Square-tiled surfaces of fixed combinatorial type: equidistribution, counting, volumes of the ambient strata

Anton Zorich (joint work with V. Delecroix, E. Goujard, P. Zograf)

Renormalization in dynamics

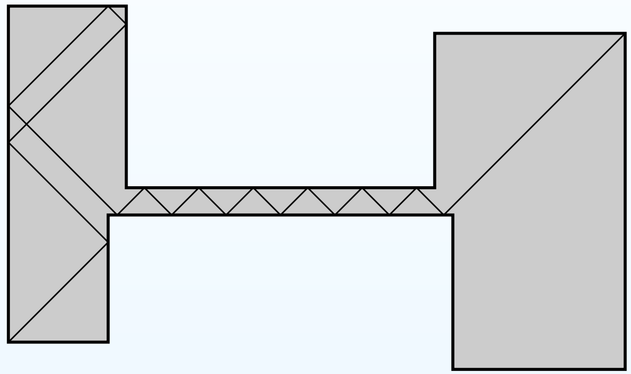
Pisa, April 7, 2016

Why one might care about volumes of moduli spaces: counting billiard trajectories and closed flat geodesics

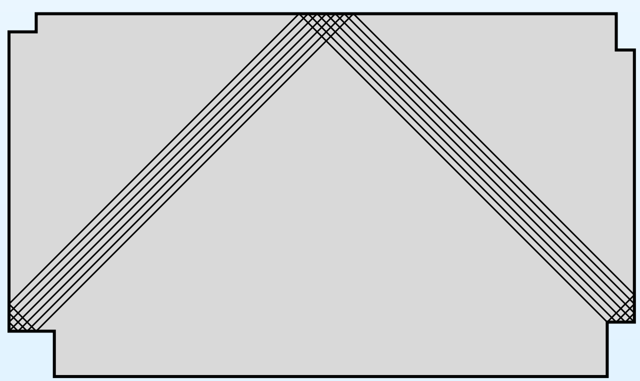
Following Moon Duchin
let us play on
right-angled billiard tables.

Closed trajectories and generalized diagonals

We count the asymptotic number of trajectories of length at most L joining a given pair of corners ("*generalized diagonals*") as the bound L tends to infinity.



We also want to count the number of periodic trajectories of length at most L, or rather the number of *bands* of periodic trajectories. We might also count the bands with the weight representing the "thickness" of the band.



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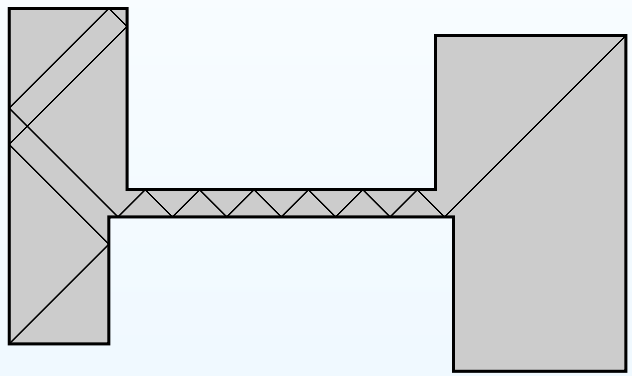

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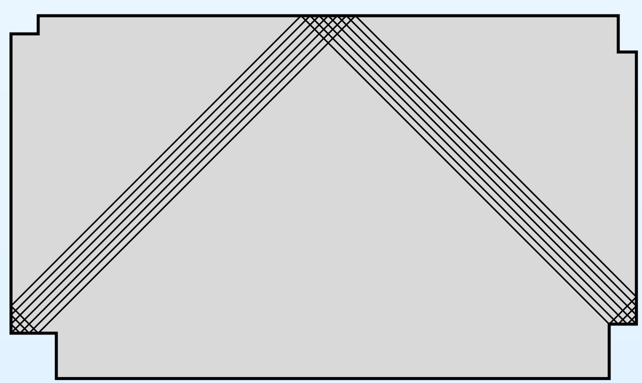
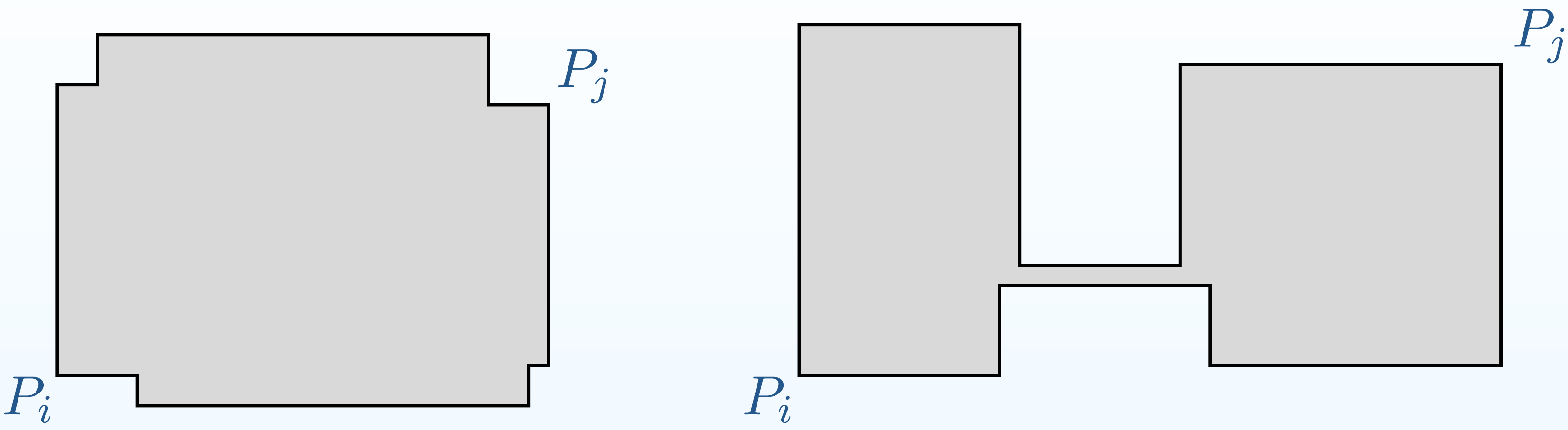


Number of generalized diagonals

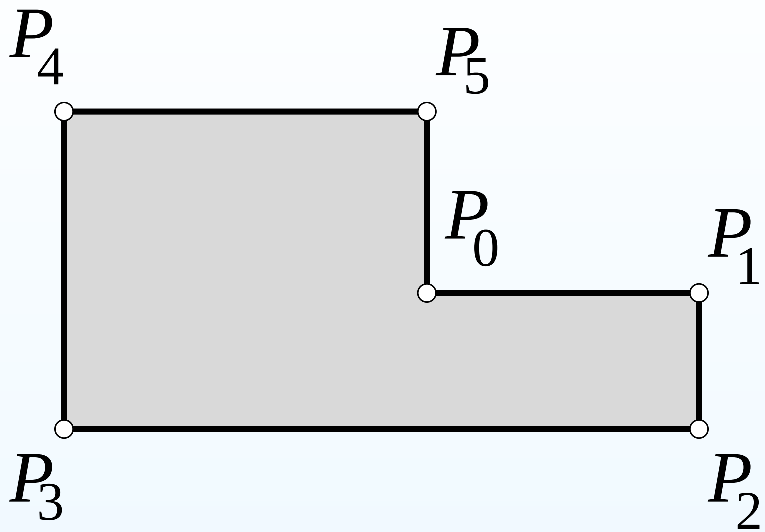


Theorem (A. Eskin, J. Athreya, A. Z., 2012). *For almost any right-angled polygon* Π *in any family* $\mathcal{B}(k_1, \dots, k_n)$ *of right-angled polygons with angles* $k_1 \frac{\pi}{2}, \dots k_n \frac{\pi}{2}$, *the number* $N_{i,j}(\Pi, L)$ *of trajectories of length bounded by* L *joining any two fixed corners with true right angles* $\frac{\pi}{2}$ *is asymptotically the same as for a rectangle:*

$$N_{i,j}(\Pi, L) \sim \frac{1}{2\pi} \cdot \frac{(\text{bound } L \text{ for the length})^2}{\text{area of the table}} \quad \text{as} \quad L \to \infty$$

and does not depend on the shape of the polygon Π.

