

CENTRO DI RICERCA MATEMATICA ENNIO DE GIORGI

Diophantine Geometry

April 12 - July 22, 2005

Final Report

The research program in "Diophantine Geometry" took place at the Centro di Ricerca Matematica Ennio De Giorgi, Pisa, from April 12 to July 22, 2005. The scientific committee consisted of Yuri Bilu (Bordeaux I), Enrico Bombieri (IAS Princeton), David Masser (Basel), Lucien Szpiro (Graduate Center-CUNY) and Umberto Zannier (SNS Pisa).

The original idea of the committee was to have updating seminars on different but interacting branches of the main topic; in particular, emphasis was mainly put on the following themes:

1 - Rational points and their distribution, especially from the point of view of local-global principles.

2 - Integral points on algebraic varieties; here one viewpoint is to confine them in a proper Zariski-closed subset; another one is to bound their number as the height increases, as uniformly as possible (with respect to some relevant parameters associated to the variety).

3 - Algebraic points on algebraic varieties; much recent work has been done here after the Bogomolov Conjecture (and its solution), predicting lower bounds for the height of algebraic points in subvarieties of group varieties, provided the points lie outside certain special sets. A related matter is to study the points in the intersection of the variety with the family of algebraic subgroups of given dimension. As was illustrated in some talks, certain problems in the subject recently reappeared in the context of Shimura varieties and the André-Oort conjecture.

4 - Diophantine Approximation and Transcendence; this has always been deeply related to Diophantine Geometry. Various aspects appeared in the lectures at the workshop, such as the abc-conjecture, the transcendence of values of special functions and the Subspace Theorem in bounded degree. Diophantine Approximation plays also a crucial role in the uniform distribution of heights defined by dynamical systems associated to the iteration of maps, a topic important also in the previous point n. 3. Some very recent results on this were presented in some talks.

Generally speaking, the talks were distributed so that the rhythm was not too diluted nor too intensive. This was appreciated by the participants as it allowed them to find time for discussions and joint work.

In addition to the participants, the talks were attended by several people from Scuola Normale and the Mathematics Department of Pisa, and also by a number of young scholars from abroad, who received partial support from Centro De Giorgi.

Finally, it is the intention of the committee and of the Centro De Giorgi to collect some material in a Proceedings volume. This possibility has already been welcomed by several participants.

LECTURES:

Francesco Amoroso, *Lower bounds for the height on a multiplicative group*

Abstract: Let V be an algebraic subvariety of a torus. Then, by a theorem of Zhang, the complement in V of the Zariski closure of the set of its torsion points is discrete for the metric

induced by the normalized height. We describe several quantitative versions of this result which are close to the conjectural bounds.

Alan Baker, *Transcendence theory and the abc-conjecture*

Francesco Baldassari, *p-adic Beta and Gamma functions in étale and crystalline cohomology*

Michael Bennett, *The factorization of $x^2 + x$*

Abstract: If we factor $x^2 + x = 2^a 3^b y$, with a, b, x and y nonnegative integers, then it has been known for almost 700 years that $y > 1$ provided $x > 8$. We will show how to derive a lower bound of the shape $y > x^{(1/4)}$, valid for all $x > 8$ and sketch (curious) connections between this result and the Inverse Galois Problem. This is joint work with Michael Filaseta and Ognian Trifonov.

Massimo Bertolini, *Hida families of modular forms and rational points on elliptic curves*

Abstract: We review some recent results, obtained in collaboration with Henri Darmon, on the construction of rational points on elliptic curves via the derivatives of certain Hida p-adic L-functions.

Daniel Bertrand, *Multiplicity estimates for q-difference equations*

Yuri Bilu, *Divisibility of class numbers and Hilbert's irreducibility theorem*

Abstract: Given positive integers m and $n > 2$, we show that for every large X there exists $>> X^a$ (totally real) number fields of degree n , of discriminant $< X$ and with class number divisible by m . Here a is a positive number depending on m and n (we show that $a = 1/2m(n-1)$ is admissible). Our result extends to the fields of degree $n > 2$ the previous work of Murty on quadratic fields. Joint with F. Luca.

