Topological substitutions and Rauzy fractals

Arnaud Hilion

joint work with Nicolas Bédaride and Timo Jolivet

Pisa – April 2016

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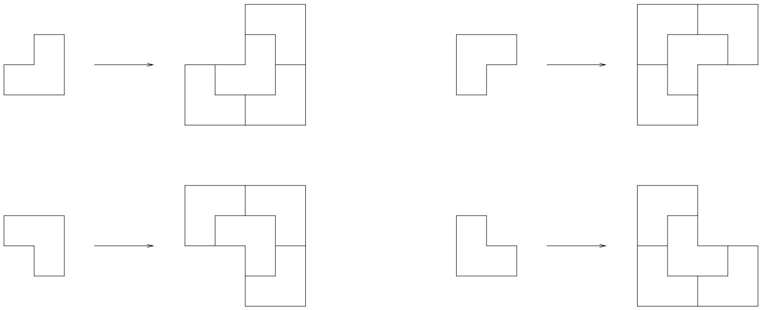
- Iterate the process

An example



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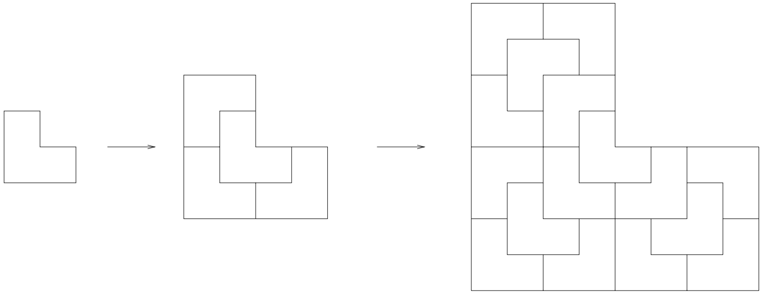

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- Less geometrical, more combinatorial

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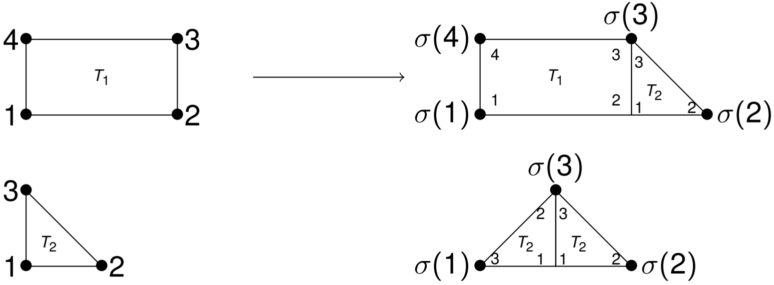

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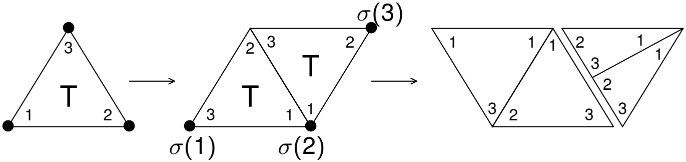

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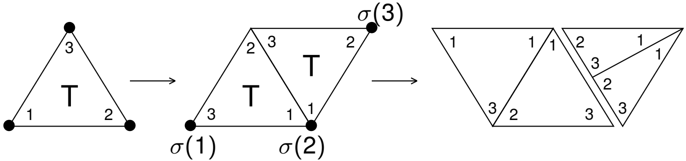

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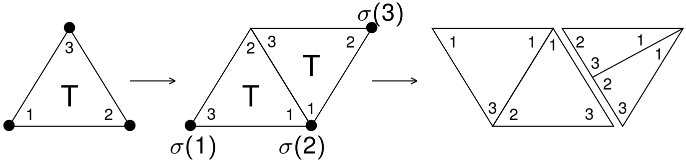

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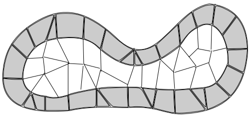

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Such a geometrical realization of $\sigma^\infty(T)$ is a *substitutive tiling*.

