Transfer Operator in Ergodic Theory associated to Farey map

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Abstract

We interest for the reasons which we study the transfer operator in ergodic theory of the Farey map. In this respect, we study the role of the transfer operator in the ergodic theory of expanding maps of the interval and what happens to the essential spectrum of the operator as the map is less and less expanding. At this point, we can introduce the Farey map, and its connections to number theory through the Farey tree and the expansion in continued fractions of real numbers and we study the behaviour of the spectrum with the matrix formulation of the operator.

Introduction

1.The role of the transfer operator in the ergodic theory of expanding maps of the interval

Definition
The operateur $\mathcal{P}$ is the dual of the operateur of composition with F,

- $$\int_0^1 g \circ F(x) f(x) dx = \int_0^1 g(x) Pf(x) dx$$

- $$\mathcal{P}f(x) = \sum_{y, F(y)=x} \frac{f(y)}{|F'(y)|}$$

Properties

⋆ $m = f\nu$ is invariant if and only if $\mathcal{P}f = f$,

⋆ $m = f\nu$ is invariant, ergodic if and only if 1 is simple as eigenvalue of $\mathcal{P}$,

⋆ If $m = f\nu$ is invariant and mixing then 1 is an eigenvalue of $\mathcal{P}$((reciprocal is true under certain hypothesis supplementary).

⋆ The mixing coefficients are related to the asymptotic behavior of the transfer operator by:

$$C_n(\varphi, \psi) = |\varphi(\mathcal{P}^n(\psi f) - \psi f) d\nu|$$

2. Essentiel spectrum (What happens to the essential spectrum of the operator as the map is less and less expanding)

Definition:Essential Spectral radius
The essentiel spectral radius $r_{ess}(P)$ is the smallest number $r_{ess} \geq 0$ such that any $\lambda \in sp(\mathcal{P})$ with modulus $|\lambda| > r_{ess}$ is an isolated eigenvalue of finite multiplicity.

Theorem [2.] The essentiel spectral of the opérator $\mathcal{P}$ of C^k is the disk of radius $r_{ess}(\mathcal{P}) = \exp f(-k)$

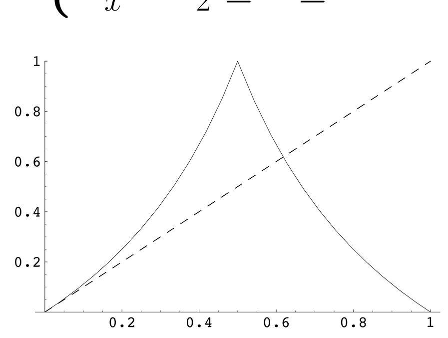
$$e^{-k\,(log\,\rho + \gamma_r)} \leq r_{ess}(\mathcal{P}) \leq e^{-k\,log\,\rho}, \qquad r \in [0,1)$$

Corollary
For $\rho = 1$ and for each fixed $k \geq 0$ the essential spectrum of $\mathcal{P} : \mathcal{C}^k \to \mathcal{C}^k$ is the unit disk.

3.The Farey map, and its connections to number theory through the Farey tree and the expansion in continued fractions of real numbers

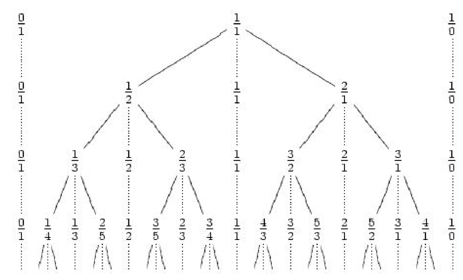
Definition
Let $F : [0,1] \to [0,1]$ be the Farey map defined by

$$F(x) = \begin{cases} \frac{x}{1-x} & \text{if } 0 \leq x \leq \frac{1}{2} \\ \frac{1-x}{x} & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}$$



The Farey map

Its name comes from the relations with the Farey fractions are studied in[1]



First four levels of the Stern-Brocot tree.