Fedor Bogomolov, *On the structure of curves and algebraic varieties over "small" closed fields*

Abstract: I will discuss a number of results on the structure of the image of a curve defined over \bar{F}_p in its jacobian. The latter is viewed as an infinite torsion group. One of the applications of these results is "rational connectedness" of all Kummer $K3$ surfaces over \bar{F}_p . Namely for any finite set of points in Kummer $K3$ surfaces there is a chain of rational curves connecting these points.

An "approximation" version of this result holds also for \bar{Q} -points for Kummer $K3$ surfaces defined over \bar{Q} .

It is a joint work with Yu. Tschinkel.

Fedor Bogomolov, *On the Couniformization Conjecture*

Abstract: The conjecture claims the existence of a correspondence between any two hyperbolic (genus > 1) curves defined over "minimal" algebraically closed field (\bar{Q} and \bar{F}_p) which is nonramified with respect to the first curve. I will discuss some partial results and related topics and questions. It is a joint work with Yu. Tschinkel.

Tim Browning, *On Manin's conjecture for singular del Pezzo surfaces*

Abstract: The conjecture in the title relates the asymptotic distribution of rational points of bounded height on Fano varieties to their geometry.

I will outline the state of affairs in dimension 2, culminating in a discussion of recent joint work with Regis de la Breteche and Ulrich Derenthal.

Paula B. Cohen, *Transcendence of special functions related to hypergeometric functions*

Abstract: We give a survey of recent results on transcendence of hypergeometric functions of 1 and several variables at algebraic points. We also discuss such questions for some related functions of 1 variable first considered by Picard.

Jean-Louis Colliot-Thélène,

Flasque resolutions of connected linear algebraic groups;

Picard and Brauer groups of homogeneous spaces

Abstract: Let X be a homogeneous space of a connected linear algebraic group over a field of characteristic zero. Under the assumption that the geometric stabilizers are connected, we show that the Galois module defined by the Picard group of a smooth compactification of X is of a very special type, namely it is a "flasque" Galois lattice. We compute the class of this module up to addition of a permutation module.

This recent joint work with B. Kunyavskii extends known results for principal homogeneous spaces under tori (Voskresenskii, Sansuc and the speaker, 1976) and principal homogeneous spaces under connected linear algebraic groups (Borovoi and Kunyavskii, 2004). There are connexions with the Brauer-Manin obstruction and with the study of R -equivalence on rational points.

Pietro Corvaja, *On integral points on surfaces*

Abstract: In a joint work with U. Zannier a new method was introduced to investigate the integral points on affine surfaces. We obtained finiteness and degeneracy results under suitable numerical conditions on the divisor at infinity. We shall discuss such results as well as some of their implications concerning divisibility problems.

Sinnou David, *Heights on group varieties*

Abstract: We shall describe some recent results on the successive minima for the height of algebraic points of a subvariety of a commutative algebraic group variety.

Roberto Dvornicich, *On the Hasse principle for the division of points in a commutative algebraic group*

Abstract: Let \mathcal{A} be a commutative algebraic group defined over a number field k . We consider the following question: $\{\text{sl Let } r \text{ be a positive integer and let } P \in \mathcal{A}(k). \text{ Suppose that for all but a finite number of primes } v \text{ of } k \text{ we have } P = rD_v, \text{ for some } D_v \in \mathcal{A}(k_v) \text{. Can one conclude that there exists } D \in \mathcal{A}(k) \text{ such that } P = rD?\}$ A complete answer for the case of the multiplicative group G_m is classical. We study other instances, mainly concerning elliptic curves and algebraic tori, obtaining results in both directions: namely, we have families of examples for which the answer is positive and families of examples for which the answer is negative. This is a joint work with U. Zannier.

Jan-Hendrik Evertse, *Distances between the conjugates of an algebraic number;*

Linear equations in unknowns from a multiplicative group in a function field

Gerd Faltings, *Kim's proof of Siegel's theorem 1, 2*

Abstract: Kim has given a new proof of Siegel's theorem about integral points on the projective line minus three points. The first lecture gives background material (fundamental groups, Galois-cohomology, unipotent local systems) while the second covers the proof itself.

