CENTRO DI RICERCA MATEMATICA ENNIO DE GIORGI

Differential Geometry and Topology 2004

Final Report

Background

The research programme in 'Differential Geometry and Topology' took place in Pisa from 6 September to 12 November 2004. The scientific committee consisted of Simon Donaldson (London), Paul Gauduchon (Palaiseau), Claude LeBrun (Stony Brook) and Simon Salamon (Turin), the latter helping with local organization.

The original idea of the committee was to centre the programme around two main, but contrasting, ideas. The study on the one hand of special Riemannian metrics, such as extremal Kähler metrics (Topic 1), and ones with reduced holonomy and their associated geometry of calibrated submanifolds (Topic 2). The study on the other hand of more generic metrics as presented in the theory of Ricci flow on 3-manifolds (Topic 3).

As the planning developed, these topics were augmented by related aspects of complex and Kähler geometry and group actions (Topic 4), and the study of Einstein metrics from other perspectives including Seiberg-Witten invariants (Topic 5). All five topics were united by certain common keywords such as `Ricci tensor', and as expected Einstein metrics played a key role in virtually all aspects of the research.

Further interactions between the various topics took place during the course of the programme, and this led to a valuable exchange of ideas between participants in different 'schools'. It was evident that the recent progress on Ricci flow had stimulated research in the other fields represented below, and it was no surprise therefore that many of the key speakers had already taken on board some of the new techniques. Examples of this included the successful use of heat and Ricci flow methods in the theory of extremal Kähler metrics and stability, and the direct study of Ricci flow itself in the setting of Kähler manifolds.

Topic 1: Extremal Kähler metrics

The theory of extremal Kähler metrics has its origin in the problem of finding `canonical' metrics on a manifold. Calabi considered critical points for the L^2 norm of the scalar curvature restricted to metrics in a given Kähler class. Such *extremal Kähler metrics* are characterized by the fact that the scalar curvature is a Killing potential; they include constant scalar curvature (CSC) Kähler metrics and, as absolute minima, scalar-flat Kähler (SFK) metrics. They also include Kähler-Einstein metrics, and Kähler metrics with vanishing Bochner tensor, all contained in the larger family of weakly Bochner-flat Kähler metrics considered within Topic 4.

In contrast to the negative or zero case, not all complex manifolds with positive first Chern class admit Kähler-Einstein metrics. Indeed, it is by now a generally accepted conjecture that the existence of Kähler-Einstein metrics, or more generally CSC or other extremal Kähler metrics, should be equivalent to some kind of *stability* condition very much in the spirit of Mumford's Geometric Invariant Theory (GIT). Although a general existence theorem is still lacking, much evidence supports this view, in particular the recent work of Donaldson and significant improvements by Mabuchi and others.

Current work focuses on the relationship between an algebraic notion of stability and existence of CSC and other extremal Kähler metrics. The deformation theory of SFK metrics and resolution of their singularities also falls into this topic.

Paul Gauduchon (3 lectures) gave an introduction to extremal Kähler metrics on the eve of the official opening. He summarized their main properties and mentioned Calabi's non-trivial examples. He subsequently gave a brief description of the Mabuchi metric on the space of Kähler metrics, and showed that the existence of geodesics between any two points in the space of Kähler metrics implies the uniqueness of extremal metrics (using arguments of Donaldson and Guan). His talks form the basis of an introductory but extensive monograph on the subject, to be published by the Centro De Giorgi.

Toshiki Mabuchi (4 lectures) spoke on stability and uniqueness of extremal Kähler metrics on a polarized algebraic manifold. He discussed Donaldson's quantization method for the stability problem (a balanced metric as a finite-dimensional analogue of a CSC Kähler metric), an obstruction to asymptotic semistability of a polarized algebraic manifold, and associated integral invariants. Attention also focussed on asymptotic weighted Bergman kernels, as well as the second fundamental form in the geometry of Kähler potentials, and its application to the uniqueness problem.

Zhang-Dan Guan (3 lectures) presented a complete classification of extremal metrics on compact almost homogeneous manifolds of cohomogeneity one. He covered existence, stability, geodesics, uniqueness, generalized Futaki invariants, geodesic stability, parallel transform and Moser vector fields.