The ergodic properties

$$\left.\begin{array}{l} The\ infinite\ measure \\ \rho = 1\ in\ the\ previous\ Corollary \end{array}\right\} \Rightarrow\ r_{ess} = 1$$

On the leading eigenvalue of transfer operators of the Farey map with real temperature

Transfer Operator

We can associate a family of *signed generalized transfer operators* $\mathcal{P}_q^{\pm}$, for a complex parameter q with $\Re(q) > 0$, whose action on a function $f(x) : [0,1] \to \mathbb{C}$ is given by a weighted sum over the values of f on the set $F^{-1}(x)$, namely letting

$$(\mathcal{P}_{0,q}f)(x) := \left(\frac{1}{x+1}\right)^{2q} f\left(\frac{x}{x+1}\right) \qquad (1)$$

$$(\mathcal{P}_{1,q}f)(x) := \left(\frac{1}{x+1}\right)^{2q} f\left(\frac{1}{x+1}\right) \qquad (2)$$

we define

$$(\mathcal{P}_q^{\pm} f)(x) := (\mathcal{P}_{0,q}f)(x) \pm (\mathcal{P}_{1,q}f)(x) \qquad (3)$$

The study of the spectral properties of the operators $\mathcal{P}_q^{\pm}$ has been considered in [1] for real values of q and in [3] for general complex q.

In [1] the spectral properties of $\mathcal{P}_q^{\pm}$ are studied on the Hilbert space $\mathcal{H}_q$ of complex functions which can be represented in terms of the integral transform

$$L^2(m_q) \ni \phi(t) \mapsto \mathbb{B}_q[\phi](x) := \frac{1}{x^{2q}} \int_0^\infty e^{-\frac{t}{x}}\, e^t\, \phi(t)\, m_q(dt) \qquad (4)$$

where m_q is the absolutely continuous measure on $(0, \infty)$ defined as

$$m_q(dt) = t^{2q-1}\, e^{-t}\, dt. \qquad (5)$$

Hence the operators $\mathcal{P}_q^{\pm}$ are studied on the space

$$\mathcal{H}_q := \{f(x) = \mathcal{B}_q[\phi](x) \;:\; \phi(t) \in L^2(m_q)\} \qquad (6)$$

endowed with the inner product

$$(f_1, f_2) := \int_0^\infty \varphi_1(t)\, \overline{\varphi_2(t)}\, m_q(dt) \quad \text{if } f_i = \mathcal{B}_q[\varphi_i] \qquad (7)$$

which makes $\mathcal{H}_q$ a Hilbert space. A first result from [1] states

Theorem[1]
For $q \in (0, \infty)$ the space $\mathcal{H}_q$ is invariant for $\mathcal{H}_q^{\pm}$ and $\mathcal{H}_q^{\pm} : \mathcal{H}_q \to \mathcal{H}_q$ are isomorphic to self-adjoint compact perturbations of the multiplication operator $M : \mathrm{L}^2(m_q) \to \mathrm{L}^2(m_q)$ given by

$$(M\varphi)(t) = e^{-t}\, \varphi(t)$$

More specifically

$$\mathcal{P}_q^{\pm}\, \mathcal{B}_q\,[\varphi] = \mathcal{B}_q\,[P_q^{\pm} \varphi]$$

where $P_q^{\pm} = M \pm N_q$ and $N_q : \mathrm{L}^2(m_q) \to \mathrm{L}^2(m_q)$ is the symmetric integral operator given by

$$(N_q\varphi)(t) = \int_0^\infty \frac{J_{2q-1}\left(2\sqrt{st}\right)}{(st)^{q-1/2}}\, \varphi(s)\, m_q(ds)$$

where J_p denotes the Bessel function of order p.

The study of the spectral properties of $\mathcal{P}_q^{\pm}$ on the space $\mathcal{H}_q$, is then reduced to study of the spectral properties of the operators $P_q^{\pm} = M \pm N_q$ defined in the previous Theorem on the space $L^2(m_q)$. To this aim we recall some properties of the Hilbert space $L^2(m_q)$. First of all the measure $m_q(dt)$ is finite, indeed

$$\int_0^\infty m_q(dt) = \Gamma(2q)$$

Second, the (generalised) Laguerre polynomials $L_n^{2q-1}(t)$, $n \geq 0$, defined by

$$e_n(t) := L_n^{2q-1}(t) = \sum_{m=0}^{n} \binom{n+2q-1}{n-m} \frac{(-t)^m}{m!}$$

form a complete orthogonal system in $L^2(m_q)$, with

$$(e_n, e_m) = \frac{\Gamma(n+2q)}{n!}\, \delta_{n,m}$$

Consider the independent family of functions $f_n(t) := \frac{t^n}{n!}$ which satisfy

$$N_q f_n = M e_n \qquad N_q e_n = M f_n$$

and let $\ell_n^{\pm}(t) := e_n(t) \pm f_n(t), \quad \zeta_n^{\pm} := e^{-t}(e_n(t) \pm f_n(t))$