Naive intuition does not help...



However, say, for almost any L-shaped polygon Π the number $N_{0,j}(\Pi, L)$ of trajectories joining the corner P_0 with the angle $3\frac{\pi}{2}$ to some other corner P_j has asymptotics

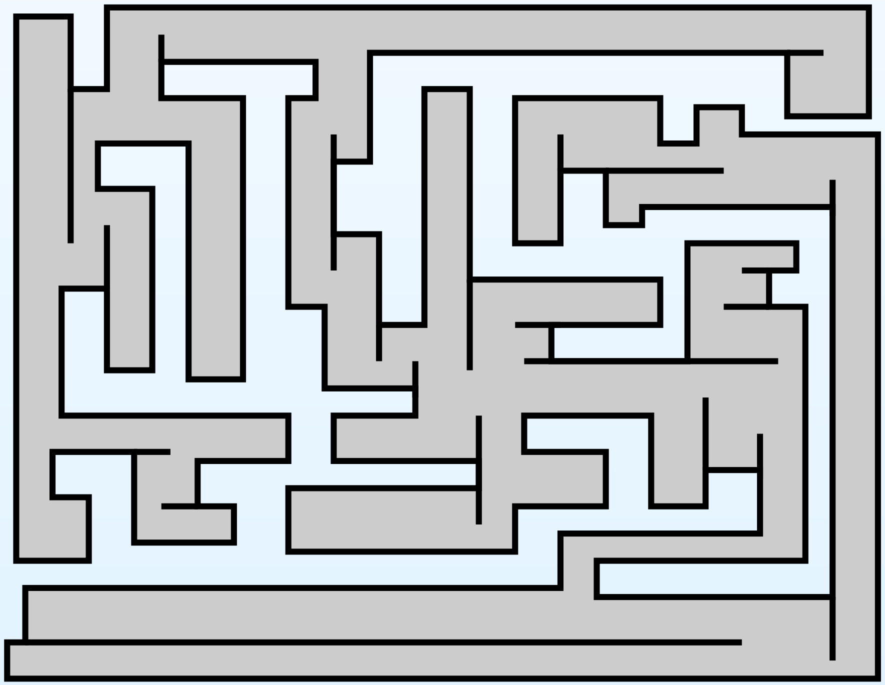
$$N_{0,j}(\Pi, L) \sim \frac{2}{\pi} \cdot \frac{(\text{bound } L \text{ for the length})^2}{\text{area of the table}} \quad \text{as} \quad L \to \infty\,,$$

which is 4 times (and not 3) times bigger than the number of trajectories joining a fixed pair of right corners...

Billiard in a right-angled polygon: general answer

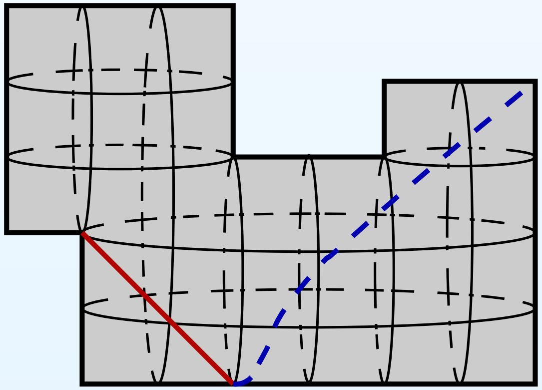
For each topological type we explicitly compute the coefficient in the exact quadratic asymptotics for the corresponding number of generalized diagonals (number of closed trajectories) of bounded length L, which is the same for almost all Π in the billiard family. Say, the coefficients in the exact quadratic asymptotics for the number of generalized diagonals joining a pair of distinct fixed vertices of one of the angles $\frac{\pi}{2}, 3\frac{\pi}{2}, 4\frac{\pi}{2}$ is described by the following table:

$angle$	$\frac{4\pi}{2}$	$\frac{3\pi}{2}$	$\frac{\pi}{2}$
$\frac{4\pi}{2}$	$\frac{9\pi}{10}$	$\frac{45\pi}{64}$	$\frac{9\pi}{32}$
$\frac{3\pi}{2}$	$\frac{45\pi}{64}$	$\frac{16}{3\pi}$	$\frac{2}{\pi}$
$\frac{\pi}{2}$	$\frac{9\pi}{32}$	$\frac{2}{\pi}$	$\frac{1}{2\pi}$



Billiards in right-angled polygons versus quadratic differentials on $\mathbb{C}\mathrm{P}^1$

The topological sphere obtained by gluing two copies of the billiard table by the boundary is naturally endowed with a flat metric. This metric has conical singularities at the points coming from vertices of the polygon, otherwise it is nonsingular. In the case of a “right-angled polygon” the flat metric has holonomy in $\mathbb{Z}/(2\mathbb{Z})$ and corresponds to a meromorphic quadratic differential with at most simple poles on $\mathbb{C}\mathrm{P}^1$. Geodesics on this flat sphere project to billiard trajectories! Thus, to count billiard trajectories we may count geodesics on flat spheres.



Families of flat surfaces with prescribed conical singularities and with trivial holonomy or holonomy in $\mathbb{Z}/(2\mathbb{Z})$ correspond to strata of holomorphic Abelian or quadratic differentials with prescribed degrees of zeroes $\mathcal{H}(m_1, \dots, m_n)$ and $\mathcal{Q}(d_1, \dots, d_n)$ correspondingly.

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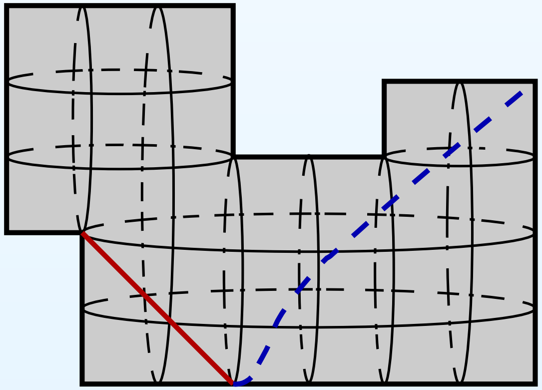

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Siegel–Veech constants

Regular closed geodesics on a flat surface appear in families filling maximal flat cylinders. Each of the two boundaries of such a cylinder contains a conical singularity. It is convenient to count families of parallel closed geodesics with a weight equal to the area of the cylinder divided by the area of the surface.

Theorem (A. Eskin, H. Masur.) *For almost any flat surface S in any stratum $\mathcal{H}(m_1,\dots,m_n)$ the number of families of parallel closed geodesics satisfies*

$$N_{area}(S,L) \sim c_{area}(\mathcal{H}(m_1,\dots,m_n)) \cdot \frac{L^2}{\text{area of the surface}} \quad \text{as} \quad L\to\infty\,.$$

Corresponding constants c_{area} are called *Siegel–Veech constants*. Factors $\frac{1}{2\pi}$, $\frac{2}{\pi}$, $\frac{16}{3\pi^2}$ are examples of *Siegel–Veech constants* for corresponding strata.