Richard Thomas (2 lectures) explained the correspondence between stable holomorphic bundles and Hermitian-Yang-Mills connections, and the conjectures of Yau, Tian, Fujiki and Donaldson on stable algebraic varieties and CSC Kähler metrics. He discussed Mumford's GIT notion of stability for algebraic varieties, and a modification, namely Tian's K-stability, as well as geometric criteria (work with J. Ross).

Xiuxiong Chen (2 lectures) presented his key work on the regularity of geodesics in the space of Kähler metrics and its application to Kähler geometry.

Gang Tian (2 lectures) talked about K-stability, its reformulations and relations to the existence of CSC Kähler metrics (joint work with S. Paul).

Michael Singer (2 lectures) turned first to the special case of zero scalar curvature in complex dimension 2, and finished with more exotic constructions of SFK metrics (work with Y. Rollin). This work involves elements of the geometry of ruled surfaces, including parabolic bundles and knowledge of Hirzebruch-Jung strings. **Claudio Arezzo** (2 lectures) spoke on new families of Kähler-Einstein manifolds (work with A. Ghigi and G. Pirola), and gluing of CSC Kähler metrics. The presence of symmetries allows one to establish Tian's analytic stability and obtain complete families of algebraic Fano manifolds and orbifolds admitting Kähler-Einstein metrics. **Max Pontecorvo** announced new existence results for anti-self-dual Hermitian metrics, and cast them in the general context of locally conformally Kähler surfaces.

Santiago Simanca discussed heat flows for extremal Kähler metrics. He set up the flow equation on the space of G-invariant Kähler metrics with fixed cohomology class, whose critical points are extremal metrics.

Topic 2: Reduced holonomy and associated geometry

From the Riemannian viewpoint, Calabi-Yau (CY) manifolds are compact manifolds with holonomy group equal to SU(n), with special focus on the case n=3 (meaning 6 real dimensions). Initially, in the complex projective setting, they are generally detected by the existence of a holomorphic n-form. By Yau's theorem, they carry a Ricci-flat Kähler metric, though the latter's explicit nature is unknown. Calabi-Yau spaces appear as the internal spaces (of tiny diameter) in string theories.

Mirror symmetry is an observed pairing in the set of CY manifolds of fixed (e.g. 6) dimensions in which Hodge numbers are interchanged to reflect a duality between complex and symplectic deformations. The Strominger-Yau-Zaslow (SYZ) conjecture attempts to explain mirror symmetry in terms of a pair of special Lagrangian (SL) foliations with a common base. The fibres are 3-tori that typically develop singularities over a knot in the base. Geometrical PDE gluing techniques are of value in analysing this picture.

The above theory can be extended, in broad terms, to the theory of 7-manifolds admitting metrics with holonomy equal to G_2 (and therefore Ricci-flat). Whilst these are obviously not complex, it was shown by Joyce that many compact example can be built up from CY spaces with holonomy SU(2) or SU(3). The role of SL submanifolds is replaced by that of 4-dimensional *coassociative* submanifolds, and Kovalev showed that a class of examples can be constructed starting from Fano 3-folds. Physical aspects of the theory of G_2 metrics (that occur in the 11-dimensional unification of various string theories) were subsumed under Topic 5.

Mark Gross (3 lectures) surveyed the approach to mirror symmetry using affine manifolds. Given a real manifold whose transition maps are integral affine linear maps, one can construct either a symplectic manifold fibred in tori over the real manifold, or a complex manifold fibred in tori. These torus fibrations are naturally dual to each other, and exhibit a simple form of mirror symmetry. In order to obtain interesting examples, such as mirror symmetry for K3 surfaces or complete intersections in toric varieties, one must study affine manifolds with singularities, but it is then much more difficult to construct symplectic or complex manifold structures on the torus fibrations. These issues were discussed and related to work of Siebert, Kontsevich and Soibelman.