We have

Proposition
Let $H^{\pm} = Span\{\ell_n^{\pm}\}_{n\geq 0}$ and $\mathcal{E}^{\pm} = Span\{\zeta_n^{\pm}\}_{n\geq 0}$

i) $H^+ \cap H^- = \{0\}$ and $\mathcal{E}^+ \cap \mathcal{E}^- = \{0\}$;

i) $ker P_q^{\pm} = H^{\mp}$;

iii) $L^2(m_q) = H^+ \oplus \mathcal{E}^- = H^- \oplus \mathcal{E}^+$ with $H^{\pm} = (\mathcal{E}^{\mp})^{\perp}$;

iv $P_q^{\pm}$ are positive operators;

v) If $P_q^{\pm}\phi = \lambda\phi$ for some λ then $\phi \in \mathcal{E}^{\pm}$;

vi) The spectrum of $P_q^{\pm}$ in $L^2(m_q)$ is real and consists of the absolutely continuous spectrum $\sigma_{ac}(P_q^{\pm}) = [0,1]$ and of the point spectrum $\sigma_p(P_q^{\pm})$.

Corollary 2.14 in [1] has then to be changed into

Proposition
In $L^2(m_q)$ the point spectrum of $P_q^{\pm}$ contains $\lambda = 0$ with infinite multiplicity and real eigenvalues contained in the set $[0, 1+\gamma^{2q}]$, where $\gamma = \frac{\sqrt{5}-1}{2}$. The eigenvalues not embedded in $[0,1]$ are isolated and have finite multiplicity.

The matrix approach

Using Theoritical and Numerical techniques, imply that $\sigma_p(P_q^+)$ consists of a single eigenvalue $\lambda_q \in (1,2]$ for $q \in [0,1)$, and is empty for $q \geq 1$. Moreover general results from [6] imply that λ_q approaches 1 like $(1-q)\log\frac{1}{1-q}$ as q approaches 1 from below.

To study the behaviour of the eigenvalues, we consider the matrix formulation of $P_q^{\pm}$ in terms of the orthogonal basis $\{e_k\}$ of $L^2(m_q)$ which satisfies (). Let $\phi \in L^2(m_q)$ be written as

$$\phi(t) = \sum_{n=0}^{\infty} \phi_n\, e_n(t) \qquad \text{with} \quad (\phi, e_n) = \phi_n\, (e_n, e_n) = \phi_n \frac{\Gamma(n+2q)}{n!}$$

then ϕ is an eigenfunction of $P_q^{\pm}$ with eigenvalue λ if and only if

$$(P_q^{\pm}\phi, e_k) = \lambda\,(\phi, e_k) = \lambda\,\phi_k \frac{\Gamma(k+2q)}{k!} \qquad \forall\, k \geq 0$$

Moreover it has been shown in [1] that

for all $n, k \geq 0$ we obtain that

$$P_q^{\pm}\phi = \lambda\phi \quad \Leftrightarrow \quad C^{\pm}\Phi = \lambda D\Phi \quad \Leftrightarrow \quad A^{\pm}\Phi = \lambda\Phi$$

where $C^{\pm}$ and D are symmetric infinite matrices given by

$$C^{\pm} = (c_{kn}^{\pm})_{n,k\geq 0}, \quad D = diag\Big(\frac{\Gamma(n+2q)}{n!}\Big)$$

$A^{\pm}$ is the infinite non-symmetric matrix

$$A^{\pm} = (\alpha_{kn}^{\pm}) = \frac{k!\Gamma(n+2q)}{2^{n+k+2q}} \sum_{m=0}^{\min\{n,k\}} \frac{1 \pm (-1)^m}{\Gamma(m+2q)m!(n-m)!(k-m)!}$$

and Φ is the infinite vector.