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Theorem (A. Eskin, H. Masur, A. Z., 2003)

$$c_{area}(\mathcal{H}_1(m_1,\dots,m_n)) = \sum_{\text{types of degenerations}} (\text{explicit combinatorial factor})\cdot$$

$$\cdot\ \frac{\prod_{j=1}^{k} \operatorname{Vol}\mathcal{H}_1(\text{adjacent simpler strata})}{\operatorname{Vol}\mathcal{H}_1(m_1,\dots,m_n)}\,.$$

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Theorem (E. Goujard, 2015)

$$c_{area}(\mathcal{Q}_1(d_1,\dots,d_n)) = \sum_{\text{types of degenerations}} (\text{explicit combinatorial factor}) \cdot$$

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Another reason to care about volumes of moduli spaces: diffusion rates and Lyapunov exponents

Diffusion for a random walk and for a brownian motion

Let $X_1, ..., X_n$ be a sequence of independent and identically distributed random variables (heads or tails, measurements in uncorrelated experiments, etc). Assume that the variance σ^2 is finite and that the expected value is 0. Let $S_n := X_1 + \cdots + X_n$. Clearly, with probability one one has

$$\frac{X_1 + \cdots + X_n}{n} = \frac{S_n}{n} \to 0 \quad \text{as } n \to +\infty\,.$$

The Central Limit Theorem describes the expected deviation of the sum S_n from 0. In a sense, it is one of the fundamental laws of Nature:

Cental Limit Theorem. *The distribution of the the sum S_n normalized by the factor $\frac{1}{\sqrt{n}}$ tends to the normal distribution with mean 0 and variance σ^2.*

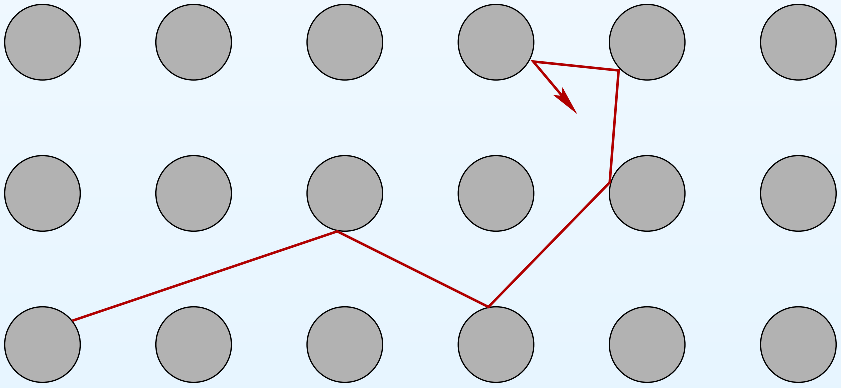
Corollary: random walk or brownian motion in the plane. *The root of the mean square of the translation distance after n steps of a random walk with zero mean is*

$$\sqrt{E|S_n^2|} = \sigma\sqrt{n} = \sigma \cdot n^{\frac{1}{2}}\,.$$

Lorentz gas: diffusion in periodic billiard. Convex obstacles

Theorem (Bunimovich, Chernov, Sinai, 1991). *For periodic configuration of convex scatterers on the plane the particle after scaling by* $\sqrt{t}$ *satisfies the Central Limit Theorem if the horizon is finite (that is, if any ray intersects a scatterer).*

Theorem (Szász, Varjú, 2007; some first ideas — Bleher 1992). *In infinite horizon case, for example, for round scatterers placed at the lattice points, the Central Limit Theorem still holds but the scaling should be by* $\sqrt{t \ln t}$*.*

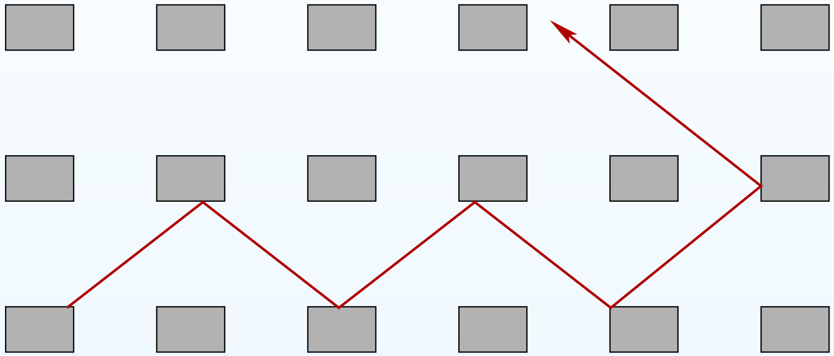


Chernov, Dolgopyat (2009): further interesting results in this direction.

In all cases the *diffusion rate* is again $\frac{1}{2}$ as for the random walk.

Diffusion in a periodic billiard (Ehrenfest "Windtree model")

Consider a billiard on the plane with $\mathbb{Z}^2$-periodic rectangular obstacles.



Theorem (V. Delecroix, P. Hubert, S. Lelièvre, 2014). *For all parameters of the obstacle, for almost all initial directions, and for any starting point, the billiard trajectory spreads in the plane with the speed* $\sim t^{2/3}$. *That is,*

$$\lim_{t\to+\infty} \log\,(\text{diameter of trajectory of length } t)/\log t = 2/3.$$

The diffusion rate $\frac{2}{3}$ *is given by the Lyapunov exponent of certain renormalizing dynamical system associated to the initial one.*

Remark. Changing the height and the width of the obstacle we get quite different billiards, but this does not change the diffusion rate!

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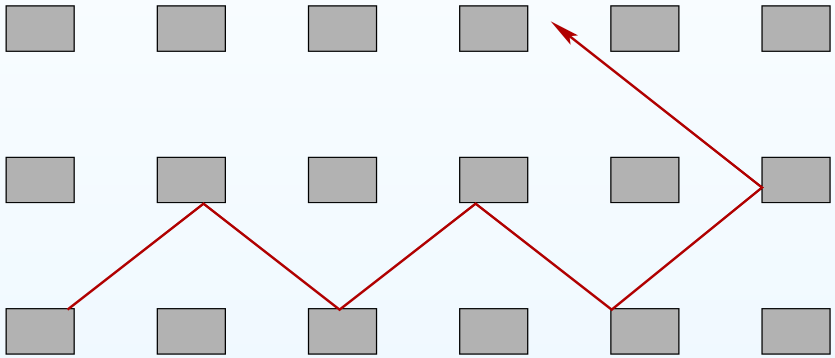

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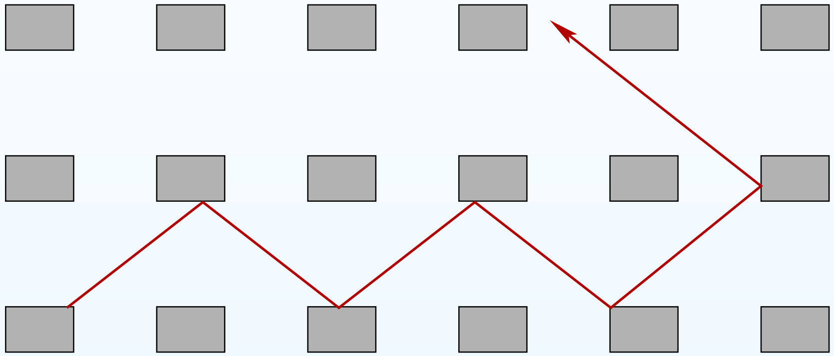

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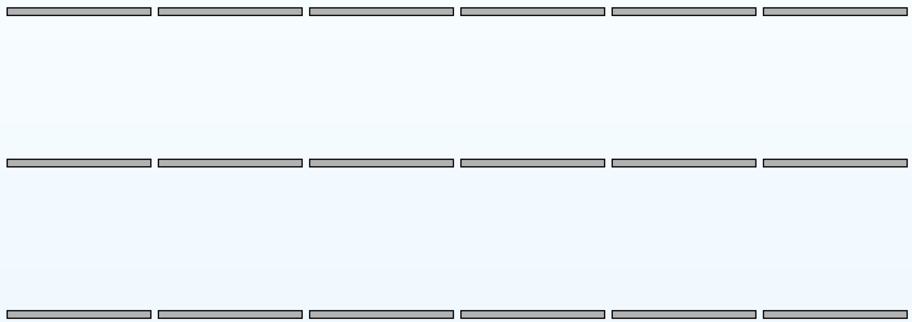