Carlo Gasbarri, *Leaves of foliations with many rational points*

Abstract: Let K be number field embedded in \mathbb{C} . Let (X, \mathcal{F}) be a foliated quasi projective variety defined over K . We will sketch the proof of the following statement: Let A be an affine variety defined over \mathbb{C} and $\gamma: A \rightarrow X(\mathbb{C})$ be an analytic map such that i) $\gamma(A)$ is a leaf of the foliation; ii) $\gamma^{-1}(X(K))$ is Zariski dense in A ; then the map γ is algebraic. This generalizes a classical transcendence theorem by Bombieri and Schneider Lang. It implies for instance the following statement: Let X be a quasiprojective variety defined over K and (E, ∇) be a vector bundle equipped with an integrable connection over it. Suppose that $\sigma: X \rightarrow E$ is an analytic horizontal section of (E, ∇) , then the restriction of σ to the Zariski closure of $\sigma^{-1}(E(K))$ is

algebraic.

Andrew Granville, *Rational and integral points on hyperelliptic curves*

Abstract: Using the abc conjecture and other tools we investigate the number of points on quadratic twists of a given curve and find various bounds which tie in well with well-known conjectures. We then make some conjectures of our own.

P. Habegger, *The equation $x+y=a$ in multiplicatively dependent unknowns*

Abstract: When α is any non-zero algebraic number we give essentially optimal upper bounds for the height of multiplicatively dependent algebraic numbers x and y with $x+y=\alpha$. We improve these bounds for special values of α and get sharp bounds in certain cases. Finally for infinitely many α we show that the maximal height value obtained is isolated in a strong sense.

Roger Heath-Brown, *Quadratic Polynomials and Norm Forms*

Abstract: We are interested in the Hasse Principle and Weak Approximation for equations of the type $P(t)=N(x_1, \dots, x_d)$, where P is a polynomial in one variable and N is a norm form produced by an algebraic number field of degree d . The talk will present a line of attack in the case in which P is an irreducible quadratic polynomial, using methods from analytic number theory. This is work in progress.

Counting Points on Algebraic Varieties

Abstract: We examine upper bounds for the number of rational points of height $\leq B$ on an algebraic variety. The talk will cover Results, Applications and Methods.

Mark Hindry, *Rational points and heights on abelian varieties.*

Abstract: The talk will be an introductory survey reviewing the Mordell-Weil theorem, Neron-Tate height. We will discuss results and conjecture concerning effectivity questions, bounds for the torsion, lower bounds for the heights of points of infinite order and upper bounds for the regulator and applications to the study of rational points on subvarieties.

Michel Laurent, *On exponents of uniform Diophantine Approximation and questions of spectra*

Abstract: Following Khintchine and Jarnik, we introduce both an usual and a uniform exponent of Diophantine Approximation associated to a given system of linear forms with real coefficients. We shall be concerned with the possible values of the four exponents thus obtained when considering a system and its dual. The problem is almost trivial in dimension one. For a couple of real numbers (α, β) , we refine the Khintchine Transference Theorem and determine exactly the spectrum in \mathbb{R}^4 of these four values when (α, β) ranges over \mathbb{R}^2 . When α is a Sturmian continued fraction and $\beta = \alpha^2$, explicit formula for the associated quadruple have been recently obtained by Roy and Bugeaud-Laurent. We shall present these works. Relations with the problem of inhomogeneous Diophantine Approximation will be also described.

Aaron Levin, *Generalizing Siegel's and Picard's Theorems*

Abstract: Let C be a complex affine irreducible curve with more than two points at infinity. Siegel's theorem states that, assuming C is defined over a number field, there are only finitely many integral points on C . On the other hand, it is a theorem of Picard that all holomorphic maps $f: \mathbb{C} \rightarrow C$ are constant. Based on recent work of Corvaja and Zannier, I will give new conjectures which generalize Siegel's and Picard's theorems to higher-dimensional varieties and describe progress I have made on proving these conjectures.

David Masser, *Counting, heights and algebraic subgroups - large height*

Abstract: Northcott showed that there are at most finitely many algebraic points in a given affine space with height and degree bounded above. Schanuel gave the asymptotic counting function for large height over a fixed number field. In this talk we describe a generalization of the latter result

and some of its applications: for example to count multiplicative algebraic subgroups, or to count points of large height and fixed degree on a line.