$$\Phi = (\phi_n)_{n\geq 0} \quad such\ that \ \sum_{n=0}^{\infty} |\phi_n|^2 \frac{\Gamma(n+2q)}{n!} < \infty$$

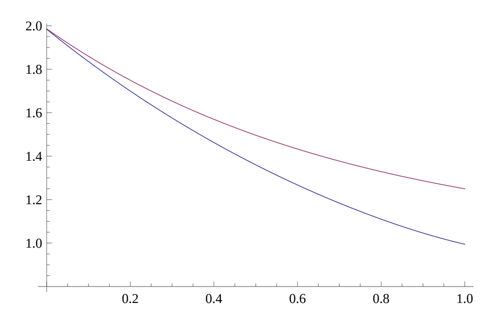
Proposition 1:[3] Let $A_N^{\pm}$ be the $N \times N$ north-west corner approximation of the matrix $A^{\pm}$ and let λ_N and ϕ^N be the dominant eigenvalue and eigenvector with $\phi_0^N = 1$. Then the sequence $\{\lambda_N\}_N$ is increasing and the sequences $\{\phi_k^N\}_N$ with fixed k are non-decreasing. Then $\lambda := \lim_{N\to\infty} \lambda_N$ is an eigenvalue of $A^{\pm}$ and $\Phi = (\phi_k)_{k\geq 0}$, defined as $\phi_k := \lim_{N\to\infty} \phi_k^N$, is an eigenvector such that $A^{\pm}\Phi = \lambda\Phi$.

A first consequence is the following proposition.

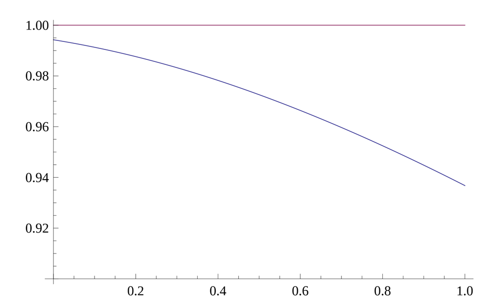
Proposition 2:

Let $\lambda_q^{\pm}$ be the eigenvalues of $A^{\pm}$ as obtained in Proposition 1 as functions of the real parameter q, and $\Phi^{\pm}$ the correspondent positive eigenvectors. Then

- $\phi_0^+ = 1$, $\phi_1^+ = \frac{1}{2}$;
- $\phi_0^- = 0$, $\phi_1^- = 1$;
- if $\phi_k^+ \leq \frac{1}{2}$ for all $k \geq 1$ then $\lambda_q^+ \leq 1 + 2^{-2q}$;
- if $\phi_k^- \leq 1$ for all $k \geq 1$ then $\lambda_q^- \leq 1$.

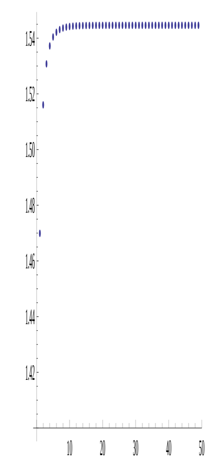


The blue line is the curve $\lambda_{50}^{+}(q)$ and the red one is the function $f^{+}(q) = 1 + 2^{-2q}$

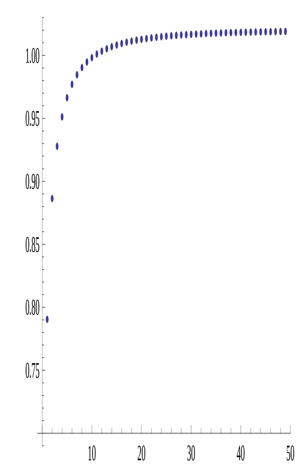


The blue line is the curve $\lambda_{50}^{-}(q)$ and the red one is the function $f^{-}(q) = 1$.

we fix $N = 50$ in Proposition 1 we obtain curves $\lambda_{50}^{\pm}(q)$ which are a lower bound for $\lambda_q^{\pm}$.