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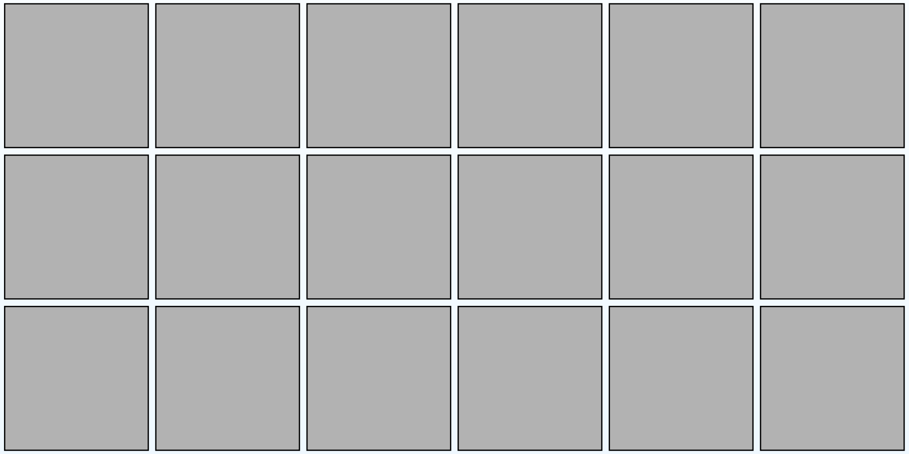

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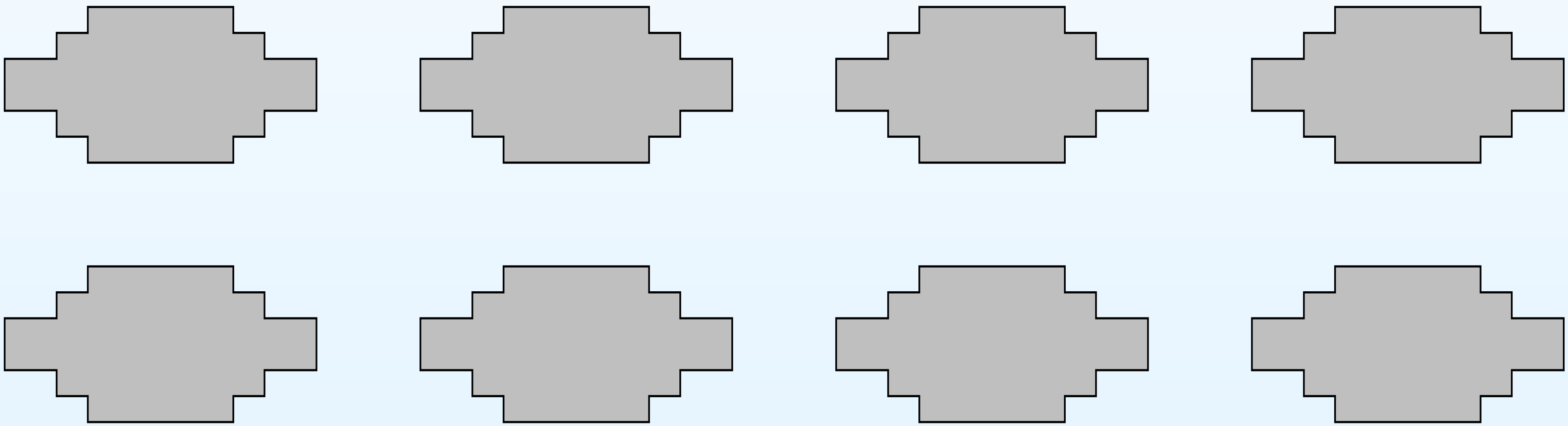
Changing the shape of the obstacle

Theorem (V. Delecroix, A. Z., 2015). *Changing the shape of the obstacle we get a different diffusion rate. Say, for a symmetric obstacle with $4m-4$ angles $3\pi/2$ and $4m$ angles $\pi/2$ the diffusion rate is*

$$\frac{(2m)!!}{(2m+1)!!} \sim \frac{\sqrt{\pi}}{2\sqrt{m}} \quad \text{as } m \to \infty\,.$$



Note that once again the diffusion rate depends only on the number of the corners, but not on the (almost all) lengths of the sides, or other details of the shape of the obstacle.

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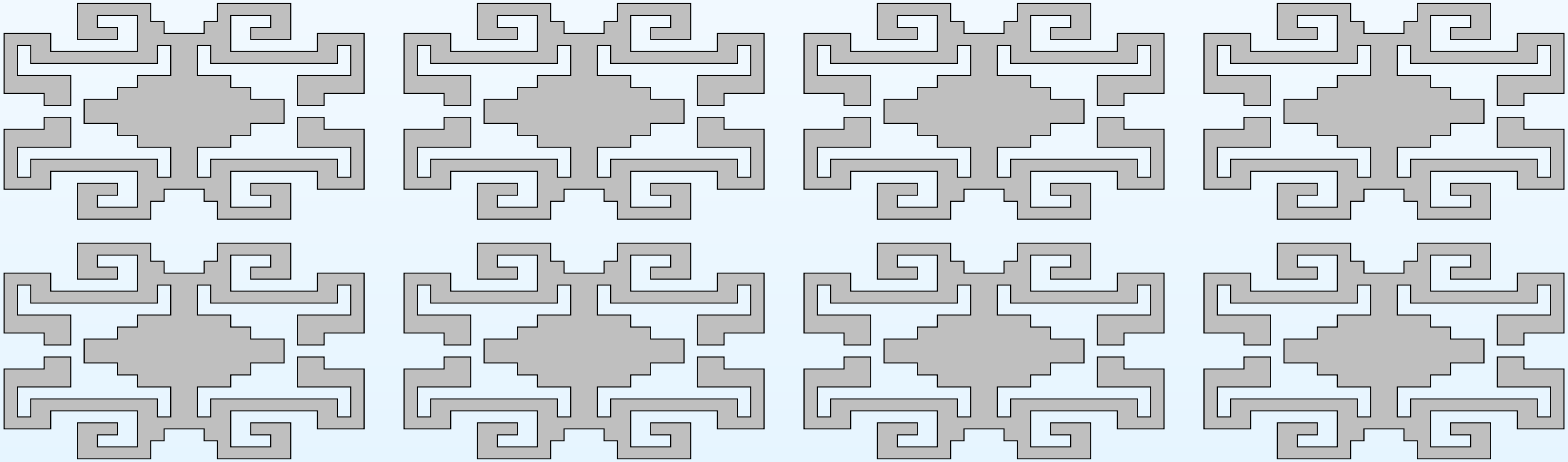

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Lyapunov exponents of the Hodge bundle over the Teichmüller flow

Consider a natural vector bundle over a stratum with a fiber $H^1(S;\mathbb{R})$ over a "point" (S,ω). This *Hodge bundle* carries a canonical *Gauss – Manin connection*: we have a lattice $H^1(S;\mathbb{Z})$ in each fiber, which tells us how we can locally identify the fibers. Thus, the Teichmüller flow on $\mathcal{H}_1(d_1,\dots,d_n)$ defines a multiplicative cocycle acting on fibers of this bundle. The monodromy matrices are symplectic which implies that the Lyapunov exponents of the cocycle are symmetric: $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_g \geq -\lambda_g \geq \dots \geq -\lambda_2 \geq -\lambda_1$.

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Theorem (A. Eskin, M. Kontsevich, A. Z., 2014) *The Lyapunov exponents λ_i of the Hodge bundle $H^1_{\mathbb{R}}$ along the Teichmüller flow restricted to an $\mathrm{SL}(2,\mathbb{R})$-invariant suborbifold $\mathcal{L} \subseteq \mathcal{H}_1(d_1,\dots,d_n)$ satisfy:*

$$\lambda_1 + \lambda_2 + \dots + \lambda_g \;=\; \frac{1}{12}\cdot\sum_{i=1}^{n}\frac{d_i(d_i+2)}{d_i+1} + \frac{\pi^2}{3}\cdot c_{area}(\mathcal{L})\,.$$

where $c_{\text{area}}(\mathcal{L})$ is the Siegel–Veech constant.

