1 Helicity in Open Volumes

So far we have considered helicity integrals inside closed volumes, where the field lines never cross the boundary. However, this requirement is far too restrictive for many physical and mathematical applications. Solar magnetic fields, for example, cross a natural boundary at the photosphere. Scientists studying the interior of the sun will see this as an outer boundary, while those who study the solar atmosphere regard the photosphere as an inner boundary. We will need a form of helicity integral which works just for the interior, or just for the exterior. Furthermore, the sum of interior and exterior helicities should sensibly relate to the helicity integral of all space. Helicity should be allowed to cross boundaries as well. Laboratory physicists studying confined fusion energy devices also deal with magnetic fields not completely enclosed inside the plasma. In fact, magnetic helicity can be injected into a plasma (to improve its stability) via the field lines that cross the boundary. Mathematicians also interest themselves in topological objects more general than just knots and links. Tangles and braids, for example, involve curves with fixed endpoints on a boundary.

Here we will consider the magnetic helicity $H(\mathbf{B}) = H(\mathbf{B}, \mathbf{B})$ inside an arbitrary volume $\mathcal{V}$. We give a definition of helicity which retains topological meaning and is gauge invariant.

Let space be divided into domains $\mathcal{V}$ and $\mathcal{V}'$, containing magnetic fields $\mathbf{B}$ and $\mathbf{B}'$. At the boundary $\mathcal{S}$, $\mathbf{B} \cdot \hat{n} = \mathbf{B}' \cdot \hat{n}$. The magnetic field defined in all space is

$$\{\mathbf{B}, \mathbf{B}'\}(\mathbf{x}) = \left\{ \begin{array}{ll} \mathbf{B}, & \mathbf{x} \in \mathcal{V}; \\ \mathbf{B}', & \mathbf{x} \in \mathcal{V}'. \end{array} \right. \tag{1}$$

Unfortunately, $H(\{\mathbf{B}, \mathbf{B}'\})$ includes information about all the magnetic structure in $\mathbf{B}'$. We need to subtract this extra information.

Simply integrating $\mathbf{A} \cdot \mathbf{B}$ over $\mathcal{V}$ as before will not do, if $\mathcal{V}$ contains CG or HK fields. The integral will no longer be gauge invariant or topologically meaningful. Instead, we measure the helicity relative to a minimal base state. This procedure is similar to measuring voltage relative to ground, or potential gravitational energy relative to sea level.

Thus we will look for some simple vector field $\mathbf{P}$ inside $\mathcal{V}$ for which we can calculate the reference helicity $H(\{\mathbf{P}, \mathbf{B}'\})$. Once we subtract this reference helicity, the dependence on the external field will vanish.

Definition: The magnetic helicity inside an arbitrary volume $\mathcal{V}$ is given

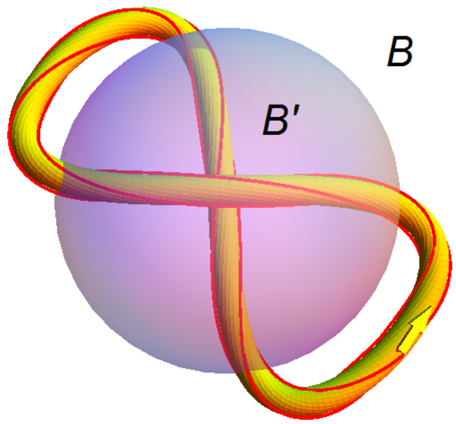


Figure 1: The magnetic field (here a flux rope in the shape of a figure-8) is separated into two pieces $\mathbf{B}$ and $\mathbf{B}'$ by a boundary surface.