Diego Matessi and Ricardo Castano-Bernard (1 lecture each) discussed Lagrangian fibrations of CY manifolds. This description of a K3 surface can be extended to deal with the quintic 3-fold using suitable models of singular Lagrangian fibrations. In the so-called negative case, a piece-wise smooth Lagrangian fibration fails to be smooth only over the singular fibres. **Mark Haskins** (3 lectures) constructed an infinite family of special Lagrangian cones over any compact orientable surface of odd genus, the first examples of SL cones whose links are surfaces of genus greater than 1 (joint work with N. Kapouleas). He subsequently discussed the theory from the viewpoint of spectral geometry and integrable systems.

Alexei Kovalev (4 lectures) started with a class of quasi-projective complex manifolds which topologically have a cylindrical end and admit a Ricci-flat Kähler metric with holonomy SU(n), building on work of Tian and Yau. This was applied to the construction of metrics with holonomy G_2 on compact 7-manifolds. Examples are obtained using the algebraic geometry of Fano 3-folds and K3 surfaces. The resulting manifolds are topologically distinct from Joyce's examples, and are fibred by coassociative submanifolds with typical fibre diffeomorphic to a K3 surface. In a similar way, the Schoen CY 3-fold was (re)constructed analytically, leading to an example of an SL fibration required in the SYZ mirror symmetry conjecture.

Daniel Huybrechts (2 lectures) explained Hitchin's notion of generalized Calabi-Yau structures, and studied their moduli for K3 surfaces. They give rise to new weight-2 Hodge structures on K3 cohomology, and are related to Brauer classes and B-fields. He also reported on derived categories of generalized K3 surfaces (work with P. Stellari). **Bert Van Geemen** (2 lectures) discussed a Hodge-theoretic approach towards the Brauer group of a K3 surface. Basic K3 facts gave way to a study of the Brauer group of an elliptic fibration.

Luca Migliorini discussed the topology of (not necessarily compact) hyperkähler manifolds, given that several known families admit a contraction to a singular variety satisfying a mysterious `semismall estimate' (work with M. de Cataldo). **Zhang-Dan Guan** discussed bounds of the Betti numbers of a compact hyperkähler manifold of complex dimension 4, using spinor representations and the minimum length property. **Ryushi Goto** spoke about deformations of CY, hyperkähler and exceptional structures defined by closed differential forms. He established a criterion for unobstructed deformations from the cohomological point of view, and discussed the smoothing problem.

Stefan Ivanov gave an answer to a question posed by Bryant, by showing that a compact 7-manifold with a calibrated G_2 structure is Einstein if and only if the holonomy reduces to G_2, and extended this result. **Anna Fino** constructed homogeneous conformally parallel G_2 structures on a class of solvmanifolds, and related it to the evolution of half-flat structures on 6-manifolds. **Pawel Nurowksi** considered geometry arising from the irreducible inclusion of SO(3) in SO(5), and an associated notion of twistor space also related to G_2. **Paul-Andi Nagy** discussed several contexts, all in the presence of a special kind of symmetry, when it is possible to perform a double reduction of a 6-dimensional nearly-Kähler manifold.

Topic 3: Ricci flow and the geometrization of 3-manifolds

A remarkable property of the Ricci curvature of a Riemannian manifold is that it is of the same tensorial type as the metric itself, hence can be regarded as a vector field on the space of all metrics. The Ricci flow was introduced by Hamilton as a mechanism to improve properties of a Riemannian metric initially assigned on a compact manifold. It is a non-linear heat equation in the metric, and he showed that on a 3-manifold an initial metric of positive Ricci curvature converges in the limit to one of constant sectional curvature. The ultimate goal of the theory is to prove Thurston's Geometrization Conjecture, whereby any compact oriented 3-manifold can be decomposed into a number of pieces, each with a well-determined geometry. This amounts to a classification of 3-manifolds, modulo the study of hyperbolic ones, and implies the Poincaré Conjecture as a special case.

The deeper analysis of Ricci flow is concerned with singularities that arise in finite time. In geometric contexts, a common way to understand singularities is to rescale the solution on a sequence converging to the singularity, so to make it bounded. In this way, one can determine a model of the singularity. Perelman's celebrated work addresses many of the technical problems relating to singularities, such as encoding them into the topology of the underlying manifold, and is expected to imply a complete solution of the Geometrization Conjecture.