Counting, heights and algebraic subgroups - large degree

Abstract: Northcott showed that there are at most finitely many algebraic points in a given affine space with height and degree bounded above. When the points are restricted to an algebraic curve, and furthermore lie on the union of all proper algebraic subgroups, Bombieri, Zannier and the speaker showed that the height is usually automatically bounded. We give fairly precise estimates for the number of points with large degree, and we present a conjecture about the asymptotic counting function.

Loïc Merel, Recent progress on a uniform version of Serre's open image theorem

Abstract: A theorem of J-P. Serre asserts that any automorphism on the p -division points of a (non-CM) elliptic curve over \mathbb{Q} comes from an element of the Galois group, when p is a large enough prime number. A uniform version would assert that the "large enough" does not depend on the elliptic curve, for instance $p > 37$. It is classical to decompose this problem in terms of rational points on modular curves associated to the maximal subgroups of $GL_2(\mathbb{F}_p)$: Borel subgroups, normalizer of split Cartan subgroups, normalizer of nonsplit Cartan subgroups. The case of Borel subgroups has been treated by B. Mazur in the 1970's. I will discuss the recent progress on the normalizer of split Cartan case due to P. Parent and M. Rebolledo

Maurice Mignotte, On powers among Fibonacci numbers

Abstract: This is a joint work with Samir Siksek (Oman) and Yann Bugeaud (Strasbourg). We prove that the powers among Fibonacci numbers are 1, 8 and 144. This talk is given for a quite general audience.

Junjiro Noguchi, Some results in the analogue of Nevanlinna theory and Diophantine approximations

Abstract: I will discuss the analogue of Nevanlinna theory in one and several variables and the theory of Diophantine approximations, centering around the so-called second main theorem and abc-conjecture. I would like to present some observations on the classical results in these subjects from this viewpoint, and then discuss some new results motivated by the analogue which have been obtained in these several years. Lang's conjecture and Kobayashi hyperbolicity will be also discussed.

Federico Pellarin, Differential structure of rings and multiplicity problems

Abstract: Given a linear differential system $DY=AY$, where A is a $m \times m$ matrix with complex rational functions entries, and where $Y=(Y_1, \dots, Y_m)$ is a vector of algebraically independent functions, it is known since the seventies that the set of non-zero homogeneous prime ideals P of $\mathbb{C}(z)[Y_1, \dots, Y_m]$ such that DP is contained in P has a minimal non-zero element (this was proved by Nesterenko). However, there does not exist a more general result yet, valid for non-linear differential systems $DY_j=P_j(Y_1, \dots, Y_m)$ too. This property is useful to obtain "multiplicity estimates". It is true for the non-linear Ramanujan differential system $DY_1=(1/12)(Y_1^2-Y_2^2)$, $DY_2=(1/3)(Y_1 Y_2-Y_3^2)$, $DY_3=(1/2)(Y_1 Y_3-Y_2^2)$, and the correspondent multiplicity estimate, proved by Nesterenko, was used to check the algebraic independence of π and e^π . In this talk we discuss about the connection between the property above and multiplicity problems, and we show multiplicity estimates for other non-linear differential systems, which might have diophantine applications.

Patrice Philippon, Counting small points on subvarieties of power of elliptic curves

Richard Pink, Relations between the conjectures of Mordell-Lang and Andre-Oort

Abstract: I will discuss a general conjecture for subvarieties of pure or mixed Shimura varieties which generalizes the André-Oort conjecture. For subvarieties contained in a fiber of a Shimura family of semi-abelian varieties this conjecture amounts to one that is being studied by Bombieri,

Masser, Zannier, Viada, Rémond, and Ratazzi.

Corentin Pontreau, *Lower bounds for the height in small (co)-dimension*

Abstract: We are interested in subvarieties of codimension 1 or 2 of a torus. We give lower bounds for the normalized height: arithmetic ones (when there are conditions on the field of definition of variety, precisely here the rationals) and geometric ones.

Nicolas Ratazzi, *Height, torsion points and algebraic subgroups in CM abelian varieties*

Abstract: We will discuss results and conjectures concerning, bounds for the torsion, lower bounds for the heights of points of infinite order. We will give an application of this results on a generalisation of the Manin-Mumford Conjecture for curves embedded in CM abelian varieties.