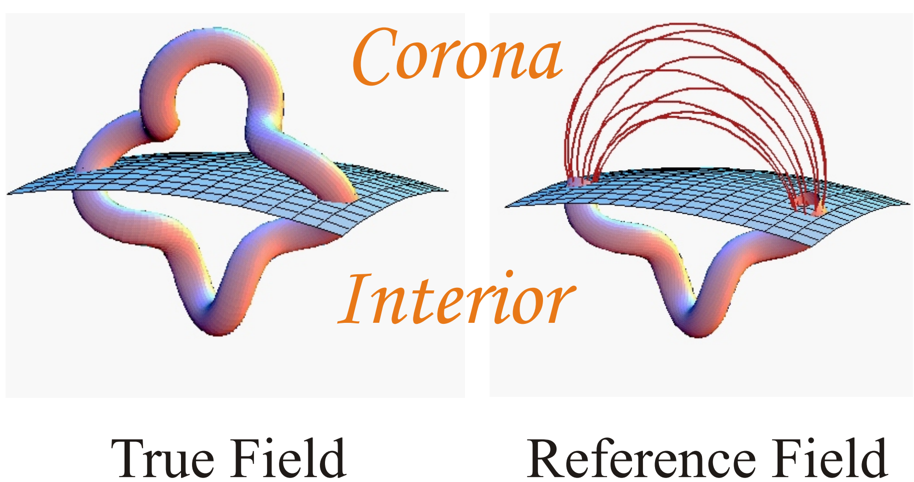


Figure 2: Calculating the helicity of the corona by subtracting the helicity of a reference potential field $\mathbf{P}$.

by

$$\boxed{H_{\mathcal{V}} = H(\{\mathbf{B}, \mathbf{B}'\}) - H(\{\mathbf{P}, \mathbf{B}'\})}. \tag{2}$$

The boundary information $\mathbf{B} \cdot \hat{n}$ tells us the distribution of flux crossing the boundary $\mathcal{S}$. It also determines a unique vector field, the vacuum (or potential) field $\mathbf{P}$:

The vacuum (potential) field $\mathbf{P}$ in $\mathcal{V}$ satisfies

$$\mathbf{P} \cdot \hat{\mathbf{n}}|_S = \mathbf{B} \cdot \hat{n}; \tag{3}$$

$$\nabla \times \mathbf{P}(\mathbf{x}) = 0, \qquad \mathbf{x} \in \mathcal{V}. \tag{4}$$

If $\mathcal{V}$ is multiply connected, the net flux of $\mathbf{P}$ through any closed curve on $\mathcal{S}$ should also be the same for $\mathbf{B}$ and $\mathbf{P}$. In other words, if $\mathbf{B}$ within $\mathcal{V}$ is decomposed (uniquely!) into FK, HK, and CG components, then $\mathbf{P} = HK + CG$.

The vacuum (or potential) field is

1. The minimum energy state consistent with the boundary data $\mathbf{B} \cdot \hat{n}$ and flux data.

2. Curl–free, i.e. zero electric currents.

Theorem

1. $H_{\mathcal{V}}$ is gauge invariant;

2. $H_{\mathcal{V}}$ can be expressed as an integral over $\mathcal{V}$ alone, and is thus independent of the field $\mathbf{B}'$ outside of $\mathcal{V}$:

$$H_{\mathcal{V}} = \int_{\mathcal{V}} (\mathbf{A} + \mathbf{A}_P) \cdot (\mathbf{B} - \mathbf{P}) \, \mathrm{d}^3x \tag{5}$$

   where $\nabla \times \mathbf{A_P} = \mathbf{P}$.

Proof

1. Both terms in $H_{\mathcal{V}}$ are integrated over all space. Thus they are gauge invariant. Gauge invariance can also be checked from equation (5).

   The vector potential for the true magnetic field $\{\mathbf{B}, \mathbf{B}'\}$ will be called $\{\mathbf{A}, \mathbf{A}'\}$. By continuity, the parallel components of $\mathbf{A}$ and $\mathbf{A}'$ coincide:

$$\hat{\mathbf{n}} \times \mathbf{A} = \hat{\mathbf{n}} \times \mathbf{A}' \tag{6}$$

The vector potential for the reference field $\{\mathbf{P}, \mathbf{B}'\}$ is $\mathbf{A}_P$ inside $\mathcal{V}$, but equals $\mathbf{A}' + \nabla\psi$ inside $\mathcal{V}'$:

$$\mathbf{A}_{\{\mathbf{P},\mathbf{B}'\}} = \{\mathbf{A}_P, \mathbf{A}' + \nabla\psi\} \tag{7}$$

(the $\nabla\psi$ term must be there so that at the boundary $\mathcal{S}$ the vector potential smoothly matches up with $\mathbf{A}_P$.) Thus

$$\hat{\mathbf{n}} \times \mathbf{A}_P = \hat{\mathbf{n}} \times (\mathbf{A}' + \nabla\psi). \tag{8}$$

