 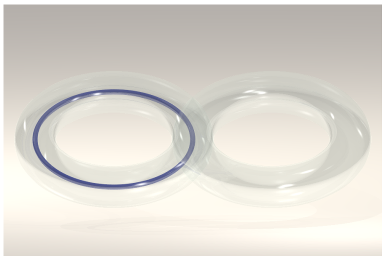

Figure 1: Simply and multiply connected volumes.

1 Magnetic helicity in general

We now consider more deeply how magnetic helicity can arise in different situations in nature and technology. We will have to deal with situations where field lines cross the boundary of the volume of interest, or the volume is multiply connected. We can also take advantage of situations where the magnetic field is contained within flux ropes, and describe the helicity in terms of the geometry of the ropes. We can also take advantage of the special properties of spherical and planar geometries, when they appear in a problem, to simplify the calculation of helicity.

1.1 Decomposition of Vector Fields

Before going further with the analysis of magnetic helicity integrals, we should turn back to vector fields in a finite three dimensional volume. We can ask questions about both the volume and the vector field.

First, we can ask about the topology of the volume. We will consider the following possibilities:

1. *simply-connected* (Very briefly: if simply connected, then any closed curve drawn within the volume can be shrunk to a point, where the shrinking stays within the volume. Example: the interior of a sphere.

2. *multiply-connected* (not simply connected). If multiply connected, some closed curves cannot be shrunk to a point; there will be some obstruction. Example: the interior of a torus.)

3. *2-connected* A volume is 2-connected if any closed surface drawn within

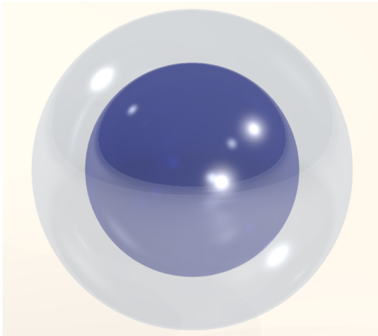

Figure 2: A spherical shell.

can be shrunk to a point. Example: the interior of a sphere is 2-connected, but a *spherical shell* is not.

What about the vector field $\mathbf{B}$ within the volume. Let the volume be Some properties it might have include:

1. divergence free: $\nabla \cdot \mathbf{B} = 0$.
2. curl-free: $\nabla \times \mathbf{B} = 0$.
3. gradient: $\exists$ a scalar function ψ such that $\mathbf{B} = \nabla\psi$.
4. closed: $\mathbf{B} \cdot \hat{n} = 0$.

Given a volume with a particular topology, what sort of vector fields can we contain inside? Also, given a vector field, can we find a unique decomposition of the field into pieces with the special properties listed above?

These questions are treated by the Hodge decomposition theorem (for a thorough review, see J. Cantarella, D. DeTurck, H. Gluck & M. Teytel, *Influence of geometry and topology on helicity*, Magnetic Helicity in Space and Laboratory Plasmas, Geophysical Monograph 111, American Geophysical Union (1999) 17-24).

We first describe five different classes of vector fields living within $\mathcal{V}$. Each will be described as either an electric field $\mathbf{E}$ or a magnetic field $\mathbf{B}$, although there will be many other possible examples we could choose (e.g. velocity fields and vortex fields). Some are called gradients because there exist scalar fields ψ inside $\mathcal{V}$ where $\mathbf{E} = \nabla\psi$ or $\mathbf{B} = \nabla\psi$.

If the volume is multiply connected, like a torus, then there can in general be *interior fluxes* (e.g. in a torus, there is one interior flux, consisting of field

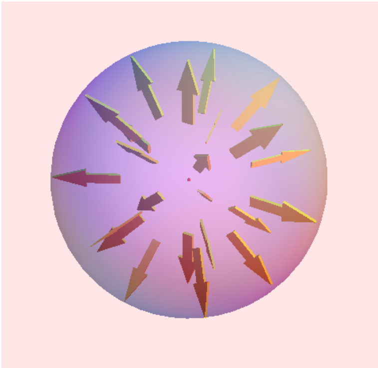

Figure 3: A field with non-zero divergence, e.g. an electrostatic field inside a conducting boundary.