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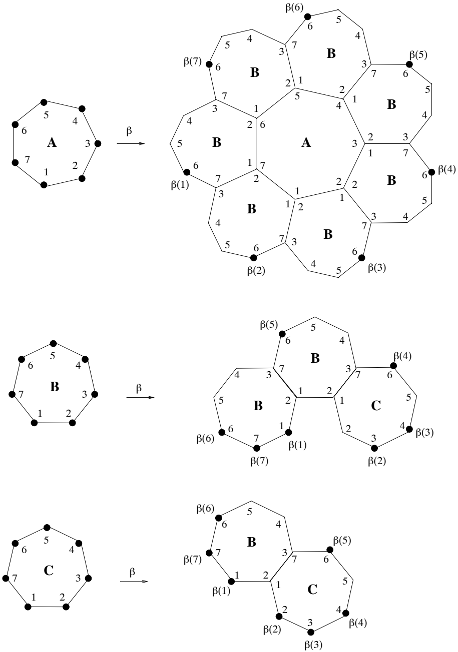
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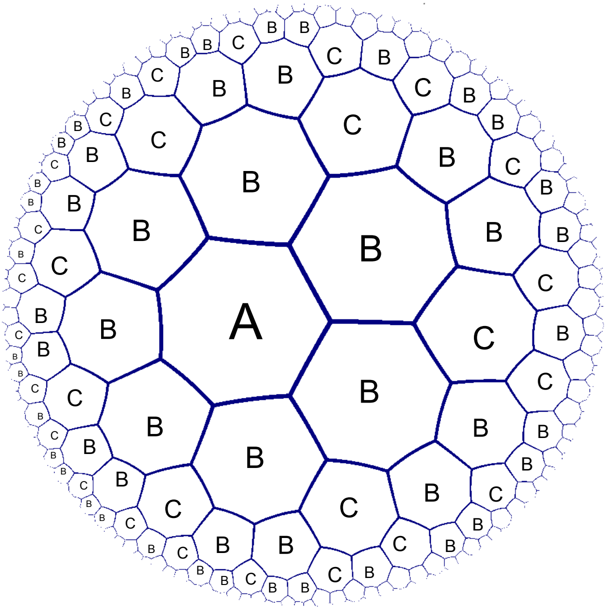
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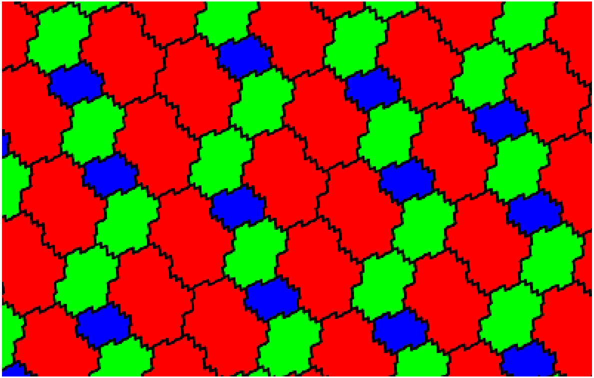

Tribonacci substitution

$$\sigma : \begin{cases} 1 \mapsto 12 \\ 2 \mapsto 13 \\ 3 \mapsto 1 \end{cases} \qquad \mathbf{M}_\sigma = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

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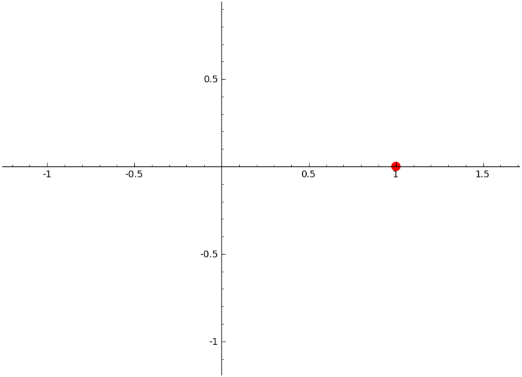
Action of $\mathbf{M}_\sigma$ on $\mathbb{R}^3$:

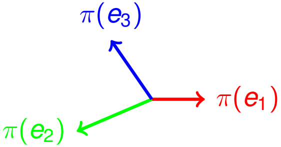
- **Expanding line $\mathbb{E}$**
- **Contracting plane $\mathbb{P}$**

Rauzy fractals

$\sigma^{\infty}(1) = 12131211213121213121 1\cdots$ $\quad \pi : \mathbb{R}^3 \to \mathbb{P}$,

$\pi(e_1)$ $\quad$ projection along $\mathbb{E}$





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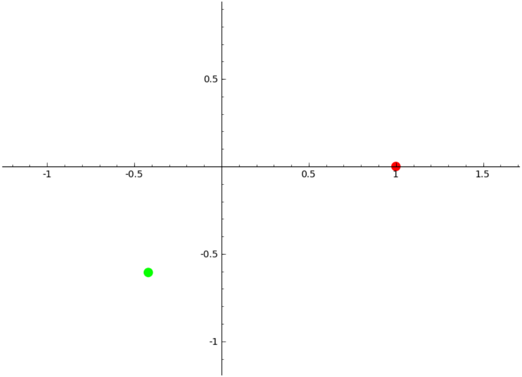

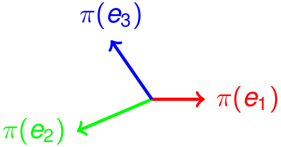

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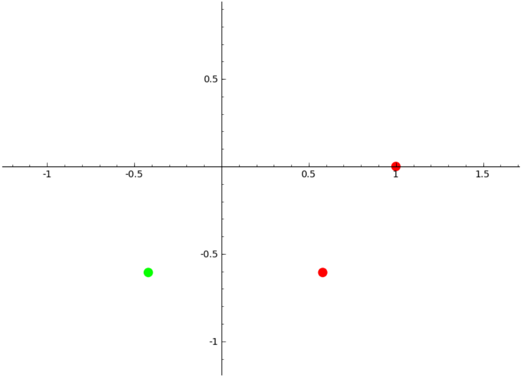

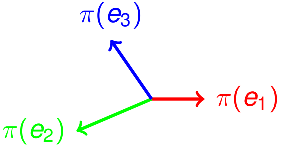

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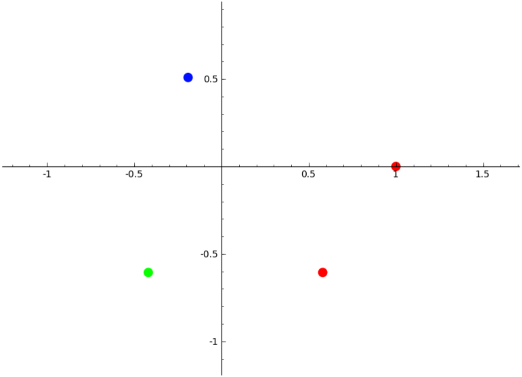