Then

$$H(\{\mathbf{B}, \mathbf{B}'\}) = \left(\int_{\mathcal{V}} \mathbf{A} \cdot \mathbf{B} + \int_{\mathcal{V}'} \mathbf{A}' \cdot \mathbf{B}' \right) \mathrm{d}^3x; \tag{9}$$

$$H(\{\mathbf{P}, \mathbf{B}'\}) = \left(\int_{\mathcal{V}} \mathbf{A}_P \cdot \mathbf{P} + \int_{\mathcal{V}'} (\mathbf{A}' + \nabla\psi) \cdot \mathbf{B}' \right) \mathrm{d}^3x; \tag{10}$$

$$\Rightarrow H_{\mathcal{V}} = \int_{\mathcal{V}} (\mathbf{A} \cdot \mathbf{B} - \mathbf{A}_P \cdot \mathbf{P}) \ \mathrm{d}^3x - \int_{\mathcal{V}'} (\nabla\psi) \cdot \mathbf{B}' \ \mathrm{d}^3x \tag{11}$$

$$= \int_{\mathcal{V}} (\mathbf{A} \cdot \mathbf{B} - \mathbf{A}_P \cdot \mathbf{P}) \ \mathrm{d}^3x + \oint_{\mathcal{S}} \psi \mathbf{B} \cdot \hat{n} \ d^2x, \tag{12}$$

using $\nabla \cdot \mathbf{B}' = 0$ in the last equation.

Meanwhile

$$\int_{\mathcal{V}} (\mathbf{A}_P \cdot \mathbf{B} - \mathbf{A} \cdot \mathbf{P}) \ \mathrm{d}^3x = \int_{\mathcal{V}} (\mathbf{A}_P \cdot \nabla \times \mathbf{A} - \mathbf{A} \cdot \nabla \times \mathbf{A}_P) \ \mathrm{d}^3x \tag{13}$$

$$= \int_{\mathcal{V}} \nabla \cdot \mathbf{A} \times \mathbf{A}_P \ \mathrm{d}^3x \tag{14}$$

$$= \oint_{\mathcal{S}} \mathbf{A} \times \mathbf{A}_P \cdot \hat{\mathbf{n}} \ d^2x \tag{15}$$

$$= \oint_{\mathcal{S}} \mathbf{A} \times (\mathbf{A}_P - \mathbf{A}) \cdot \hat{\mathbf{n}} \ d^2x. \tag{16}$$

Thus

$$\oint_{\mathcal{S}} \mathbf{A} \times (\mathbf{A}_P - \mathbf{A}) \cdot \hat{\mathbf{n}} \ d^2x = \oint_{\mathcal{S}} \mathbf{A} \times \nabla\psi \cdot \hat{\mathbf{n}} \ d^2x \tag{17}$$

$$= \oint_{\mathcal{S}} (\psi \nabla \times \mathbf{A} - \nabla \times \psi \mathbf{A}) \cdot \hat{\mathbf{n}} \ d^2x \tag{18}$$

$$= \oint_{\mathcal{S}} \psi \mathbf{B} \cdot \hat{n} \ d^2x. \tag{19}$$

(The last term in equation (18) vanishes by Stoke's theorem.) Putting together equations (12), (16), and (19) yields the helicity formula.

1.1 Addition of Helicities in Different Regions of Space

Let $H_{total}(\{\mathbf{B}, \mathbf{B}'\})$ be the helicity of all space. Also let $\mathbf{P}$ be the potential field in $\mathcal{V}$, and $\mathbf{P}'$ be the potential field in $\mathcal{V}'$ (given the boundary data $\mathbf{B} \cdot \hat{n}$ on $\mathcal{S}$ and the magnitude of any interior fluxes if $\mathcal{V}$ is multiply connected and contains an HK field).

The total helicity of space equals the sum of the helicities contained in $\mathcal{V}$ and $\mathcal{V}'$, plus a term involving the potential fields on either side of $\mathcal{S}$:

$$H_{total}(\{\mathbf{B}, \mathbf{B}'\}) = H_{\mathcal{V}}(\mathbf{B}) + H_{\mathcal{V}'}(\mathbf{B}') + H(\{\mathbf{P}, \mathbf{P}'\}). \qquad (20)$$

Proof Write

$$\begin{aligned} H_{\mathcal{V}}(\mathbf{B}) &= H_{total}(\{\mathbf{B}, \mathbf{B}'\}) - H_{total}(\{\mathbf{P}, \mathbf{B}'\}), && (21)\\ H_{\mathcal{V}'}(\mathbf{B}') &= H_{total}(\{\mathbf{P}, \mathbf{B}'\}) - H_{total}(\{\mathbf{P}, \mathbf{P}'\}). && (22) \end{aligned}$$

1.2 Helicity with planar or spherical boundaries

Theorem

If the boundary $\mathcal{S}$ is a plane or a sphere, then the potential term $H(\{\mathbf{P}, \mathbf{P}'\})$ vanishes.