Although the Ricci flow is not a conventional gradient flow, it *is* a gradient flow of the metric on the subspace of those of fixed volume and harmonic representative. This may be used to seek a Kähler-Einstein metric on a given Kähler manifold with appropriate first Chern class. Such a process was analysed by Cao, following work the work of Aubin and Yau on the Calabi conjecture, and provides an important link with Topic 1 that helped to integrate these two subjects.

Richard Hamilton gave a special lecture by way of introducing the subject of Ricci flow on 3-dimensional manifolds, and Perelman's achievements within it.

Huai-Dong Cao (5 lectures) gave 2 talks on the basic features of Ricci flow. This included the differential Harnack estimates of Li-Yau (for the scalar heat equation) and Hamilton (for the Ricci flow). He subsequently explained how Perelman's estimate for the conjugate backward heat equation can be derived by a similar method. One can then relate the L-function of Perelman to the optimal path integral of Li-Yau type. Other topics included Gaussian densities and the stability of Ricci solitons (work with R. Hamilton and T. Ilmanen); using these tools, one can sometimes predict or limit the formation of singularities in the Ricci flow. His final talk addressed the Ricci flow on Kähler manifolds and some open problems.

Michael Anderson (4 lectures) surveyed Grisha Perelman's work towards the geometrization conjecture. He began by presenting Thurston's theory, and then moved on to discuss Hamilton's results on Ricci flow. The remaining time was devoted to the following topics: the entropy functional, proof of the non-collapse and no-cigar theorems, the structure of blow-up limits, and the canonical neighborhood theorem. Finally, he examined Ricci flow with surgery, and its long-time behaviour.

Bruce Kleiner (2 lectures) addressed mean curvature flow of surfaces in 3-manifolds, in the context of Perelman's work. He discussed the singular structure that may arise, that is broadly in line with the expectation that the only problems are neck pinches and components which collapse to asymptotically round spheres (work with T. Colding).

Christophe Margerin (2 lectures) provided a somewhat different viewpoint, emphasizing what is needed to achieve Thurston's geometrization programme for 3-manifolds.

Alessandro Savo studied the heat equation evolution of a smooth function on a Riemannian manifold with Dirichlet boundary conditions. He obtained an explicit recursive formula which computes all invariants related to the asymptotic behaviour of the total heat content.

Topic 4: Kähler geometry, topology and generalizations

This topic brought together a number of facts and theories pertaining to Kähler geometry in relation with various questions of topology, algebraic geometry, special holonomy, extended Teichmüller spaces and symplectic geometry.

It also incorporated applications of the theory of Generalized Complex Geometry (GCG), introduced and developed by Gualtieri and Hitchin, that succeeds in placing complex and symplectic structures on an equivalent setting. It consequently provides a natural formalism for aspects of mirror symmetry, and was used by other speakers in Topic 2.

Some lecturers discussed explicit hyperkähler metrics, and others links between geometrical sructures and topology in 3 and 4 dimensions.

Claire Voisin (2 lectures) spoke about the Kodaira problem. There are numerous restrictions on the topology of Kähler manifolds, and she explained her result asserting that being projective implies supplementary restrictions. In any dimension at least 4, there exist Kähler compact manifolds which do not have the homotopy type of projective complex manifolds, and in particular do not *deform* to a projective one. The examples are obtained by blowing-up complex tori, though simply-connected such examples exist starting from dimension 6.

Vestislav Apostolov, David Calberbank, Christina Tonnesen-Friedman (5 lectures in total) spoke on various aspects of their joint work (also with P. Gauduchon) on torus actions on a Kähler manifold. So-called Hamiltonian 2-forms lead to *orthotoric* (a special case of toric) geometry and new constructions of Kähler-Einstein orbifolds. A classification of a large class of compact manifolds admitting rigid Hamiltonian torus actions was presented, as well as a study of weakly Bochner-flat Kähler metrics that can be regarded as special types of extremal Kähler metrics.