The sequences $\{\lambda_N^+(q)\}$ for $q = 1/3$ with $N = 1, \ldots, 50$.



The sequences $\{\lambda_N^+(q)\}$ for $q = 0.95$ with $N = 1, \ldots, 50$.

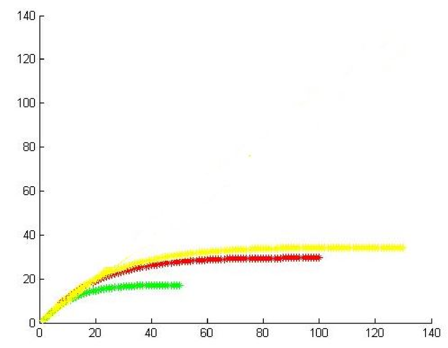
The convergence of $\lambda_N^{\pm}(q)$ to $\lambda_q^{\pm}$ is quite fast, as shown for λ_N^+ in the last Figure for the cases $q = \frac{1}{3}$ and $q = 0.95$ with $N = 1, \ldots, 50$. Moreover the sequences $\{\lambda_N^{\pm}(q)\}$ are increasing, hence we can determine $\lambda_q^{\pm}$ with arbitrary accuracy by increasing N.

We plot the functions $k \mapsto S(k)$ with $k \in [0, N]$ for different values of N,

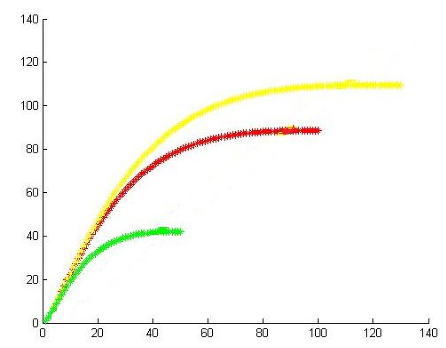
$$S_N(k) := \sum_{n=0}^{k} |\phi_n^N|^2 \frac{\Gamma(n+2q)}{n!}, \quad k = 0, \cdots, N-1$$

in particular $N = 50, 100, 130$ (notice that in the captions of the figures, it is N which is 50, 100, 130).

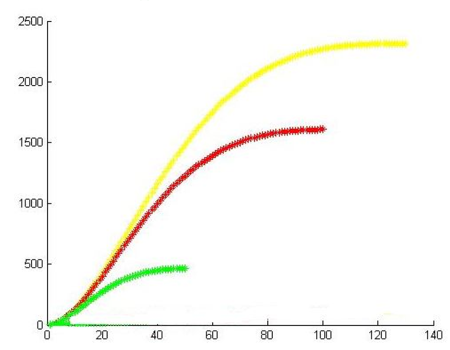
we should explain why we change N. The reason is that by Proposition 2, the eigenvector depends on N and it is non-decreasing.



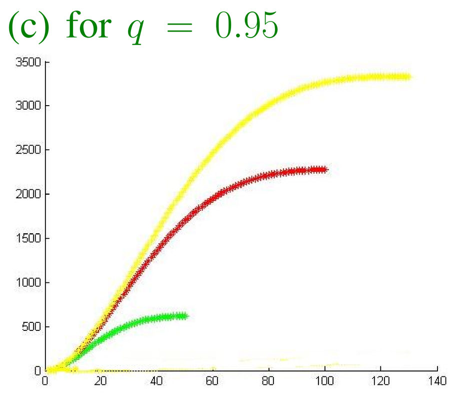
(a) for $q = 0.3$



(b) for $q = 0.5$



(c) for $q = 0.95$



(d) for $q = 1$

3: The sequences (a) **The curve of** $S(k = 50, 100, 130)$ **with** $q = 0.3$, (b) **the curve of** $S(k = 50, 100, 130)$ **with** $q = 0.5$, (c) **the curve of** $S(k = 50, 100, 130)$ **with** $q = 0.95$ **and (d) the curve of** $S(k = 50, 100, 130)$ **with** $q = 1$.

Using different N we want to measure the changes in the components of the eigenvector.

What we find is that for q small, the changes are small (for $q = 0.3$ the curves $S(k)$ are close to each other), and increasing q the changes are bigger. This is reasonable, since we know that for $q = 1$ there is no convergence of the sum $S(k)$.