Proof for a planar boundary

The Coulomb vector potential $\mathbf{A}$ for a vector field $\mathbf{V}$ can be written in the Biot-Savart form already shown, or in the form (obtained after integrating by parts)

$$\mathbf{A}_{\mathbf{V}}(\mathbf{x}) = -\frac{1}{4\pi} \int_{\mathcal{V}} \frac{1}{r} \nabla_{\mathbf{y}} \times \mathbf{V}(\mathbf{y})\ d^3y \qquad (\mathbf{r} = \mathbf{y} - \mathbf{x}). \qquad (23)$$

Thus

$$\mathbf{A}_P(\mathbf{x}) = -\frac{1}{4\pi} \int_{\mathcal{V}} \frac{1}{r} \nabla_{\mathbf{y}} \times \mathbf{P}(\mathbf{y})\ d^3y. \qquad (24)$$

But a potential field has zero curl. The only place the global field $\{\mathbf{P}, \mathbf{P}'\}$ has curl is in the boundary surface. For a plane boundary, this means that $\nabla \times \mathbf{P}$ is parallel to the plane. Thus

$$\mathbf{A}_P \cdot \hat{\mathbf{n}} = 0. \qquad (25)$$

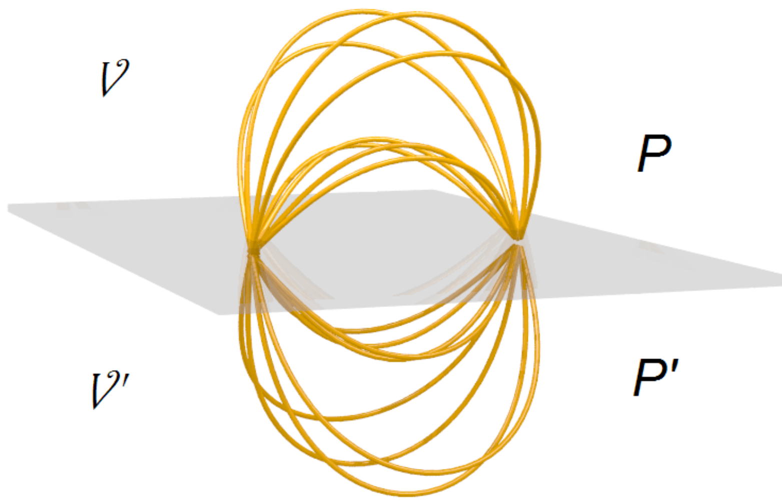


Figure 3: Suppose a magnetic field crosses a planar boundary at just two points. The potential fields on either side resemble dipole fields.

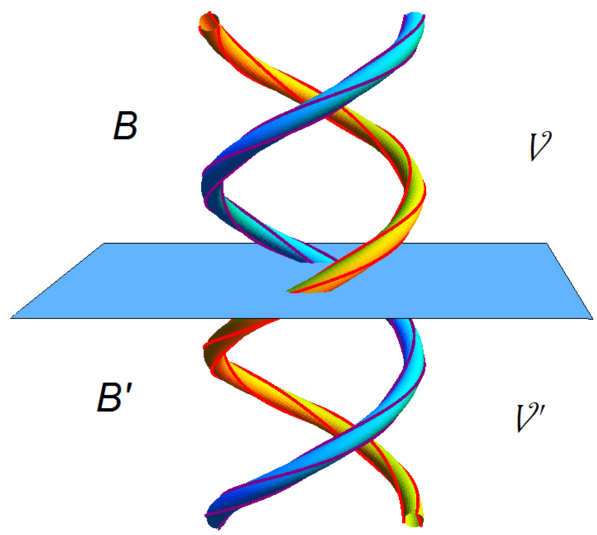


Figure 4: With a planar boundary, the helicity of all space equals the sum of the helicity below and the helicity above.