Calculation of volumes of moduli spaces of Abelian and quadratic differentials

Period coordinates, volume element, and unit hyperboloid

The moduli space $\mathcal{H}(m_1,\dots,m_n)$ of pairs (C,ω), where C is a complex curve and ω is a holomorphic 1-form on C having zeroes of prescribed multiplicities $m_1,\dots,m_n$, where $\sum m_i = 2g-2$, is modelled on the vector space $H^1(S,\{P_1,\dots,P_n\};\mathbb{C})$. The latter vector space contains a natural lattice $H^1(S,\{P_1,\dots,P_n\};\mathbb{Z}\oplus i\mathbb{Z})$, providing a canonical choice of the volume element $d\nu$ in these *period coordinates*.

Flat surfaces of area 1 form a real hypersurface $\mathcal{H}_1 = \mathcal{H}_1(m_1,\dots,m_n)$ defined in period coordinates by equation

$$1 = \text{area}(S) = \frac{i}{2}\int_C \omega\wedge\bar\omega = \sum_{i=1}^{g}(A_i\bar B_i - \bar A_i B_i)\,.$$

Any flat surface S can be uniquely represented as $S=(C, r\cdot\omega)$, where $r>0$ and $(C,\omega)\in\mathcal{H}_1(m_1,\dots,m_n)$. In these “polar coordinates” the volume element disintegrates as $d\nu = r^{2d-1}dr\,d\nu_1$ where $d\nu_1$ is the induced volume element on the hyperboloid $\mathcal{H}_1$ and $d=\dim_{\mathbb{C}}\mathcal{H}(m_1,\dots,m_n)$.

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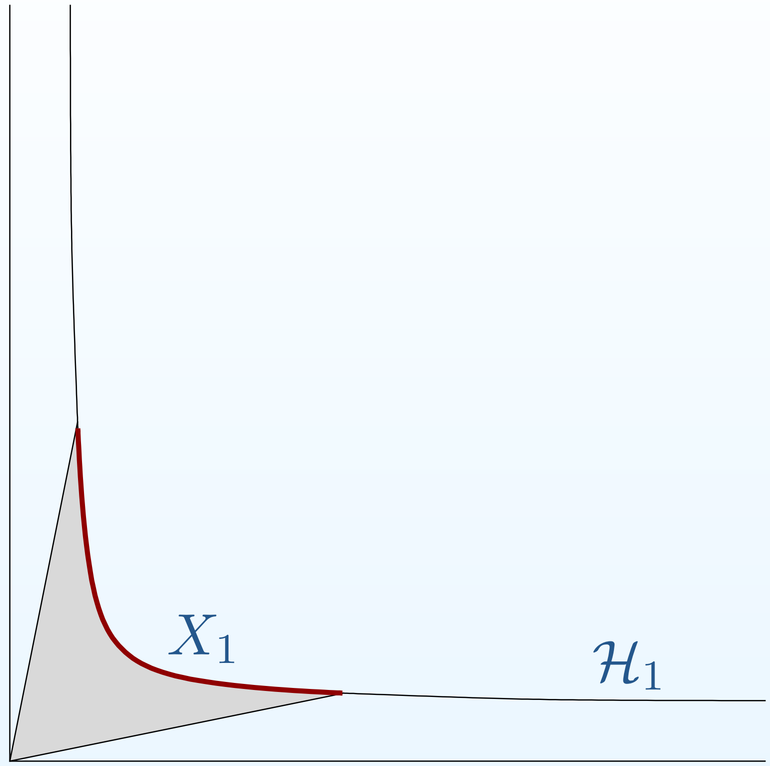
Counting volume by counting integer points in a large cone



ν_1-volume of a domain X_1 in a unit hyperboloid $\mathcal{H}_1$ is related to ν-volume of a cone $C(X_1) = \{r \cdot S | S \in X_1, r \leq 1\}$ over X_1 as $\nu_1(X_1) = 2d \cdot \nu(C(X_1))$.

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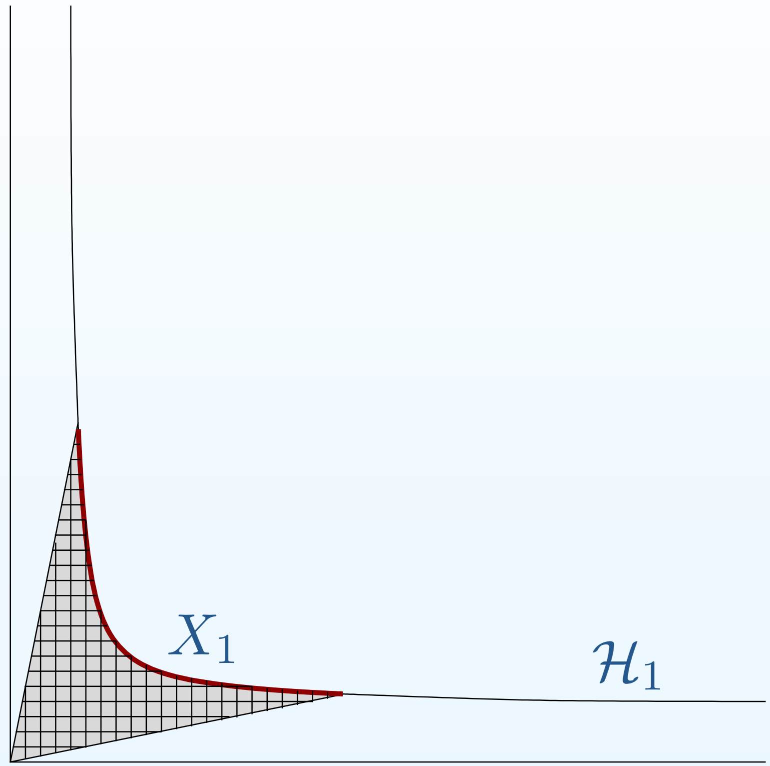

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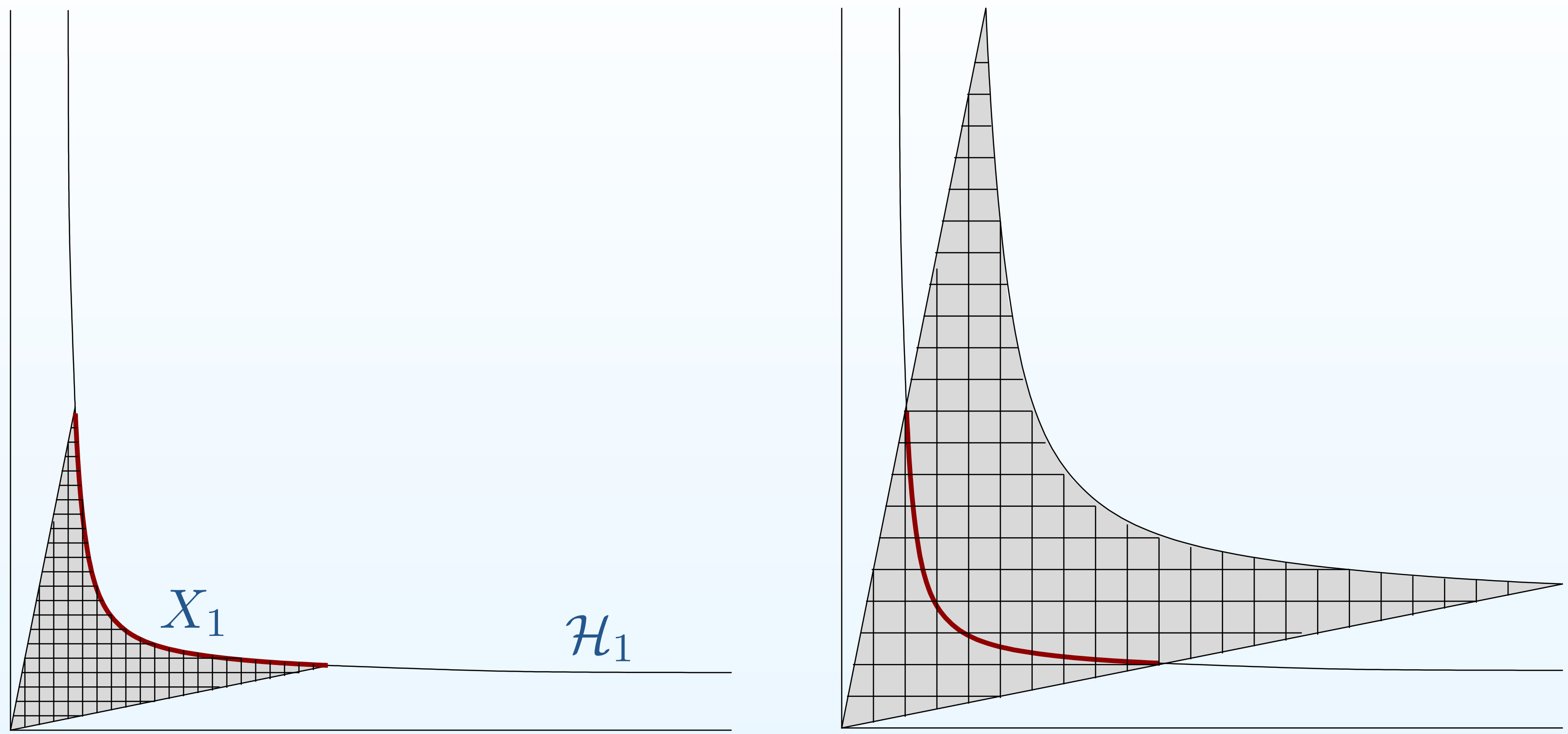