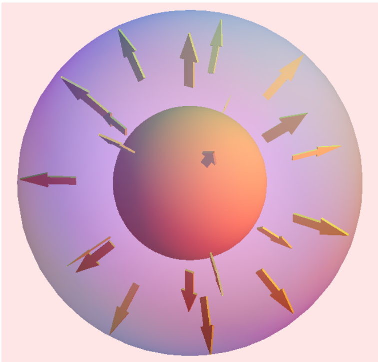

Figure 4: An HG harmonic gradient field inside a spherical shell.

lines parallel to the toroidal axis). We give both Cantarella's terminology as well as some common terminology used for electromagnetic fields.

The first two types explicitly or implicitly involve charges so are more appropriate to electric fields rather than magnetic fields:

1. *GG (Grounded Gradients)* – electrostatic fields: $\nabla \cdot \mathbf{E} \neq 0$, $\mathbf{E} = \nabla\psi$, and $\psi|_{\mathcal{S}} = 0$. This is the only type with a non-zero divergence. It corresponds to an electrostatic field generated by charge (e.g. $\nabla \cdot E = \rho_{\text{charge}}$) within $\mathcal{V}$, where the boundary is grounded (so that ψ is a constant which we can set to 0).

2. *HG (Harmonic Gradients)*: $\nabla \cdot \mathbf{E} = 0$, $\mathbf{E} = \nabla\psi$, and ψ is constant on each component of the boundary $\mathcal{S}$. HG fields correspond to electric fields in a volume with a set of perfectly conducting boundary surfaces.

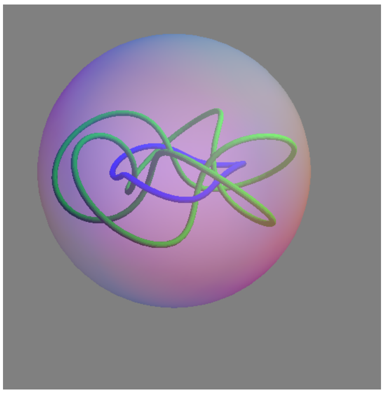

Figure 5: A closed magnetic field inside a sphere.

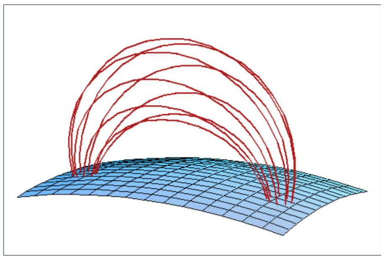

Figure 6: A potential magnetic field.

For example, the boundary of a spherical shell has two components – spheres at radii r_1 and r_2, with potentials ψ_1 and ψ_2). If $\psi_1 \neq \psi_2$ then we can infer that electric charges are hidden inside $r < r_1$. These vector fields are called 'harmonic' by Cantarella because the associated potential ψ satisfies the Laplace equation $\nabla^2\psi = 0$.

The next three correspond to magnetic fields. The first type, FK, has source current inside $\mathcal{V}$ while Types CG and HK have source currents outside of $\mathcal{V}$.

3. *FK (Fluxless knots)* – closed magnetic fields: $\nabla \cdot \mathbf{B} = 0$, $\mathbf{B} \cdot \hat{n} = 0$, all interior fluxes $= 0$.

4. *CG (Curly gradients)* – potential magnetic fields: $\nabla \cdot \mathbf{B} = 0$, $\mathbf{B} = \nabla\psi$, $\mathbf{B} \cdot \hat{n} \neq 0$ but the net flux through each boundary component is 0.

5. *HK (Harmonic knots)*: $\nabla \cdot \mathbf{B} = 0$, $\nabla \times \mathbf{B} = 0$, $\mathbf{B} \cdot \hat{n} = 0$. Such fields only exist in multiply connected volumes.

