb{R})/\mathcal{I}$, where \mathcal{I} is the ideal of functions with compact support.

Corollary If A is finitely-presented, and $a \in A$ satisfies $p(a) \neq 0$ for all points $p: A \rightarrow \mathbb{R}$, then a is invertible in A .

We define an open inclusion of (finitely-presented) C^∞ -rings to be one of the form $A \rightarrow A[a^{-1}]$ where $a \in A$. (N.B.: $A[a^{-1}]$ denotes the C^∞ -ring of fractions, not the ordinary ring of fractions.) Example: $C^\infty(M) \rightarrow C^\infty(U)$ where $U \subseteq M$ is open (take a to be a smooth characteristic function for U).

Lemma A composite of open inclusions is an open inclusion.

Hence we get a Grothendieck topology \mathcal{D} (the Dubuc topology)

on $\mathcal{C} = (C^\infty\text{-Rng}_{fp})^{\text{op}}$ by saying that a sieve R covers A iff it contains a family of open inclusions $(A \rightarrow A[a_i^{-1}] \mid i \in I)$ which are jointly surjective on points (i.e. every $p: A \rightarrow \mathbb{R}$ factors through at least one $A \rightarrow A[a_i^{-1}]$).

Remark: A finite family $(A \rightarrow A[a_i^{-1}] \mid 1 \leq i \leq n)$ generates a \mathcal{D} -covering sieve iff it's \mathbb{Z} -covering. For if it's \mathcal{D} -covering then $\sum_{i=1}^n a_i^{-2}$ is invertible, by the Corollary above; but if there exist b_i such that $\sum_{i=1}^n a_i b_i = 1$, then for each p we must have at least one $p(a_i) \neq 0$.

However, for \mathcal{D} it suffices to consider countably-generated sieves.

For any A , we can identify $Pt(A)$ with a subspace of \mathbb{R}^n for some n , so it has the Lindelöf property; and $Pt(A[a^{-1}]) \rightarrow Pt(A)$ is the inclusion of an open subset.

In fact \mathcal{D} can be generated by adding just one infinite covering family to the Zariski topology, namely $(C^\infty(\mathbb{R}) \rightarrow C^\infty((-n, n)) \mid n=1, 2, 3, \dots)$.

Thus $Sh(\mathcal{C}, \mathcal{D})$ is the classifying topos for the theory of archimedean local C^∞ -rings, i.e. those satisfying

$$(T \vdash_x \bigvee_{n=1}^{\infty} (\exists y)(y^2 = n \cdot 1 - x)).$$

Lemma The representable functors on \mathcal{C} are \mathcal{D} -sheaves; more generally, for any C^∞ -ring A , the functor $C^\infty\text{-Rng}(A, -) \mid_{\mathcal{C}^\text{op}}$ is a \mathcal{D} -sheaf yA .

Thus we have a full & faithful functor

$$i: Mf \xrightarrow{C^\infty(-)} (C^\infty\text{-Rng})^\text{op} \xrightarrow{y} \mathcal{E} = Sh(\mathcal{C}, \mathcal{D}).$$

Moreover, i maps open covers in Mf to \mathcal{D} -covering families, and hence to epimorphic families in \mathcal{E} .

The argument given earlier in the 'algebraic case' shows that $yC^\infty(\mathbb{R})$ (the generic archimedean local C^∞ -ring) satisfies the Kock-Lawvere axiom for all Weil algebras over \mathbb{R} .

It remains to verify that $C^\infty(-)$, and hence i , preserves transversal pullbacks. We do this in stages:

- $C^\infty(\mathbb{R}^{m+n})$ is the coproduct $C^\infty(\mathbb{R}^m) \otimes_\infty C^\infty(\mathbb{R}^n)$ in $C^\infty\text{-Rng}$. Note that it's not the ordinary tensor product $C^\infty(\mathbb{R}^m) \otimes_{\mathbb{R}} C^\infty(\mathbb{R}^n)$!
- For any manifolds M and N , $C^\infty(M \times N) \cong C^\infty(M) \otimes_\infty C^\infty(N)$.

- If $f: M \rightarrow \mathbb{R}^P$ is a smooth map having 0 as a regular value, then the pullback $N \rightarrow M$ is preserved, i.e.

$$\begin{array}{ccc} N & \longrightarrow & M \\ \downarrow & & \downarrow f \\ 1 & \xrightarrow{0} & \mathbb{R}^P \end{array}$$

$$C^\infty(N) \cong C^\infty(M)/(f_1, \dots, f_P).$$

- $C^\infty(-)$ preserves pullbacks of open inclusions.
- If $f: M \rightarrow N$ is transversal to a closed submanifold $N' \hookrightarrow N$, then the pullback $M' \rightarrow N'$ is preserved.

$$\begin{array}{ccc} M' & \longrightarrow & N' \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & N \end{array}$$

- We reduce the general case of a pullback $P \rightarrow M_1$

$$\begin{array}{ccc} P & \longrightarrow & M_1 \\ \downarrow & & \downarrow f_1 \\ M_2 & \xrightarrow{f_2} & N \end{array}$$

to the last one by considering

$$\begin{array}{ccc} P & \longrightarrow & N \\ \downarrow & & \downarrow \Delta \\ M_1 \times M_2 & \xrightarrow{f_1 \times f_2} & N \times N \end{array} .$$

Finally, we've proved

Theorem Let \mathcal{C} be the dual of the category of finitely-presented C^∞ -rings, and let \mathcal{D} be the Dubuc topology on \mathcal{C} . Then $Sh(\mathcal{C}, \mathcal{D})$ is a well-adapted model of synthetic differential geometry.

Variants: could enlarge \mathcal{C} to contain all finitely-generated germ-determined C^∞ -rings, or cut it down to rings of the form $C^\infty(M) \otimes W$ where W is a Weil algebra.