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To count volume of the cone $C(X_1)$ one can take a small grid and count the number of lattice points inside it. Counting points of the $\frac{1}{N}$-grid in the cone $C(X_1) = \{r \cdot S | S \in X_1, r \leq 1\}$ is the same as counting integer points in the larger proportionally rescaled cone $C_N(X_1) = \{r \cdot S | S \in X_1, r \leq N\}$.

Integer points as square-tiled surfaces

Integer points in period coordinates are represented by *square-tiled surfaces*. Indeed, if a flat surface S is defined by a holomorphic 1-form ω such that $[\omega] \in H^1(S, \{P_1, \dots, P_n\}; \mathbb{Z} \oplus i\mathbb{Z})$, it has a canonical structure of a ramified cover p over the standard torus $\mathbb{T} = \mathbb{R}^2/(\mathbb{Z} \oplus i\mathbb{Z})$ ramified over a single point:

$$S \ni P \mapsto \left(\int_{P_1}^{P} \omega \mod \mathbb{Z} \oplus i\mathbb{Z} \right) \in \mathbb{T}, \text{ where } P_1 \text{ is a zero of } \omega\,.$$

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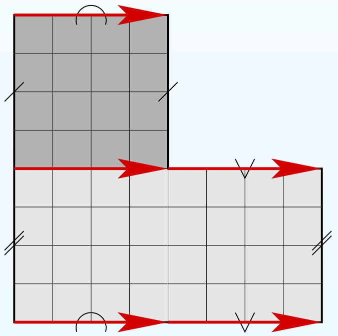
Integer points in the strata $\mathcal{Q}(d_1, \dots, d_n)$ of quadratic differentials are represented by analogous “pillowcase covers” over $\mathbb{C}\mathrm{P}^1$ branched at four points.

Thus, counting volumes of the strata is similar to counting analogs of Hurwitz numbers.

Critical graphs (separatrix diagrams)

All leaves of the horizontal (vertical) foliation on a square-tiled surface are closed. The *critical graph* Γ (aka *separatrix diagram*) is the union of all horizontal critical leaves. Vertices of Γ are represented by the conical points; the edges of Γ are formed by horizontal saddle connections.



Actually, Γ is an *oriented ribbon graph* with the following extra stucture:
1) The orientation of edges at any vertex is alternated with respect to the cyclic order of edges at this vertex.
2) The complement $S - \Gamma$ is a finite disjoint union of flat cylinders foliated by oriented circles. Thus, the set of boundary components of the ribbon graph is decomposed into pairs: to each pair of boundary components we glue a cylinder, and there is one positively oriented and one negatively oriented boundary component in each pair.

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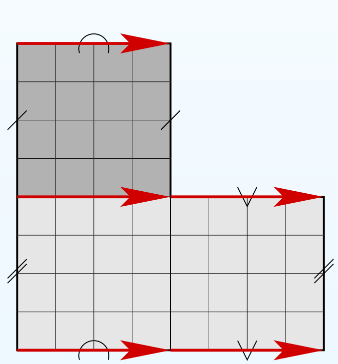

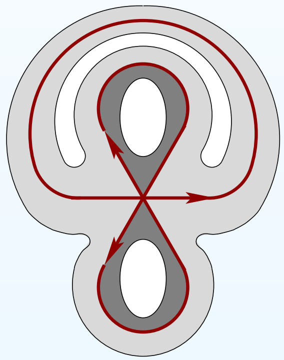

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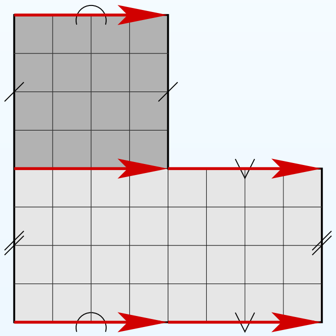

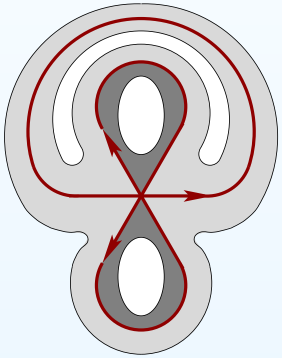

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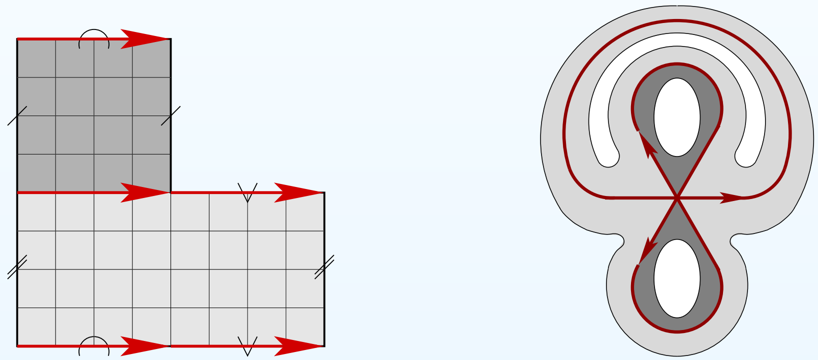

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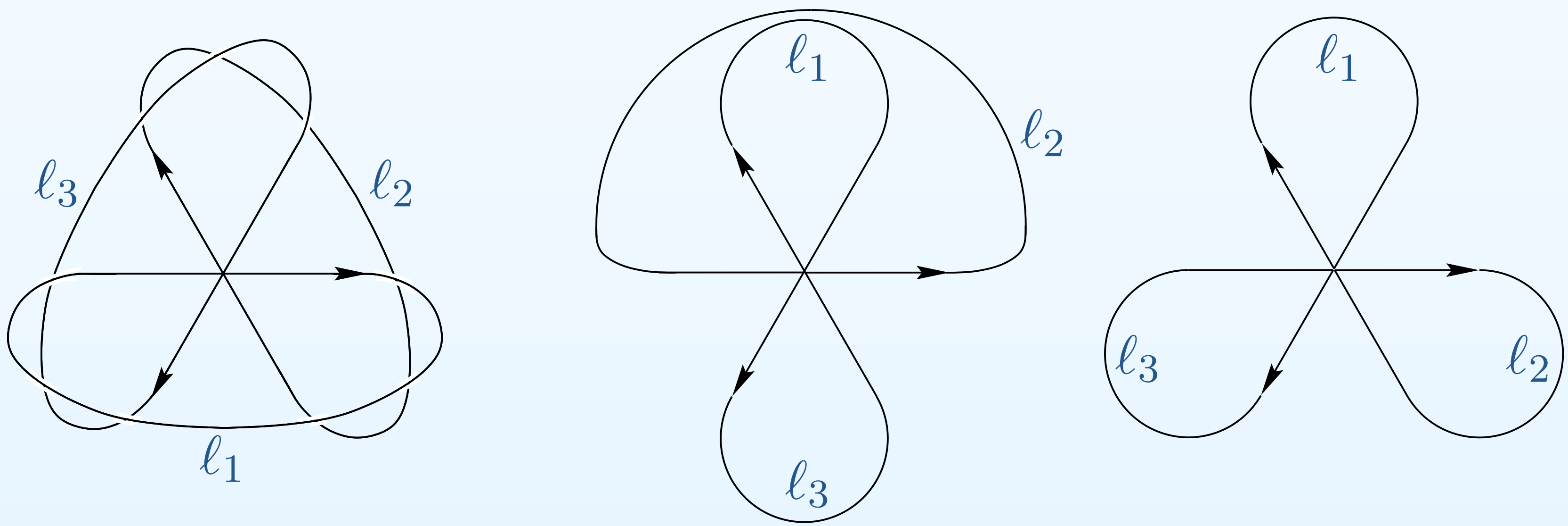
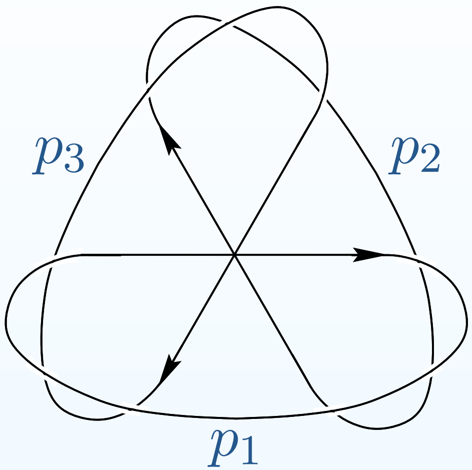
Realizable separatrix diagrams

