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Luigi Ambrosio. *Well-posedness of ODE's relative to nonsmooth vector fields.*

Olivier Druet. *Moving planes, moving spheres and symmetry.*

Stefano Marmi. *An introduction to small divisors problems.*

Petru Mironescu. *Inversion of the divergence.*

Fulvio Ricci. *Different kinds of Riesz transforms on the Heisenberg group and their L^p-boundedness.*

Denis Serre. *L^1-contracting semi-groups and the stability of shock profiles.*

Abstracts.

Luigi Ambrosio. *Well-posedness of ODE's relative to nonsmooth vector fields.*
Abstract. In the lectures we will illustrate the basic ingredients of the DiPerna-Lions theory, which provides well-posedness and stability results for ODE's $\dot{\gamma}(t) = b(t, \gamma(t))$ associated to a nonsmooth vector fields. First we will revisit a few classical facts, namely the Cauchy-Lipschitz theory, the duality between transport and continuity equation and the method of characteristics. In the last part of the lectures I will show how well-posedness results can be transferred from the PDE (continuity or transport equation) to the ODE.

Olivier Druet. *Moving planes, moving spheres and symmetry.*
Abstract. We shall introduce the technique of moving planes, relying on the invariance of an elliptic PDE by reflection with respect to a plane and the maximum principle, to get symmetry of solutions of this PDE. We shall give many examples, starting from the simplest and most geometric ones to more refined ones. We will also discuss the more recent method of moving spheres, based on the invariance of the equation with respect to an inversion. This second method permits to treat the case of systems of elliptic PDEs, for instance, and works in some cases where the moving planes technique is difficult or impossible to use.

Stefano Marmi. *An introduction to small divisors problems.*
Abstract. In the lectures we will illustrate some of the results of Kolmogorov, Arnol'd, Moser, Herman, Yoccoz and others on stability of quasiperiodic motions for various kinds of dynamical systems. We will emphasize the interplay between analytical assumptions and the number-theoretical conditions needed for the proofs. We will also possibly address some of the most recent developments of the theory, including the problem of uniqueness of KAM curves and the linearization of perturbations of linear flows on higher genus surfaces.

Petru Mironescu. *Inversion of the divergence.*
Abstract. The equation (1) div $X = f$ plays a central role in continuum mechanics. We will discuss the regularity of X in terms of f and of the domain $\Omega \subset \mathbf{R}^n$ in which the equation is solved. One of the difficulties stems in the fact that the equation is underdetermined, and that in the quest of regularity one needs to select a good solution among all candidates.
To start with, we will briefly explain elliptic regularity theory, which roughly speaking implies that X gains a derivative with respect to f. For example, if $f \in L^p$, with $1 < p < \infty$, and if Ω is not too rough, than we may pick X with derivatives in L^p.
We next discuss what happens when Ω is rough.
When $f \in L^1$ (resp. $f \in L^\infty$), we will see that there may be no solution X of (1) with derivatives in L^1 (resp. L^∞).
Another interesting situation occurs when $f \in L^n$ (with n the space dimension). In that case, one may pick X not only with derivatives in L^n (as explained above), but also continuous.
The first part of the min-course is devoted to the proofs of the above results. The arguments make use of standard functional analysis, and on elementary harmonic analysis results that

will be recalled during the course. The last result requires more involved analysis, and the course will be an opportunity to present a glimpse of the classical harmonic analysis.
Finally, we will explain how these results extend to more general setting, the one of Hodge systems, and how this is related to Hardy inequalities and some geometric inequalities.

Fulvio Ricci. *Different kinds of Riesz transforms on the Heisenberg group and their L^p-boundedness.*
Abstract. On the $(2n+1)$-dimensional Heisenberg group H_n we consider two left-invariant metrics and two associated "laplacians". One of them is the Carnot-Carathéodory metric with its sub-Laplacian L, and the other is a Riemannian metric, for which the Laplace-Beltrami operator is the "full" Laplacian $\Delta = L + \partial_t^2$, where t denotes the vertical coordinate. In each metric we have a family of Riesz transforms, $XL^{-1/2}$ and $X\Delta^{-1/2}$ respectively, where X is a left-invariant vector field. The nature of the Riesz transforms is well understood, as they are Calderón-Zygmund operators satisfying the standard estimates in the appropriate metric.
There are several reasons to study the operators which arise from compositions of Riesz transforms (or more general C-Z operators) of the two kinds and their L^p-boundendess properties. One of them is that both metrics intervene in the analysis of the (Riemannian) Hodge Laplacian Δ_k acting on differential forms of order k and, in particular of L^p-boundedness of the Hodge-Riesz transforms $d\Delta_k^{-1/2}$.
We will describe the two metrics with the corresponding Riesz transforms, motivate their rôle in the analysis of the Hodge-Riesz transforms, and finally describe the class of operators generated by the interaction of the two metrics.

Denis Serre. *L^1-contracting semi-groups and the stability of shock profiles.*
Abstract. The course deals with a non-linear variant of Markov semi-groups; they conserve the mass and satisfy a maximum principle. We focus on the example of a diffusion-convection equation

$$\partial_t u + \partial_x f(u) = \partial_x^2 u,$$

where f is a given non-linear function.
This equation admits fronts of the form $u(x,t) = U(x - ct)$ (c the front velocity), where U has distinct limits at $\pm\infty$. We consider the asymptotics stability of the solution associated with an initial data $U(x) + \phi(x)$ where $\phi \in L^1 \cap L^\infty(\mathbf{R})$. The stability is understood in terms of an L^1-distance, and in the sense of 'orbital stability'. This question involves tools from dynamical systems and from functional analysis.
We shall present a list other examples (unless we get short of time) where the same tools can be employed.