Now write the potential fields as, well, gradients of potentials, $\mathbf{P} = \nabla\phi$ and $\mathbf{P}' = \nabla\phi'$:

$$\begin{aligned} H_{total}(\{\mathbf{P}, \mathbf{P}'\}) &= \int_{\mathcal{V}} \mathbf{A}_P \cdot \mathbf{P}\ \mathrm{d}^3x + \int_{\mathcal{V}} \mathbf{A}_P{}' \cdot \mathbf{P}'\ \mathrm{d}^3x & (26) \\ &= \int_{\mathcal{V}} \mathbf{A}_P \cdot \nabla\phi\ \mathrm{d}^3x + \int_{\mathcal{V}} \mathbf{A}_P{}' \cdot \mathbf{P}'\ \mathrm{d}^3x & (27) \\ &= \int_{z=0} \phi \mathbf{A}_P \cdot \hat{\mathbf{n}}\ d^2x + \int_{z=0} \phi' \mathbf{A}_P{}' \cdot \hat{\mathbf{n}}'\ d^2x & (28) \\ &= 0 + 0. & (29) \end{aligned}$$

Thus for a planar or spherical boundary,

$$H_{total}(\{\mathbf{B}, \mathbf{B}'\}) = H_{\mathcal{V}}(\mathbf{B}) + H_{\mathcal{V}'}(\mathbf{B}'). \quad (30)$$

1.3 Solar Magnetic Fields

The ability to calculate helicity of subvolumes of space has special relevance for magnetic fields in the solar atmosphere. The boundary surface here is

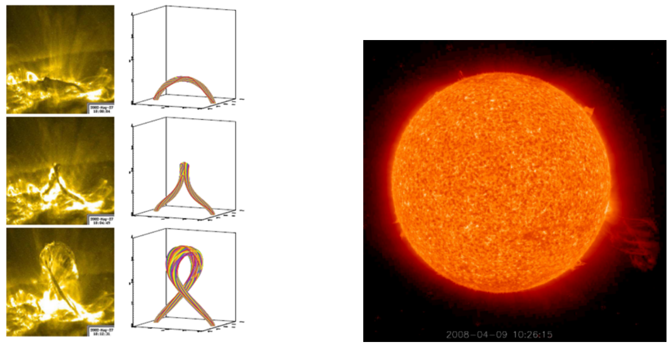
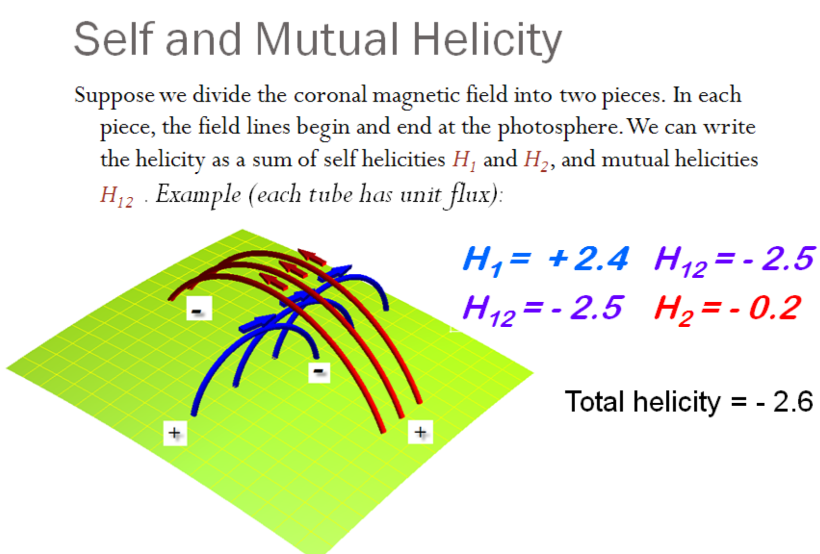


Figure 5: Erupting prominences in solar atmosphere. (left: *from Kliem & Török 2005)*. Right: Stereo image 2008.

the photosphere (see figures).



1.4 The time derivative of Helicity

The time derivative of $H_{\mathcal{V}}$ can be shown to be

$$\boxed{\frac{\mathrm{d}H_{\mathcal{V}}}{\mathrm{d}t} = -2\int_{\mathcal{V}} \mathbf{E}\cdot\mathbf{B}\,\mathrm{d}^3x - 2\oint_{\mathcal{S}} \mathbf{A}_P \times \mathbf{E}\cdot\hat{\mathbf{n}}\,d^2x} \tag{31}$$

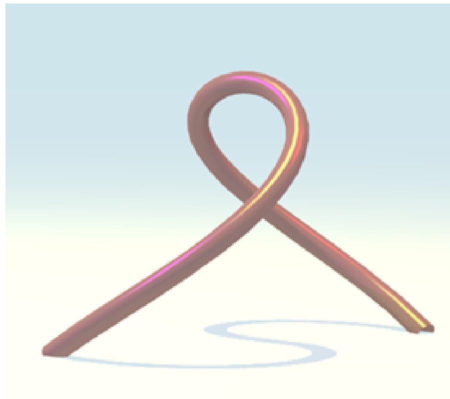


Figure 6: a loop of magnetic flux. With this shape, the magnetic helicity integral would equal $-0.2\Phi^2$, where Φ is the net magnetic flux along the loop, plus a contribution from internal twist of the field lines within the loop.