Note, however, that not all ribbon graphs as above correspond to actual flat surfaces. A flat metric endows saddle connections with positive lengths ℓ_i. The left graph is realizable for any lengths ℓ_1, ℓ_2, ℓ_3. The middle one — only when $\ell_1 = \ell_3$. The rightmost one is never realizable.



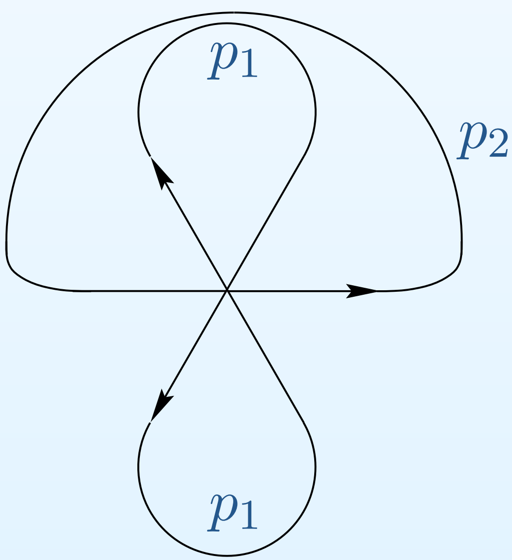
Lemma. *The set of all square-tiled surfaces (respectively pillowcase covers) sharing the same realizable separatrix diagram provides a nontrivial contribution to the volume of the corresponding stratum.*

Volume computation for $\mathcal{H}(2)$



Single cylinder

$$\frac{1}{3}\sum_{\substack{p_1,p_2,p_3,h\in\mathbb{N}\\(p_1+p_2+p_3)h\le N}}(p_1+p_2+p_3)\approx\frac{N^4}{24}\cdot\zeta(4)$$



Two cylinders

$$\sum_{\substack{p_1,p_2,h_1,h_2\\p_1h_1+(p_1+p_2)h_2\le N}}p_1(p_1+p_2)$$

$$=\frac{N^4}{24}\left[2\cdot\zeta(1,3)+\zeta(2,2)\right]=\frac{N^4}{24}\cdot\frac{5}{4}\cdot\zeta(4)$$

$$\mathrm{Vol}(\mathcal{H}_1(2))=\lim_{N\to\infty}\frac{2\cdot 4}{N^4}\cdot(\text{Number of surfaces})=\frac{\pi^4}{120}.$$

Contribution of k-cylinder square-tiled surfaces to $\operatorname{Vol}\mathcal{H}(3,1)$

$$0.19 \approx p_1(\mathcal{H}(3,1)) = \frac{3\,\zeta(7)}{16\,\zeta(6)}$$

$$0.47 \approx p_2(\mathcal{H}(3,1)) = \frac{55\,\zeta(1,6) + 29\,\zeta(2,5) + 15\,\zeta(3,4) + 8\,\zeta(4,3) + 4\,\zeta(5,2)}{16\,\zeta(6)}$$

$$\begin{aligned}0.30 \approx p_3(\mathcal{H}(3,1)) = \frac{1}{32\,\zeta(6)}\Big(&12\,\zeta(6) - 12\,\zeta(7) + 48\,\zeta(4)\,\zeta(1,2) + 48\,\zeta(3)\,\zeta(1,3)\\
&+ 24\,\zeta(2)\,\zeta(1,4) + 6\,\zeta(1,5) - 250\,\zeta(1,6) - 6\,\zeta(3)\,\zeta(2,2)\\
&- 5\,\zeta(2)\,\zeta(2,3) + 6\,\zeta(2,4) - 52\,\zeta(2,5) + 6\,\zeta(3,3) - 82\,\zeta(3,4)\\
&+ 6\,\zeta(4,2) - 54\,\zeta(4,3) + 6\,\zeta(5,2) + 120\,\zeta(1,1,5) - 30\,\zeta(1,2,4)\\
&- 120\,\zeta(1,3,3) - 120\,\zeta(1,4,2) - 54\,\zeta(2,1,4) - 34\,\zeta(2,2,3)\\
&- 29\,\zeta(2,3,2) - 88\,\zeta(3,1,3) - 34\,\zeta(3,2,2) - 48\,\zeta(4,1,2)\Big)\end{aligned}$$

$$0.04 \approx p_4(\mathcal{H}(3,1)) = \frac{\zeta(2)}{8\,\zeta(6)}\Big(\zeta(4) - \zeta(5) + \zeta(1,3) + \zeta(2,2) - \zeta(2,3) - \zeta(3,2)\Big).$$

Results of Eskin, Okounkov, and Pandharipande

Theorem (A. Eskin, A. Okounkov, R. Pandharipande). *For every connected component $\mathcal{H}^c(d_1,\dots,d_n)$ of every stratum, the generating function*

$$\sum_{N=1}^{\infty} q^N \sum_{\substack{N\text{-square-tiled}\\ \text{surfaces } S}} \frac{1}{|Aut(S)|}$$

is a quasimodular form, i.e. a polynomial in Eisenstein series $G_2(q)$, $G_4(q)$, $G_6(q)$. Volume $\operatorname{Vol}\mathcal{H}_1^c(d_1,\dots,d_n)$ of every connected component of every stratum is a rational multiple $\frac{p}{q}\cdot\pi^{2g}$ of π^{2g}, where g is the genus.

A. Eskin implemented this theorem to an algorithm allowing to compute $\frac{p}{q}$ for all strata up to genus 10 and for some strata (like the principal one) up to genus 200.

A. Eskin, A. Okounkov and R. Pandharipande proved analogous result for the volumes $\operatorname{Vol}\mathcal{Q}_1(d_1,\dots,d_k)$ of the moduli spaces of quadratic differentials. However, an implementation is more painful (with exception for genus 0 where there is a close formula conjectured by Kontsevich and proved recently by Athreya, Eskin, and Zorich). Values for the first 300 low-dimensional strata in $g>0$ were obtained only in 2015 by E. Goujard.

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Large genus asymptotics

Conjecture on asymptotics of volume for large genera

Let $\mathbf{m} = (m_1, \ldots, m_n)$ be an unordered partition of an even number $2g-2$, $|\mathbf{m}| = m_1 + \cdots + m_n = 2g-2$. Denote by Π_{2g-2} the set of all partitions.