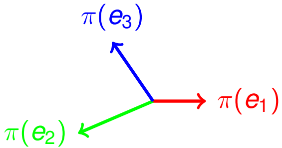

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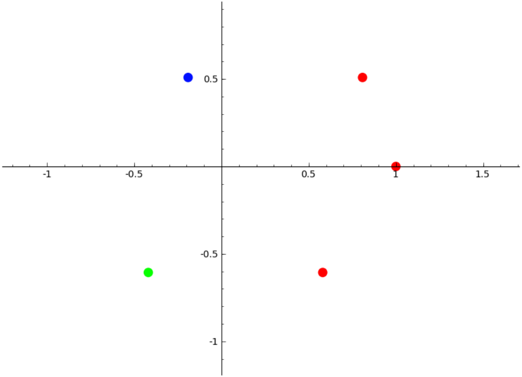

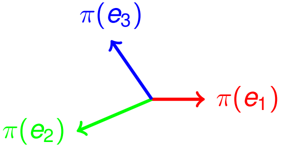

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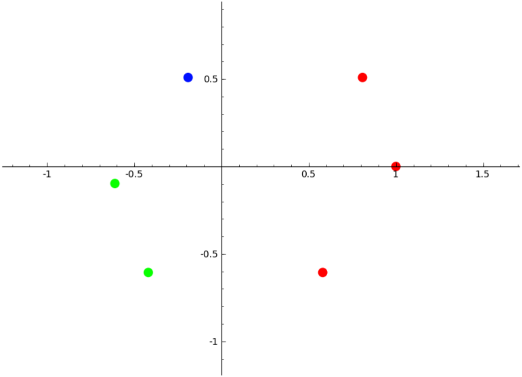

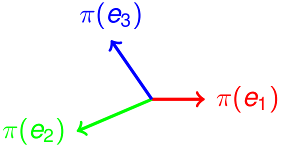

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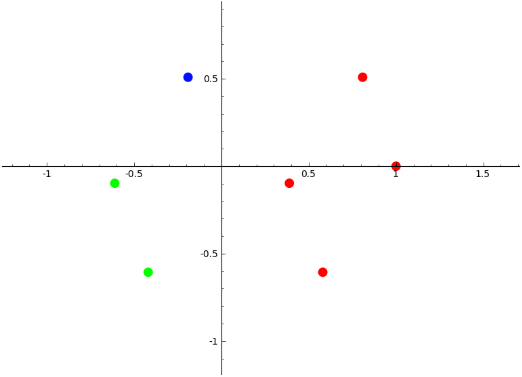

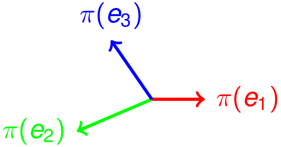

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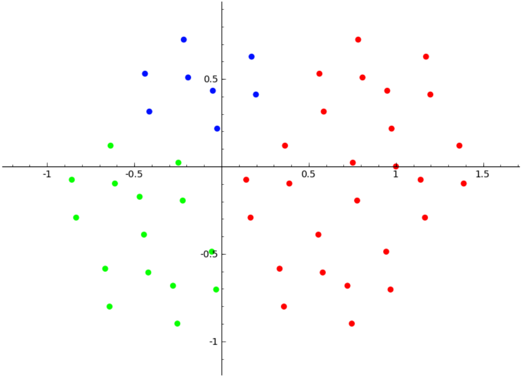

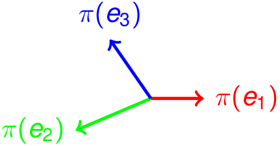

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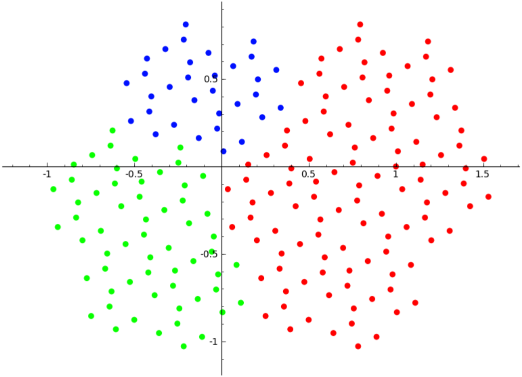

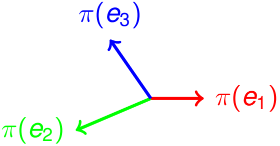

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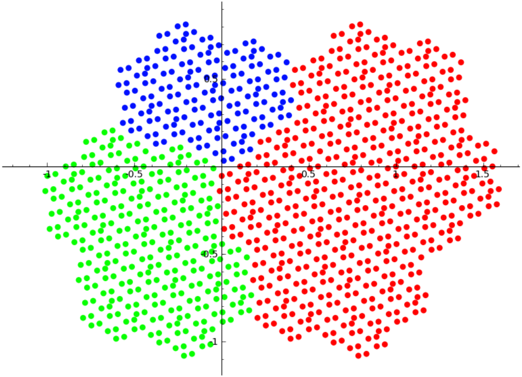

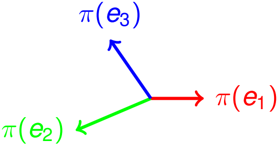

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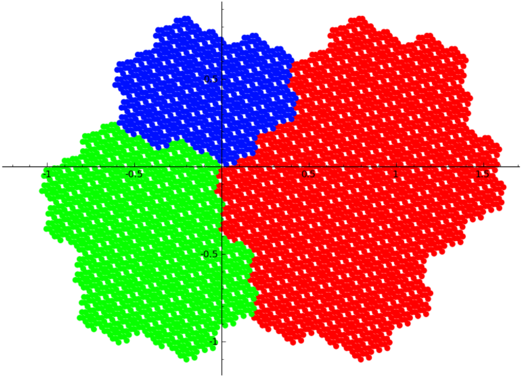