Here $\mathbf{A}_P$ is a vector potential uniquely defined by

$$\nabla \times \mathbf{A}_P = \mathbf{P}, \tag{32}$$

$$\hat{\mathbf{n}} \cdot \nabla \times \mathbf{A}_P = B_n, \tag{33}$$

$$\nabla \cdot \mathbf{A}_P = 0, \tag{34}$$

$$\mathbf{A}_P \cdot \hat{\mathbf{n}} = 0. \tag{35}$$

1.5 Helicity Dissipation

Suppose $\mathbf{E} = \eta \mathbf{J}$ where η is the resistivity and $\mathbf{J}$ is the electric current. Then the first term measures resistive (Ohmic) dissipation of helicity. We can compare this to Ohmic dissipation of magnetic energy. Let E_M be the magnetic energy and μ_0 the SI unit correction factor (the 'vacuum magnetic permeability'). Then

$$E_M = \frac{1}{2\mu_0} \int B^2 \, \mathrm{d}^3x; \tag{36}$$

$$\frac{\mathrm{d}H}{\mathrm{d}t} = -2 \int \eta \mathbf{J} \cdot \mathbf{B} \, \mathrm{d}^3x; \tag{37}$$

$$\frac{\mathrm{d}E_M}{\mathrm{d}t} = -\int \eta J^2 \, \mathrm{d}^3x. \tag{38}$$

(39)

A Schwartz inequality gives

$$\left|\frac{\mathrm{d}H}{\mathrm{d}t}\right| \leq \sqrt{8\eta\mu_0 E_M \left|\frac{\mathrm{d}E_M}{\mathrm{d}t}\right|}. \tag{40}$$

As η is effectively tiny for astrophysical magnetic fields, this suggests that helicity dissipation occurs much more slowly than energy dissipation.

1.6 Helicity Flux

The boundary term gives helicity flow from one region of space into another. For ideal magnetohydrodynamics, $\mathbf{E} = \mathbf{B} \times \mathbf{V}$, giving

$$\boxed{\frac{\mathrm{d}H_{\mathcal{V}}}{\mathrm{d}t} = -2 \oint_{\mathcal{S}} \left((\mathbf{A}_P \cdot \mathbf{V})\mathbf{B} - (\mathbf{A}_P \cdot \mathbf{B})\mathbf{V}\right) \cdot \hat{\mathbf{n}}\ d^2x}. \tag{41}$$

One example of helicity flux occurs over the timescale of a solar cycle (typically about 11 years). The equator of the sun rotates faster than the poles, so magnetic features near the equator move with respect to polar features. This leads to a twisting up of field lines in each hemisphere, injecting negative helicity into the Northern hemisphere of the sun, and positive helicity into the Southern hemisphere. At the same time, positive helicity streams out of the Northern surface into the Northern solar wind and into Northern interplanetary space, with negative helicity streaming out into the Southern half of interplanetary space.

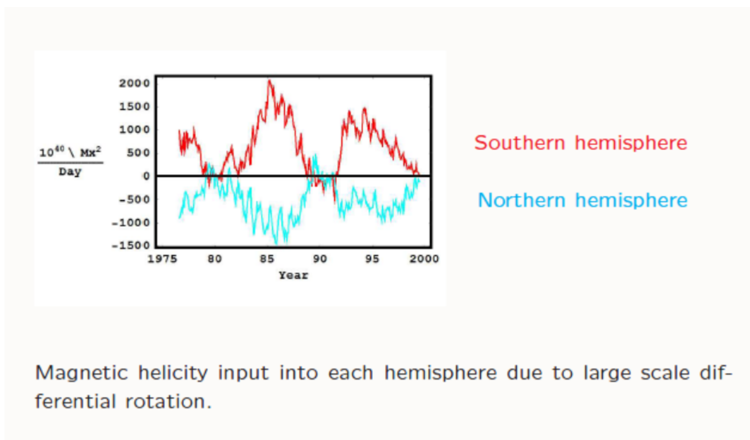


Figure 7: Observations of helicity flux into the Northern and Southern solar interior