Roger Bielawski (2 lectures) explained the construction of hyperkähler metrics by the generalised Legendre transform, its relation with flows on Jacobians and how monopole metrics fit into the scheme. The asymptotic monopole metrics, corresponding to a cluster decomposition, were viewed as a deformation of the product of monopole metrics of lower charges which captures the interaction of the clusters. **Laura Barberis** considered geometrical structures related to a particular class of invariant complex structures on Lie groups, namely the abelian ones, and gave applications to hyperkähler metrics with torsion. **Stefano Marchiafava** surveyed results on complex and Kaehler submanifolds of quaternionic Kaehler manifolds.

Gil Cavalcanti discussed the strong Lefschetz property and dd^c lemmas in the context of GCG, providing a direct link between geometry and topology in accordance with the programme's title. This theory exhibits these two properties as instances of a common phenomenon, whose implications were studied. **Antonella Nannicini** discussed generalized geometry from a different point of view, related to submanifold theory. **Adriano Tomassini** gave some examples of generalized Calabi-Yau structures on solvmanifolds, with special reference to complex dimension 3.

Francois Labourie (2 lectures) spoke on aspects of Teichmüller theory, explaining first how it can be thought of as a dictionary between various fields of mathematics, including hyperbolic geometry, representations of surfaces groups, and combinatorics of trivalent graphs. He then explained how the theory extends (at least conjecturally) to dimension $n>2$, with work of Hitchin and Fock-Goncharov completing some pages of the dictionary. Certain aspects survive as n tends to infinity. **Yann Rollin** (2 lectures) spoke on *monopole fillable* contact manifolds, meaning that the Kronheimer-Mrowka-Seiberg-Witten invariant (for a 4-manifold with a contact boundary) is non-vanishing (joint work with T. Mrowka). In this set-up, there is a strong interplay between the 3-dimensional contact geometry and the 4-dimensional geometry of the filling.

Vincenzo Ancona (2 lectures) discussed a simplified approach to mixed Hodge theory, based on a new treatment of differential forms on singular complex spaces. **Gian Pietro Pirola** discussed the de Franchis problem that asserts that a compact Riemann surface admits only finitely many nonconstant holomorphic maps onto a surface of genus >1. Using Hodge structure, he found bounds for the case of algebraic surfaces of general type. **Paolo de Bartolomeis** (2 lectures) described a general deformation machinery, enhancing relationships between the Z and Z_2 theories, with applications to (super) holomorphic and symplectic manifolds. **Fabio Podestà** considered the class of homogeneous toric bundles over a flag manifold and provided a simple algebraic criterion which characterizes the Fano manifolds in this class. **Fiammetta Battaglia** described toric spaces associated with nonrational, nonsimple convex polytopes, from both the symplectic and complex viewpoint. **Ugo Bruzzo** developed an equivariant cohomology setting for Lie algebroids and proved a related localization formula.

Topic 5: Curvature, Einstein metrics and all that

One of the central problems of modern differential geometry is to determine which compact manifolds admit an Einstein metric (that is, a metric of constant Ricci curvature). But this is merely one facet of the more general problem of trying to relate the geometry of Riemannian manifolds to their topology.

In recent years, it has also become increasingly clear that one should pay close attention to the existence problem for complete Einstein metrics on non-compact manifolds. In particular, the study of *asymptotically hyperbolic* Einstein metrics turns out to be intimately related to the study of conformal structures of compact manifolds. This has led to new results, not only in mathematics, but also in physics, where the relation between conformal metrics on the boundary of a compact manifold-with-boundary and Einstein metrics on the interior is viewed as a typical example the so-called AdS/CFT correspondence, a topic of central interest in string theory.

Olivier Biquard (3 lectures) gave an introduction to the subject of asymptotically symmetric Einstein manifolds, involving the correspondence between Einstein metrics and their conformal boundaries. This leads to a new eta invariant for CR 3-manifolds (joint work with M. Herzlich). The last lecture explained results characterizing boundaries of self-dual Einstein metrics.

Kris Galicki (3 lectures) spoke on Sasakian structures and Einstein metrics. He began by focussing on the relation between Sasakian and canonical Kähler metrics. He proved that both standard and exotic spheres admit large families of Sasakian-Einstein metrics, and that all homotopy spheres in dimension 7 and 11 admit such metrics. Many compact, simply-connected contact 5-manifolds do not admit any Sasakian structure and even fewer have positive Sasakian structures (work with C. Boyer and J. Kollar).