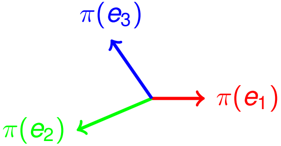

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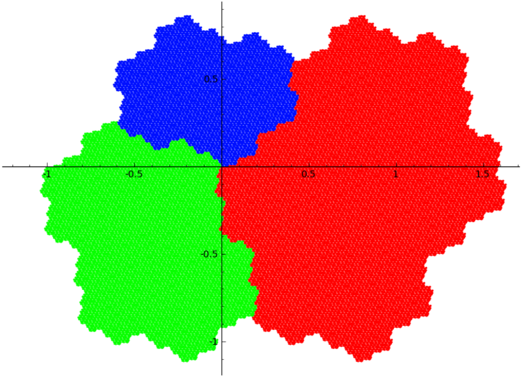
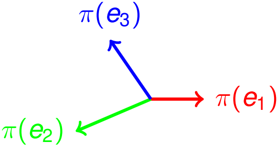
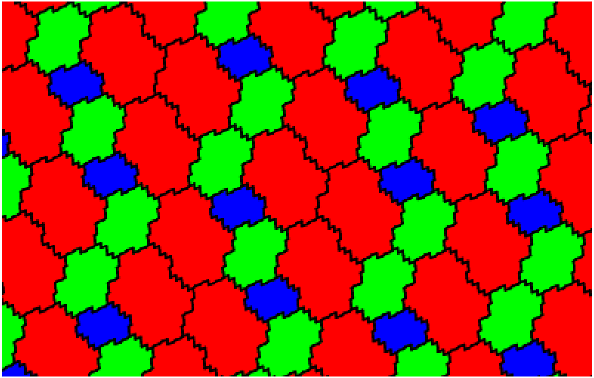
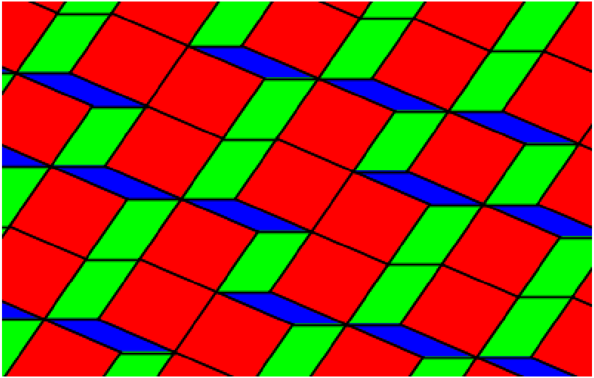




First tiling $\mathcal{T}_{\text{frac}}$: with fractal tiles



Second tiling $\mathcal{T}_{\text{step}}$: with rhombi



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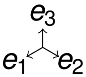

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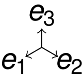

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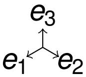

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$$\mathbf{E} : \begin{cases} [\mathbf{x}, 1]^* \mapsto \mathbf{M}_\sigma{}^{-1}\mathbf{x} + \big([\mathbf{0}, 1]^* \cup [\mathbf{0}, 2]^* \cup [\mathbf{0}, 3]^*\big) \\ [\mathbf{x}, 2]^* \mapsto \mathbf{M}_\sigma{}^{-1}\mathbf{x} + [\mathbf{e}_3, 1]^* \\ [\mathbf{x}, 3]^* \mapsto \mathbf{M}_\sigma{}^{-1}\mathbf{x} + [\mathbf{e}_3, 2]^* \end{cases}$$

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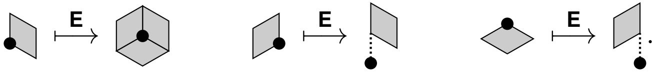

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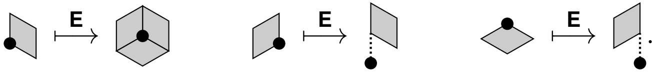
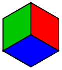


in the preimages : black dots stand for $\mathbf{x}$
in the images : black dots stand for $\mathbf{M}_\sigma{}^{-1}\mathbf{x}$

Dual substitution E



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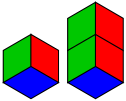

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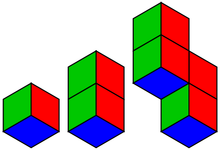
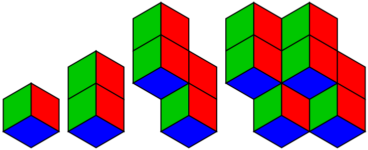
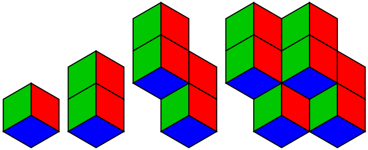
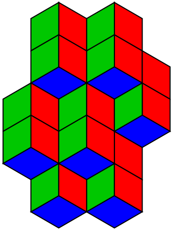