Gaël Rémond, *On the number of rational points on curves*

Abstract: We address the problem of finding a bound for the number of rational points on a curve of genus at least two over a number field.

We describe bounds which apply to all curves and also much simpler bounds valid only for certain "generic" curves.

Andrzej Schinzel, *On weak automorphs of binary forms over an arbitrary field*

Abstract: A weak automorph of a binary form defined over a field K is a linear transformation T defined over K such that $f(T(x,y)) = rf(x,y)$, r in K . For $K = \mathbb{Q}$ weak automorphs had been studied by B.Segre in 1946. It is shown how an extension of Segre's theorem leads to an improvement of the recent bound for the number of strict automorphs of f over \mathbb{C} due to I.Berchenko and P.Olver.

Hans Peter Schlickewei, *Approximation of algebraic numbers by algebraic numbers of bounded degree*

Abstract: We discuss what should be the correct exponent in a generalization of Roth's theorem to simultaneous approximations by numbers of bounded degree.

Wolfgang Schmidt, *Approximation by hypersurfaces, and classifications of Mahler and Koksma*

Abstract: Approximation by algebraic hypersurfaces is a common generalization of Dirichlet's theorem on linear forms, interpreted as a result on approximation by rational hyperplanes, and Wirsing's question on approximation by algebraic numbers. We will generalize the Mahler and Koksma classifications of transcendental numbers, and will also briefly deal with approximation by algebraic varieties of given dimension.

Joseph Silverman, *Divisibility Sequences and Algebraic Groups*

Abstract: A classical divisibility sequence is a sequence of integers $D(n)$ with the property that if m divides n , then $D(m)$ divides $D(n)$. Famous divisibility sequences include $a^n - b^n$ and the Fibonacci sequence. A very general construction associates a divisibility sequence to a rational point of infinite order in an algebraic group. In this talk I will describe this construction with many explicit examples, explain how Vojta's conjecture predicts growth bounds (some cases of which have been proven by Bugeaud, Corvaja, and Zannier), and discuss and give some numerical evidence for conjectural "lower bounds".

Alexei Skorobogatov, *Global points on Shimura curves*

Abstract: It is an open question whether all counterexamples to the Hasse principle on smooth projective curves over number fields are due to the Manin obstruction. Following Mazur's approach Bruce Jordan proved in the 1980-s that global points don't exist on certain Shimura curves, producing counterexamples to the Hasse principle. I'll show how these can be explained using descent with respect to the so called 'Shimura covering'. I'll also review other known counterexamples to the Hasse principle on Shimura curves and their relation to the Manin obstruction.

Andrea Surroca, *Integral points on curves and the abc conjecture*

Abstract: We will discuss some links between the abc conjecture of Masser and Oesterle and effective versions of Siegel's theorem about integral points on algebraic curves.

L. Szpiro, *Algebraic Dynamical Systems on the sphere (generalized Mahler measures, equidistribution and symmetries)*

Peter Swinnerton-Dyer, *Rational points on certain pencils of curve;
Counting rational points: Manin's conjecture*

Rob Tijdeman, *On arithmetic progressions of which the product of the terms is a perfect power*

Yuri Tschinkel, *Rational points on Fano varieties*

Abstract: I will discuss recent results and conjectures concerning the geometry and arithmetic of Fano varieties over nonclosed fields.

Thomas Tucker, *Generalized Mahler measures, equidistribution, and symmetric relations on canonical heights*

Abstract: We show that for any map rational map f on the projective line with coefficients in a number field or function field, the canonical height $h_f(x)$ for an algebraic point of x can be computed by taking the average of the log of the minimal polynomial for x on the periodic points of f . Similarly, we are able to use the fact that periodic points are equidistributed to show the following symmetry relation between h_f and the usual height h : the average of h_f on roots of unity is equal to the average of h on the periodic points of f . This average is related to the Zhang-Arakelov pairing on certain metrized line bundles and hence leads us to conjecture that twice this average is an upper bound on $\limsup |h(x) - h_f(x)|$ as x varies over all algebraic numbers. This represents joint work with Lucien Szpiro.