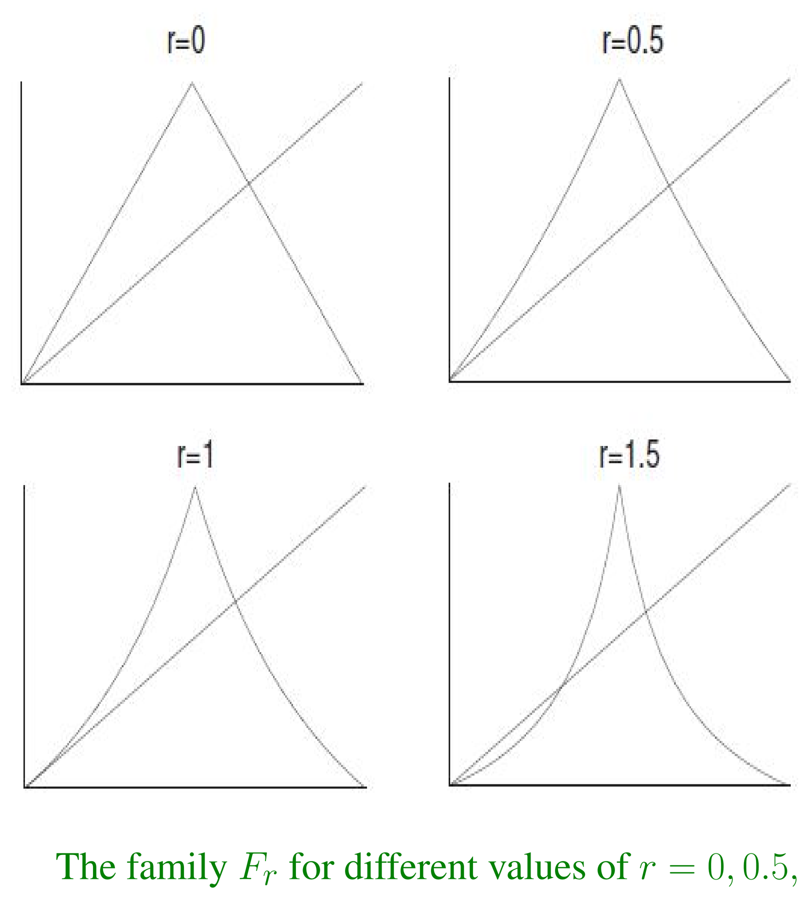
Therefor, the curves show that for $q = 0.3$, $q = 0.5$ and $q = 0.95$, i.e for $q < 1$ there is a convergence to a real eigenvector of the infinite matrix but for $q > 1$ there is no convergence to a real eigenvector of the infinite matrix.

On the spectrum of the transfer operators of a one-parameter family with intermittency transition

Definition:The family F_r

Let a one parameter family $F_r : [0,1] \to [0,1]$ with expanding maps with $r \in [0,1]$ qui interpolating between T and F, namely $F_0 = T$ et $F_1 = F$, the family F_r is defined by

$$F_r(x) = \begin{cases} \frac{(2-r)x}{1-rx} & \text{si } 0 \leq x \leq \frac{1}{2} \\ \\ \frac{(2-r)(1-x)}{1-r+rx} & \text{si } \frac{1}{2} \leq x \leq 1 \end{cases}$$



The family F_r for different values of $r = 0, 0.5, 1, 1.5$

Transfer operator

The tranfer operator $\mathcal{P}_r$ associated to the map F_r acts on functions $f : [0,1] \to \mathbb{C}$ as

$$\mathcal{P}_r(f)(x) := (\mathcal{P}_{r,0} + \mathcal{P}_{r,1})f(x)$$

where $\rho \equiv 2 - r$

$$\mathcal{P}_{r,0}f(x) := \frac{\rho}{(\rho + rx)^2} f(\frac{x}{\rho + rx})$$

and

$$\mathcal{P}_{r,1}f(x) := \frac{\rho}{(\rho + rx)^2} f(1 - \frac{x}{\rho + rx})$$