Dual substitution E



Dual substitution E





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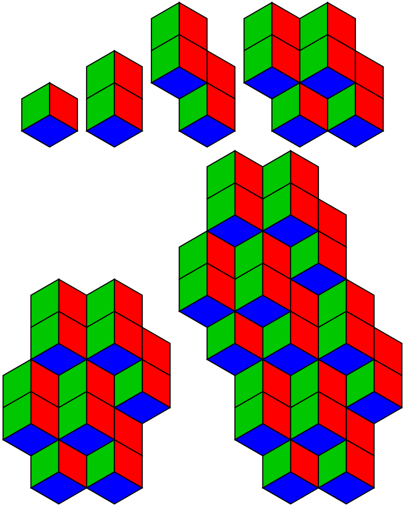
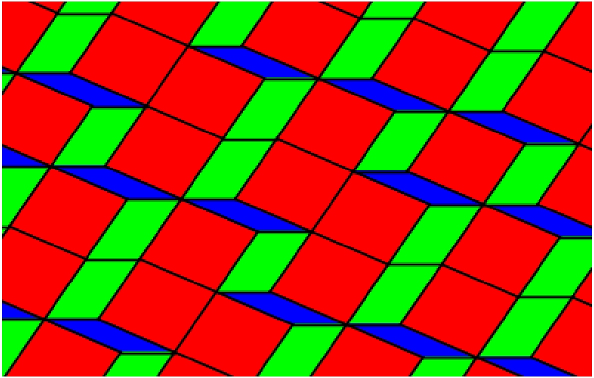
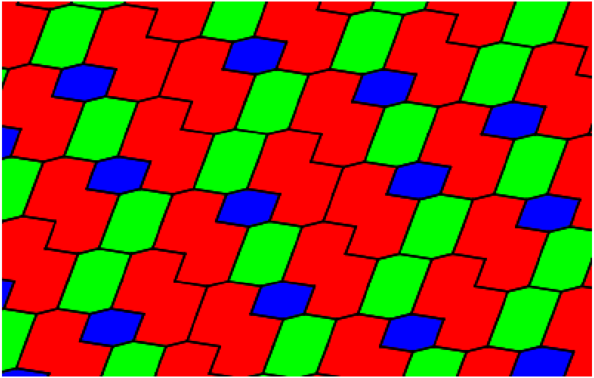


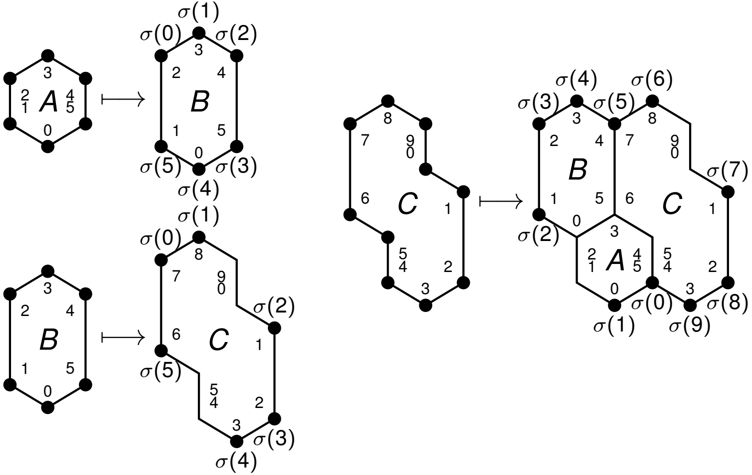
Second tiling $\mathcal{T}_{\text{step}}$: with rhombi



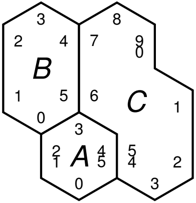
Third tiling $\mathcal{T}_{\text{top}}$



Tribonacci topological substitution



Tribonacci topological substitution



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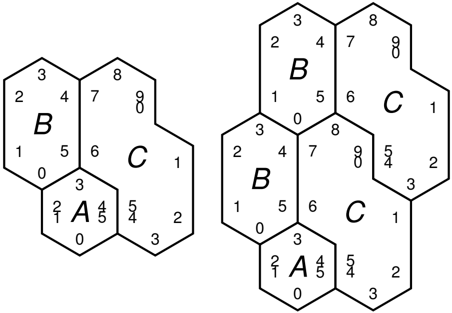
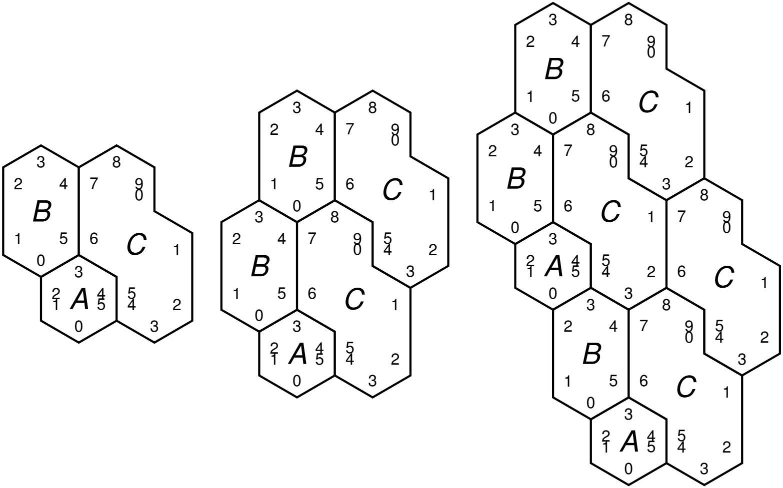
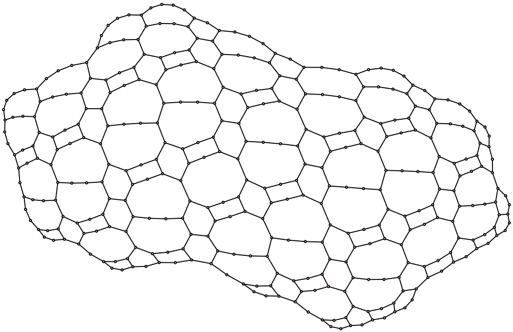