Paolo Piccinni discussed self-dual Einstein metrics with 2-dimensional isometry group, and their relation to 7-dimensional 3-Sasakian manifolds. **Andrew Swann** reported on Einstein metrics of neutral signature, with applications to the classification of so-called hypersymplectic and para-quaternionic structures.

Gary Gibbons (2 lectures) explained the theory of consistent reductions of higher dimensional supergravity theories, D-branes and new metrics of exceptional holonomy. He also constructed infinitely many Einstein metrics on a product of sphere from Kerr-de-Sitter black holes. **George Papadopoulos** reported on the attempts that have been made to classify the solitons of 11-dimensional supergravity, including a classification of solitons with maximal supersymmetry.

Jeff Viaclovsky (2 lectures) obtained a compactness result for various classes of Riemannian metrics in dimension 4, including anti-self-dual and CSC Kähler metrics. With certain geometric assumptions, the moduli space can be compactified by adding metrics with orbifold singularities (joint work with G. Tian). The analysis differs substantially from the known Einstein case in that it avoids the use of a point-wise Ricci curvature bound.**Matt Gursky** (2 lectures) introduced the problem of prescribing symmetric functions of the eigenvalues of the Ricci tensor of a conformal metric. After a technical discussion of ellipticity and other notions, he gave a general existence theorem (work with J. Viaclovsky). Applications to Ricci curvature pinching and `sphere' theorems followed.

Claude LeBrun (3 lectures) explored the problem of determining which 4-manifolds admit optimal metrics, in both the Einstein and non-Einstein context, and with applications to differential topology. The L^2 norm of the Riemann curvature tensor may be considered as a functional on the space of Riemannian metrics on a given smooth compact 4-manifold and an *optimal metric* is by definition an absolute minimum.

Alice Chang and Paul Yang (3 lectures in total) spoke about the Q-curvature. This consists of polynomials formed by the contraction of curvatures and their derivatives, and gives rise to a family of higher order conformally covariant operators generalizing the Laplacian. Their role in the study of conformally compact Einstein manifolds was discussed (using work of R. Graham and Zworski), as was its relation to the renormalized volume of conformally compact Einstein manifolds (work with J. Qing), and various PDE aspects.

Andrea Sambusetti showed that any closed negatively curved manifold is growth tight, in the sense that its universal covering has an exponential growth rate which is strictly greater than the exponential growth rate of any other normal covering. This has applications to periodic geodesics and spectral geometry.

Last, but by no means least, **Marcel Berger** gave a mid-term colloquium on convex bodies, for which the answers to some simple 3-dimensional questions concerning the area of plane sections and volume have been obtained only recently. The asymptotic version of these relations in higher dimensions is still partly conjectural.

Conclusion

One of the programme's goals was to provide a forum for non-experts, or at least those working on the periphery of a given area within Riemannian and complex geometry, to learn about work in progress and applications to topology. The events purposefully brought together experts and younger researchers, and the Centro awarded a total of 17 grants to support junior participants.

The exchange of knowledge was accomplished by a mixture of short courses (typically only 3 or 4 lectures in self-contained weekly units) and regular seminars. Roughly half of the ten weeks were subject to more intensive activity, though leaving ample time for individual collaboration. Full details of the activities are still available on the Centro's website.

The programme incorporated a total of 110 lectures, each of an hour or more, and over one hundred mathematicians participated. Roughly a quarter of these were day-visitors from nearby universities, whilst more than half had travelled from institutions outside of Italy, many in the United States. These proportions maximized contacts between research groups in different countries with common interests.

The list of participants included a total of 58 lecturers, of whom

- 11 gave short courses of between 3 and 5 lectures,
- 21 gave presentations of 2 lectures,
- 26 gave a single seminar.

A notable feature of the programme was that the key speakers were all able to report on recent major advances in their field. This was especially true for the theory of extremal Kähler metrics (Topic 1), and it is planned to devote a short follow-up workshop to this topic at the Centro De Giorgi in 2006.