Definition: trace-class

A bounded linear operator A over a separable Hilbert space H is said to be in the trace class if for some (and hence all) orthonormal bases $\{e_k\}_k$ of H the sum of positive terms

$$\|A\|_1 = \mathrm{Tr}|A| := \sum_k \langle (A^*A)^{1/2} e_k, e_k \rangle,$$

where $(A * A)^{\frac{1}{2}}$ means the square root of the positive operator $A * A$. In this case, the sum

$$\mathrm{Tr}A := \sum_k \langle Ae_k, e_k \rangle$$

is absolutely convergent and is independent of the choice of the orthonormal basis. This value is called the trace of A. When H is finite-dimensional, every operator is trace class and this definition of trace of A coincides with the definition of the trace of a matrix.

By extension, if A is a non-negative self-adjoint operator, we can also define the trace of A as an extended real number by the possibly divergent sum

$$\sum_k \langle Ae_k, e_k \rangle.$$

Theorem [4]

For all $r \in [0,1)$ the space $\mathcal{H}$ is invariant for $\mathcal{P}_r$ and

$$\mathcal{P}_r\, \mathcal{B}[\varphi] = \mathcal{B}[(M_r + N_r)\varphi]$$

for all $\varphi \in L^2(\mathbb{R}^+, m)$, where $M_r, N_r : L^2(m) \to L^2(m)$ are defined as

$$(M_r\varphi)(t) = \frac{1}{\rho} e^{-\frac{r}{\rho}t} \varphi\Big(\frac{t}{\rho}\Big) \qquad \text{and}$$

$$(N_r\varphi)(t) = \frac{1}{\rho} e^{\frac{1-r}{\rho}t} \int_0^{\infty} J_1\left(2\sqrt{st/\rho}\right) \sqrt{\frac{\rho}{st}}\, \varphi(s)\, dm(s)$$

where J_q denotes the Bessel function of order q.

i) the operators M_r and N_r on $\mathbb{H}$ are of trace-class, hence the same holds for $\mathcal{P}_r$;

ii) the spectra of M_r and N_r contain only simple eigenvalues, in particular

$$\mathrm{sp}(M_r) = \{0\} \cup \left\{\rho^{-k}\right\}_{k\geq 1} \quad \text{and}$$

$$\mathrm{sp}(N_r) = \{0\} \cup \left\{(-1)^{k-1} \left(\frac{4\rho}{(1+\sqrt{1+4\rho})^2}\right)^k\right\}_{k\geq 1};$$

iii) the trace of $\mathcal{P}_r$ can be explicitly computed and is given by

$$\mathrm{trace}\,(\mathcal{P}_r) = \frac{1}{1-r} + \frac{\sqrt{1+4\rho}-1}{2\sqrt{1+4\rho}}.$$

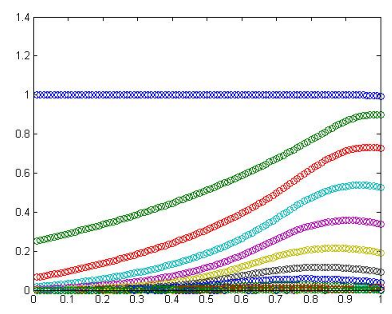
The matrix approach

Use the definitions of the operators M_r and N_r to compute **Proposition**

For all $r \in (0,1)$ we have

$$a_{kn}^r = C_{n+k+1}^n \frac{(2-r)r^k}{2^{n+k+2}}\, {}_2F_1\left(-k, -n; -k-n-1, \frac{2(r-1)}{r}\right) +$$
$$+ \sum_{l=0}^{n} (-1)^l\, C_{n+l}^{n-1}\, C_{l+k+1}^{l} \frac{(2-r)r^k}{2^{l+k+2}} *$$
$$*_2 F_1\left(-k, -l; -k-l-1, \frac{2(r-1)}{r}\right)$$

for all $k, n \geq 0$, where ${}_2F_1$ denotes the hypergeometric function.



The eigenvalues of $A_{r,N}$ for $N = 50$ as functions of $r \in (0,1)$.