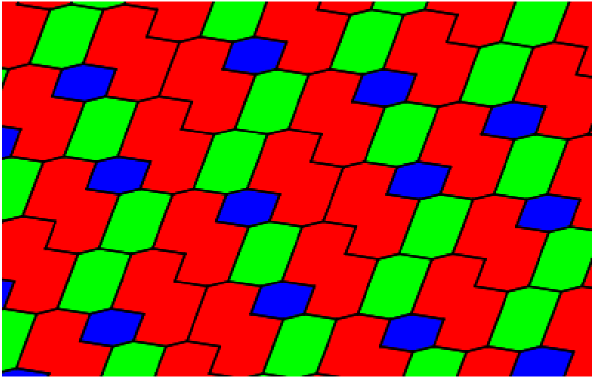
Tribonacci topological substitution



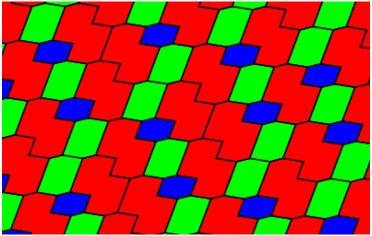
Tribonacci topological substitution : $\sigma^6(C)$

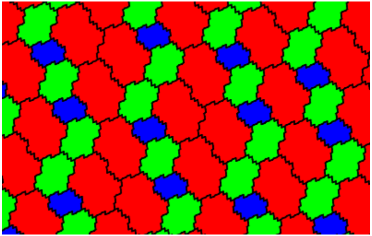


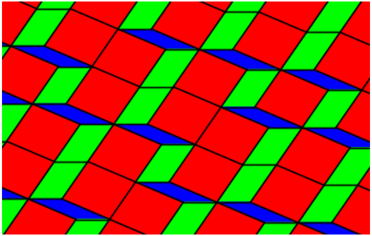
Third tiling $\mathcal{T}_{\text{top}}$: geometrization of $\sigma^\infty(C)$



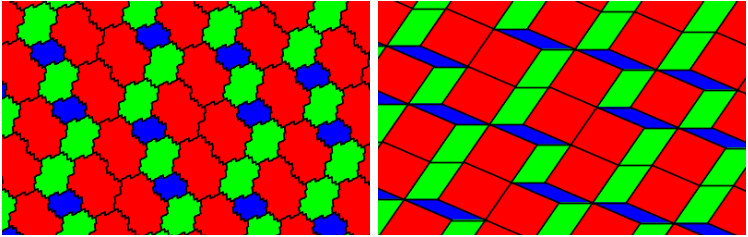
How the 3 tilings are related ?







Link between $\mathcal{T}_{\text{frac}}$ and $\mathcal{T}_{\text{step}}$



Link between $\mathcal{T}_{\text{frac}}$ and $\mathcal{T}_{\text{step}}$

Theorem (Ei-Ito, Arnoux-Ito, Arnoux-Berthé-Ito, Arnoux-Bethé-Siegel)

The set of tiles of $\mathcal{T}_{\text{step}}$ is :

$$\bigcup_{i \in \{1,2,3\}} \{\pi_\beta([\mathbf{0}, i]^*) + \pi_\beta(\mathbf{x}) : \mathbf{x} \in \mathbb{Z}^3, 0 \leq \langle \mathbf{x}, \mathbf{v}_\beta \rangle < \langle \mathbf{e}_i, \mathbf{v}_\beta \rangle\}.$$

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Theorem (Rauzy, Canterini-Siegel)

The set of tiles of $\mathcal{T}_{\text{frac}}$ is :

$$\bigcup_{i\in\{1,2,3\}} \{\mathcal{R}_i + \pi_\beta(\mathbf{x}) : \mathbf{x} \in \mathbb{Z}^3, 0 \leq \langle \mathbf{x}, \mathbf{v}_\beta \rangle < \langle \mathbf{e}_i, \mathbf{v}_\beta \rangle\}.$$

$\mathcal{T}_{\text{frac}}$ and $\mathcal{T}_{\text{step}}$ share the same addresses

$$\mathcal{V}_i = \{\mathbf{x} \in \mathbb{Z}^3 \ : \ 0 \leq \langle \mathbf{x}, \mathbf{v}_\beta \rangle < \langle \mathbf{e}_i, \mathbf{v}_\beta \rangle\}$$

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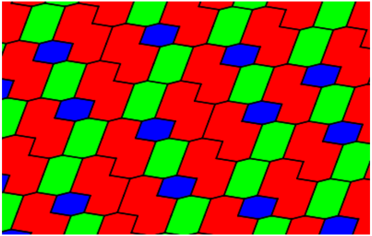

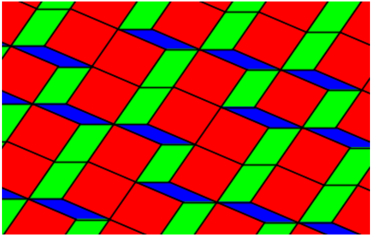

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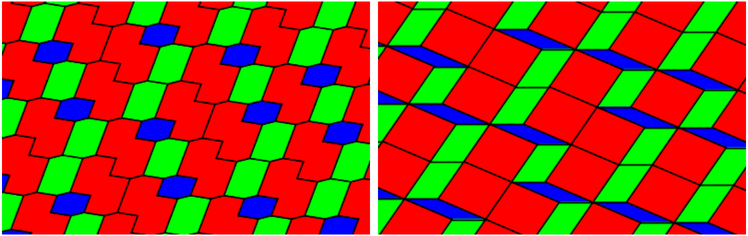