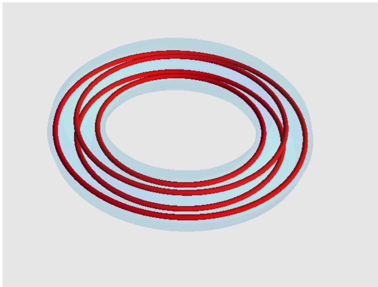

Figure 7: Current-free field lines within a torus.

Since magnetic monopoles are not known to exist in nature, both HG and GG should not be relevant in magnetic problems. Also, the HG fields are only relevant in volumes that are not 2-connected. The FK and HK fields are called 'knots' because their field lines either close upon themselves in $\mathcal{V}$ or come infinitesimally close to doing so; a closed curve in three-dimensions is a 'knot' in mathematical terminology. (A circle is an 'unknot').

The Hodge decomposition theorem simply states that any vector field within $\mathcal{V}$ can be uniquely decomposed as a sum of up to five of these basic fields, one for each type. A magnetic field existing in a simply connected region will divide into a unique FK field and a unique CG field.

There should be a warning given at this point. A commonly used volume does not easily fit into this discussion.

Many magnetic field studies employ periodic boundary conditions. Periodic boxes simplify both theory and numerics, because there are no true boundaries, and because Fourier series can be readily employed. However, periodic boxes cannot exist as sub-volumes of space. They are neither simply connected or 2-connected. Consider a constant magnetic field, say $\mathbf{B} = B_0\hat{z}$ pointing in the z direction. This has no source currents. Since there is no 'outside' to a periodic box, there can be no external sources either.

1.2 Simplest Case: Closed magnetic fields (FK) in a simply connected volume

Consider two fields $\mathbf{V}$ and $\mathbf{W}$. Then we will define

$$H(\mathbf{V}, \mathbf{W}) = \int_{\mathcal{V}} A_{\mathbf{V}} \cdot \mathbf{W} \, \mathrm{d}^3x, \qquad \nabla \times A_{\mathbf{V}} = \mathbf{V}. \tag{1}$$

We can write this in a more symmetric form by employing the *Biot-Savart* *integral* for finding a vector potential $A_{\mathbf{V}}$ corresponding to the field $\mathbf{V}$:

$$\mathbf{A}_{\mathbf{V}}(\mathbf{x}) = \frac{1}{4\pi}\int_{\mathcal{V}} \frac{\mathbf{r}}{r^3} \times \mathbf{V}(\mathbf{y})\; d^3y \qquad (\mathbf{r} = \mathbf{y} - \mathbf{x}) \tag{2}$$

Then

$$H(\mathbf{V},\mathbf{W}) = \frac{1}{4\pi}\int_{\mathcal{V}}\int_{\mathcal{V}} \mathbf{V}(\mathbf{x}) \cdot \frac{\mathbf{r}}{r^3} \times \mathbf{W}(\mathbf{y})\; d^3y\; \mathrm{d}^3x \tag{3}$$

We can now state an essential property of the helicity integral. Helicity is symmetric and bi-linear:

$$H(\mathbf{V},\mathbf{W}) = H(\mathbf{W},\mathbf{V}) \tag{4}$$

$$H(\alpha\mathbf{V} + \beta\mathbf{X},\mathbf{W}) = \alpha H(\mathbf{V},\mathbf{W}) + \beta H(\mathbf{X},\mathbf{W}). \tag{5}$$

1.3 Toroidal and Poloidal fields

In Cartesian or spherical geometries it is often useful to decompose a magnetic field into toroidal and poloidal components. Let $\mathcal{L}$ be an operator similar to the angular momentum operator in quantum theory:

$$\mathcal{L} \equiv \begin{cases} -\hat{z} \times \nabla, & \text{Cartesian geometry;} \\ -\mathbf{r} \times \nabla, & \text{Spherical geometry.} \end{cases} \tag{6}$$

Then we can write

$$\mathbf{B} = \mathcal{L}T + \nabla \times \mathcal{L}P, \tag{7}$$

where T is the toroidal function and P is the poloidal function.