Emmanuel Ullmo, *Rational points on Shimura varieties*

Evelina Viada, *About curves embedded in a product of elliptic curves*

Abstract: We are considering an irreducible curve C defined over a number field. We suppose that the curve is embedded in a product of elliptic curves E^g , or more in general we consider the curve inside its Jacobian J_C . We discuss several subset F of the algebraic points of E^g such that C intersected with F is a finite set of points. Examples of this kind are the Manin-Mumford and Mordell-Lang Conjecture. We extend this statements and give some more general results.

Carlo Viola, *The permutation group method in diophantine approximation*

Abstract: In joint work with G. Rhin, we introduced a new method to deal with qualitative and quantitative irrationality problems for constants related to multiple integrals of rational functions. Our method is based on the construction of groups of birational transformations and of permutation groups acting on suitable families of multiple integrals.

Paul Vojta, *Jets via Hasse-Schmidt Derivations*

Abstract: This will be an introductory talk on defining jet bundles for arbitrary morphisms of schemes. It will basically reprise the content of the paper of the same name, available on the arXiv preprint server, math.AG/0407113. Jet bundles are analogous to generalizations of the relative tangent bundle, adding information on higher derivatives.

Arithmetic Jet Spaces

Abstract: In his 1995 talk at Santa Cruz, J.-P. Demailly discussed compactified quotient jet spaces due originally to J. G. Semple and others. These correspond to certain closed subspaces of the iterated space of lines in the tangent bundle of a complex manifold: $\mathbb{P}(\Omega_1$

$\{X/\mathbb{C}\}$, $\mathbb{P}(\Omega^1_{\mathbb{P}(\Omega^1_{X/\mathbb{C}})/\mathbb{C}})$, etc. I tried to generalize the Semple-Demailly jet spaces to arbitrary characteristic, but was not successful. Instead, though, I have found another definition of jet space, isomorphic to the Semple-Demailly spaces away from the "vertical" part, but which is generally nonisomorphic for 3-jets and higher. This definition will be discussed.

Michel Waldschmidt, *Does there exist quadratic relations among logarithms of algebraic numbers?*

Abstract: The answer to the title question is conjecturally "no, apart from trivial ones". We give a survey on the state of the art on this topic: very few results are known, they deal with homogeneous relations. We add the following item to the very limited collection of answers to this question, dealing with non homogeneous relations. The Strong Six Exponentials Theorem of D.~Roy deals with 2×3 matrices whose entries are linear combinations, with algebraic coefficients, of 1 and logarithms of algebraic numbers. Under suitable assumptions, such a matrix has rank 2. Here we investigate the three 2×2 determinants: under suitable assumptions, one at least is not a linear combination, with algebraic coefficients, of 1 and logarithms of algebraic numbers.

Gisbert Wuestholz, *Hypergeometric Functions and Transcendence*

Andrei Yafaev, *The Andre-Oort conjecture*

Abstract: The Andre-Oort conjecture predicts that the irreducible components of the Zariski closure of a set of special points in a Shimura variety are of Hodge type. In this talk we present some results on this conjecture.

Yuri Zarhin, *Homomorphisms of abelian varieties with special reference to abelian threefolds and superelliptic jacobians*

Abstract: We'll compute the group of homomorphisms between abelian varieties in "general position". A plenty of explicit examples of superelliptic jacobians in general position will be given. We'll also discuss another class of explicit examples in dimension three that arise from Del Pezzo surfaces of degree 2.

Shou-Wu Zhang, *Euler system of Heegner points on Shimura curves*

Abstract: I will explain a joint work with Ye Tian in which we showed that Kolyavagin's work on BSD for elliptic curves over rationals can be generalized to totally real fields except some CM-case. This result have some applications to the uniform bound conjecture of Caporaso-Harris-Mazur on some families of hyper or super elliptic curves.

Equidistribution of CM-points on quaternion Shimura varieties

Abstract: I will explain a recent work about equidistribution about the Galois orbits of CM-points with maximal Hodge-Tate group on quaternions Shimura varieties provided some conjectured bounds in L-series and class groups. This work have some application to Andre-Oort conjecture, and transcendence of periods of abelian varieties and hypergeometric functions.