- not the same underlying complex

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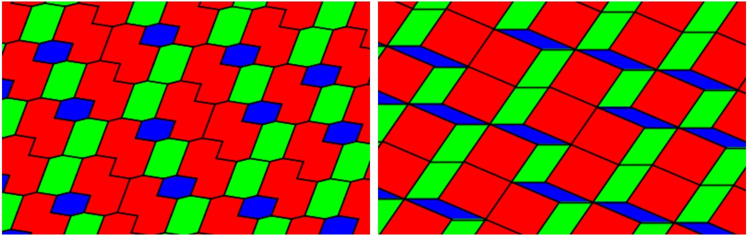

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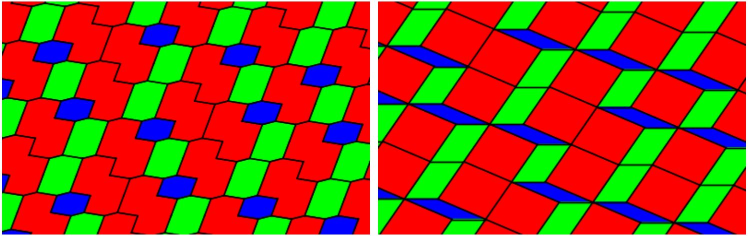

- not the same underlying complex
- no basic trick, such as taking some edges of length 0
- but we guess the "obvious" bijection between tiles

Addressing map

P a patch in $\sigma^\infty(C)$

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A *path of tiles* γ of length n in P is a sequence $T_0, \ldots, T_n$ of tiles of P such that, $\forall i$:

- $T_i \neq T_{i+1}$
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If moreover $T_0 = T_n$, γ is a *loop of tiles*.

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$\gamma = T_0, \ldots, T_n$ and $\gamma' = T'_0, \ldots, T'_m$ can be concatenated if $T'_0 = T_n$. The *concatenation* of these paths is the path of tiles :

$$\gamma\gamma' = T_0, \ldots, T_n, T'_1, \ldots, T'_m.$$

Addressing map

A map $\omega : \mathfrak{P} \to \mathbb{Z}^3$ is *additive* if for every $\gamma, \gamma' \in \mathfrak{P}$ that can be concatenated :

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There exists a finite subset $\mathfrak{P}_0 \subset \mathfrak{P}$ of loops such that if an additive map ω vanishes of $\mathfrak{P}_0$, then ω vanishes on any loop in $\mathfrak{P}$.

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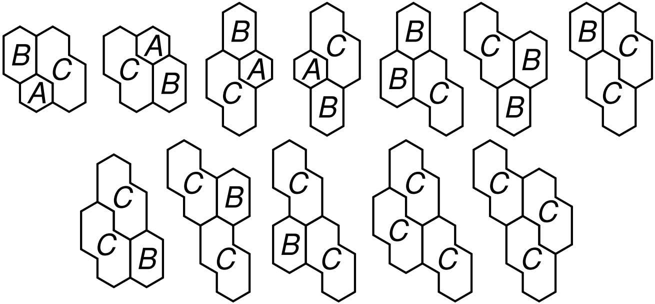
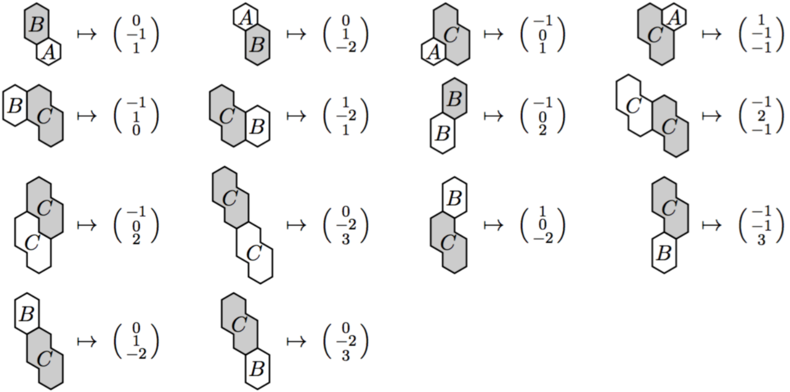