Here we examine the Cartesian case.

The operator $\mathcal{L}$ has two important properties:

$$\hat{z} \cdot \mathcal{L} = 0; \qquad \nabla \cdot \mathcal{L} = 0. \tag{8}$$

The vector potentials of the toroidal field $\mathbf{B}_T = \mathcal{L}T$ and the poloidal field $\mathbf{B}_P = \nabla \times \mathcal{L}P$ are

$$\mathrm{curl}^{-1}\mathbf{B}_T = T\hat{z} + \nabla\Psi_T; \tag{9}$$

$$\mathrm{curl}^{-1}\mathbf{B}_P = \mathcal{L}P + \nabla\Psi_P. \tag{10}$$

where $\Psi_T(x,y,z)$ and $\Psi_P(x,y,z)$ are gauge functions.

The functions P and T can be obtained from $\mathbf{B}$ by solving the equations

$$\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} = B_z; \tag{11}$$

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = (\nabla \times \mathbf{B})_z. \tag{12}$$

Two toroidal fields do not link each other, nor do two poloidal fields: First, the helicity integrals are gauge invariant, so we can ignore Ψ_T and Ψ_P. Next, since $\hat{z} \cdot \mathcal{L} = 0$,

$$H(\mathbf{B}_{T1}, \mathbf{B}_{T2}) = \int T_1 \hat{z} \cdot \mathcal{L} T_2 \, \mathrm{d}^3 x = 0. \tag{13}$$

Next

$$\begin{aligned} H(\mathbf{B}_{P1}, \mathbf{B}_{P2}) &= \int \mathcal{L} P_1 \cdot \mathbf{B}_{P2} \, \mathrm{d}^3 x \\ &= \int \mathcal{L} P_1 \cdot \left(\nabla \frac{\partial P_2}{\partial z} - \hat{z} \nabla^2 P_2 \right) \mathrm{d}^3 x = \int \mathcal{L} P_1 \cdot \left(\nabla \frac{\partial P_2}{\partial z} \right) \mathrm{d}^3 x \\ &= \int \nabla \times P_1 \hat{z} \cdot \left(\nabla \frac{\partial P_2}{\partial z} \right) \mathrm{d}^3 x = \int \nabla \cdot P_1 \hat{z} \times \nabla \frac{\partial P_2}{\partial z} \, \mathrm{d}^3 x \\ &= \oint \hat{z} \cdot P_1 \hat{z} \times \nabla \frac{\partial P_2}{\partial z} \, \mathrm{d}^2 x \\ &= 0. \end{aligned} \tag{14}$$

The last integral vanishes by the divergence theorem.

Theorem

Consider a magnetic field $\mathbf{B} = \mathbf{B}_T + \mathbf{B}_P$ in a region $\mathcal{V}$. Assume that $\mathcal{V}$ is either 1) all space, 2) a half space bounded by a plane, 3) a layer bounded by two planes, 4) the interior or exterior of a sphere, or 5) a spherical shell bounded by two concentric spheres.

Then

1. A purely poloidal field ($T = 0$) has $H(\mathbf{B}, \mathbf{B}) = 0$.
2. A purely toroidal field ($P = 0$) has $H(\mathbf{B}, \mathbf{B}) = 0$.
3. In general,

$$H(\mathbf{B}, \mathbf{B}) = 2 \int_{\mathcal{V}} \mathcal{L} T \cdot \mathcal{L} P \, \mathrm{d}^3 x. \tag{15}$$

Proof:

$$H(\mathbf{B}, \mathbf{B}) = H(\mathbf{B}_T + \mathbf{B}_P, \mathbf{B}_T + \mathbf{B}_P) = 2H(\mathbf{B}_P, \mathbf{B}_T) = 2 \int_{\mathcal{V}} \mathcal{L} T \cdot \mathcal{L} p \, \mathrm{d}^3 x. \tag{16}$$