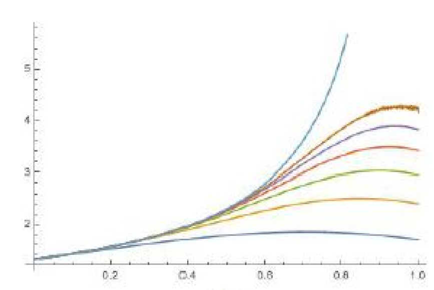
We can now numerically approximate solutions using the standard north-west corner approximation of the matrices A_r. For $N \geq 1$ we let $A_{r,N}$ denote the $N \times N$ matrix defined as

$$A_{r,N} = (a_{kn}^{r,N})k, n = 0, \cdots, N1 \ \ with \ \ a_{kn}^{r,N} = a_{kn}^r.$$

For each $r \in (0,1)$ we find N eigenvalues for $A_{r,N}$ which approximate the eigenvalues of $P_r = M_r + N_r$, whose spectrum consists only of the point spectrum as stated in the previous Theorem. In this Figure we have plotted the eigenvalues of $A_{r,N}$ for $N = 50$ as functions of $r \in (0,1)$. First of all we notice that for all r we find $\lambda = 1$ as leading eigenvalue. Second we conclude from the results that the approximation of the eigenvalues of P_r gets worse and worse as r approaches 1. This is evident in the leading eigenvalue, which is a curve very close to 1, but slightly decreasing as $r \to 1^-$, and is probably the reason for the other curves to be non-increasing as r approaches 1.

We can ask how good is the approximation given by the eigenvalues in the previous Figure.

To quantify the goodness of the approximation we can use the only analytical result about the spectrum of P_r, namely the computation of its trace given in the previous Theorem (iii).



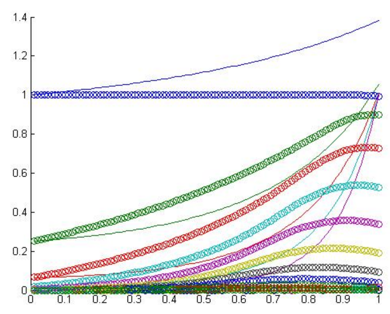
The trace of the operators P_r (upper most curve) and in increasing order, the trace of maytrix $A_{r,N}$ for $N = 10, 20, 30, 40, 50, 60$, as functions of $r \in (0,1)$.

We have plotted the function given in the previous Theorem (iii), which is the upper most curve diverging as $r \to 1^-$, and the traces of the matrices $A_{r,N}$ as functions of $r \in (0,1)$ for different values of N. In particular we have chosen $N = 10, 20, 30, 40, 50, 60$.

We see that as N increases we get a better and better approximation of the trace of P_r, and the two almost coincide for $r < 0.6$. However as r approaches 1, we see that the approximation of the trace becomes poor, and in particular for $N = 60$, the small oscillations in the curve show numerical instabilities in the computations. Hence we believe that the computation of the eigenvalues of Ar,N in Figure 1 is a very good approximation of the eigenvalues of P_r for $r < 0.6$. For example for $r = 0$, the eigenvalues of P_0 are the set $\{2^{2k}\}_{k\geq 0} \cup \{0\}$, and these values coincide for what we find in the previous Figure 1. It is instead unclear what happens as r approaches 1.

To try to understand this point we have plotted in the below Figure the eigenvalues of of $A_{r,N}$ together with the sums of the eigenvalues of M_r and Nr for $k = 1, 3, 5, 7, 9$ find in the previous Theorem (iii), that is the curves

$$\rho^k + (1)^{k1}\Big(\frac{4\rho}{1+\sqrt{1+4\rho}}\Big)^k, k = 1, 3, 5, 7, 9$$



The eigenvalues of $A_{r,N}$ for $N = 50$ compared with with the sums of the eigenvalues of M_r and N_r for $k = 1, 3, 5, 7, 9$, as functions of $r \in (0, 1)$.

Which coincide with the first five eigenvalues of P_r for $r = 0$. The behavior of the trace of $A_{r,N}$ with respect to that of P_r, and the last Figure suggest that all the eigenvalues of P_r converge to 1 as $r \to 1^-$, so that for $r = 1$ the spectrum of P_r would consist of the eigenvalues 0 and 1 and of the purely continuous spectrum $(0, 1)$.

References

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