Addressing map ω_0



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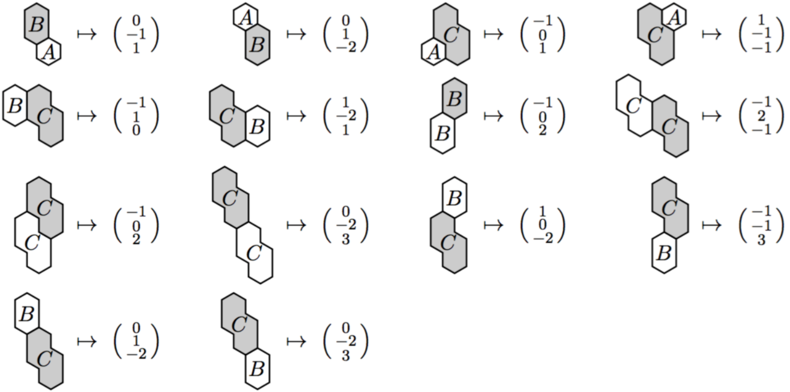
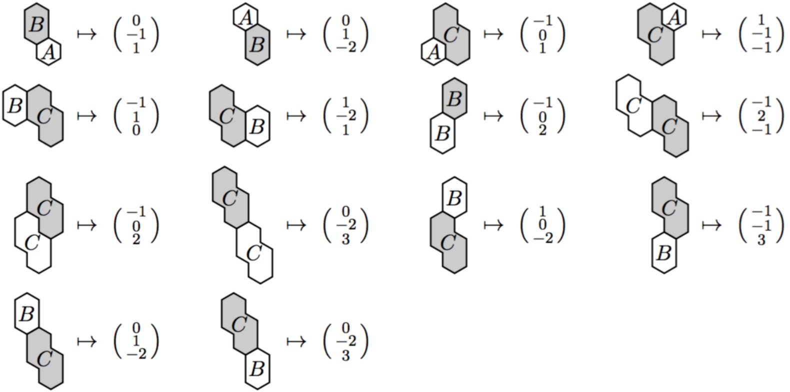


- Extend ω by additivity,

Addressing map ω_0



- Extend ω by additivity,
- Check that ω vanishes on loops.

Addressing map ω_0

T, T' tiles in a patch $P \subset \sigma^{\infty}(C)$

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γ a path of tiles from T to T'

Addressing map ω_0

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γ a path of tiles from T to T'

Then

$$\omega_0(T, T') = \omega_0(\gamma)$$

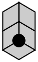
only depends on T and T'.

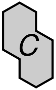
The maps Φ and φ_0

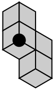
$\Phi\Big($ 

$\Big) =$  $= \mathbf{E}([\mathbf{0}, 1]^*) = \mathbf{E}^3([\mathbf{0}, 3]^*) + \mathbf{e}_3 - \mathbf{e}_1,$

$\Phi\Big($ 

$\Big) =$  $= \mathbf{E}^2([\mathbf{0}, 1]^*) = \mathbf{E}^3([\mathbf{0}, 2]^*) + \mathbf{e}_2 - \mathbf{e}_1,$

$\Phi\Big($ 

$\Big) =$  $= \mathbf{E}^3([\mathbf{0}, 1]^*).$

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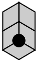
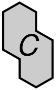
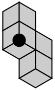
$\Bigg) =$  $= \mathbf{E}^2([\mathbf{0}, 1]^*) = \mathbf{E}^3([\mathbf{0}, 2]^*) + \mathbf{e}_2 - \mathbf{e}_1,$

$\Phi\Bigg($ 

$\Bigg) =$  $= \mathbf{E}^3([\mathbf{0}, 1]^*).$

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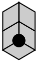
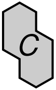
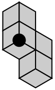
$\Big) =$  $= \mathbf{E}^2([\mathbf{0},1]^*) = \mathbf{E}^3([\mathbf{0},2]^*) + \mathbf{e}_2 - \mathbf{e}_1,$

$\Phi\Big($ 

$\Big) =$  $= \mathbf{E}^3([\mathbf{0},1]^*).$

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Theorem (Bédaride-H-Jolivet)

Let T be a tile in a patch $P \subset \sigma^\infty(C)$. Then :

$$\varphi_0 \circ \sigma(P,T) = \mathbf{E} \circ \varphi_0(P,T).$$

Link between $\mathcal{T}_{\text{top}}$ and *Tstep*

For all n :

$$\varphi_0 \circ \sigma^n(C) = \mathbf{E}^n \circ \varphi_0(C, C).$$

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$$\varphi_0 \circ \sigma^n(C) = \mathbf{E}^n \circ \varphi_0(C, C).$$

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- $\sigma^n(C)$ converges to $\sigma^\infty(C)$
- φ_0 induces a map, still denoted by φ_0 such that :
  - for every tile T of $\sigma^\infty(C)$, $\varphi_0(T, C)$ is a subset of $\mathcal{T}_{\text{step}}$,
  - $\mathcal{T}_{\text{step}} = \bigcup_{T \text{ tile of } \sigma^\infty(C)} \varphi_0(T, C).$

Lemma

For every tile T of $\sigma^\infty(C)$, there exists a unique tile $\pi([\mathbf{x}, i]^*)$ in $\mathcal{T}_{\text{step}}$ such that

$$\varphi_0(T, C) = \mathbf{E}^3([\mathbf{x}, i]^*).$$

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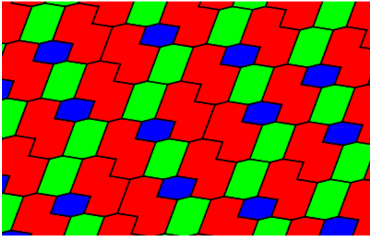
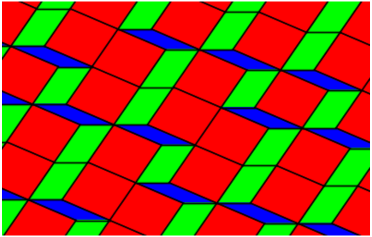
- In $\mathcal{T}_{\text{step}}$, the corresponding tiles $\Psi(T)$ and $\Psi(T')$ will differ from the vector $\pi_\beta(\omega_0(T, T'))$.

Link between $\mathcal{T}_{\text{top}}$ and $\mathcal{T}_{\text{step}}$





Ψ as a "two-dimensional sliding block code"