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$$\text{where } |\varepsilon(\mathbf{m})| \le \frac{\text{const}}{\sqrt{g}}\,.$$

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Asymptotic Siegel–Veech constant for large genera

We have seen that the Siegel–Veech constant c_{area} can be expressed as:

$$c_{area} = \sum_{\substack{\text{types of}\\ \text{degenerations}}} (\text{explicit factor}) \cdot \frac{\prod_{j=1}^{k} \operatorname{Vol} \mathcal{H}_1(\text{adjacent simpler strata})}{\operatorname{Vol} \mathcal{H}_1(m_1, \dots, m_n)}\,.$$

Conditional Theorem. *For all series of connected components $\mathcal{H}^c(\mathbf{m})$ for which the conjecture on the volume asymptotic holds, one has:*

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In particular this is true for the principal stratum and for the similar ones for which D. Chen, M. Möller, D. Zagier have already proved the volume conjecture. D. Chen, M. Möller, D. Zagier have an independent proof of this Theorem based on quadimodularity of the associated generating function.

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Contribution of 1-cylinder diagrams. Equivalent conjecture

Theorem. *The contribution c_1 of 1-cylinder square-tiled surfcaes to the volume $\operatorname{Vol}\mathcal{H}_1(m_1,\dots,m_n)$ of any nohyperelliptic stratum of Abelian differentials satisfies*

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where $d=\dim_{\mathbb{C}}\mathcal{H}(m_1,\dots,m_n)$.

Theorem. *Conjecture on volume asymptotics is (almost) equivalent to the following statement: the relative contribution of 1-cylinder square-tiled surfaces to the volume of the stratum is of the order $1/($dimension of the stratum$)$ when $g\gg 1$,*

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Theorem. *There is no correlation between statistics of the number of horizontal and vertical maximal cylinders.*

Similarly to a square-tiled surface, a rational interval exchange transformation might have $1, 2, \ldots, k$ bands of periodic orbits. Fix a nondegenerate permutation π of d elements and an open ball B in $\mathbb{R}_+^d$. Consider an ε-grid $B \cap (\varepsilon\mathbb{N})^d$ in B and collect statistics of frequencies of $1, 2, \ldots$-bands interval exchange transformations $(\pi, \lambda_\varepsilon)$ with λ_ε in the grid.

Theorem. *The limiting statistics as $\varepsilon \to 0$ does not depend on B: it is the same as the limiting statistics of k-cylinder square-tiled surfaces for the stratum of Abelian differentials corresponding to the Rauzy class of π.*

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Similarly to a square-tiled surface, a rational interval exchange transformation might have $1, 2, \ldots, k$ bands of periodic orbits. Fix a nondegenerate permutation π of d elements and an open ball B in $\mathbb{R}^d_+$. Consider an ε-grid $B \cap (\varepsilon\mathbb{N})^d$ in B and collect statistics of frequencies of $1, 2, \ldots$-bands interval exchange transformations $(\pi, \lambda_\varepsilon)$ with λ_ε in the grid.

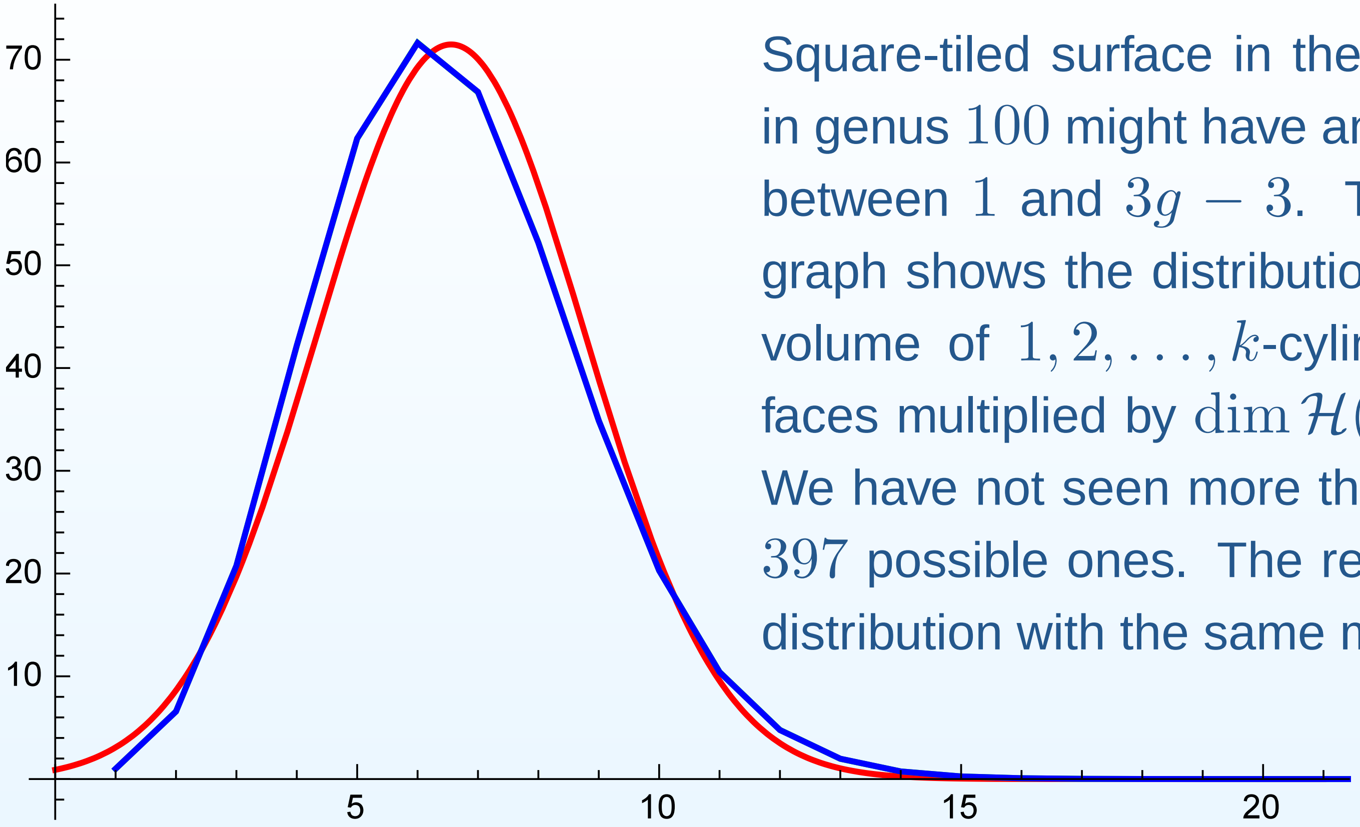
Theorem. *The limiting statistics as $\varepsilon \to 0$ does not depend on B: it is the same as the limiting statistics of k-cylinder square-tiled surfaces for the stratum of Abelian differentials corresponding to the Rauzy class of π.*

Experiments and further open problems

Frequencies of $1, \ldots, k$-cylinder square-tiled surfaces in large genera

Square-tiled surface in the stratum $\mathcal{H}(1, \ldots, 1)$ in genus 100 might have any number of cylinders between 1 and $3g-3$. How often a “random” square-tiled surface has $1, 2, ..., 297$-cylinders? What distribution do you expect?

Frequencies of $1, \ldots, k$-cylinder square-tiled surfaces in large genera



Square-tiled surface in the stratum $\mathcal{H}(1,\ldots,1)$ in genus 100 might have any number of cylinders between 1 and $3g-3$. The experimental blue graph shows the distribution of the impact to the volume of $1,2,\ldots,k$-cylinder square-tiled surfaces multiplied by $\dim \mathcal{H}(1,\ldots,1) = 4g-3$. We have not seen more than 23 cylinders out of 397 possible ones. The red graph is the normal distribution with the same mean and variance.

Conjecture. *For any nonhyperellyptic component of a stratum of Abelian differentials, the mean of the distribution is asymptotically located at* $\mathrm{const} + \log($*dimension of the stratum*$)$, *where* const *is a universal constant.*

Question. *Does the distribution tends to the normal one or to a skew one? If it is skew, is it some known distribution (like Tracy–Widom distribution)?*

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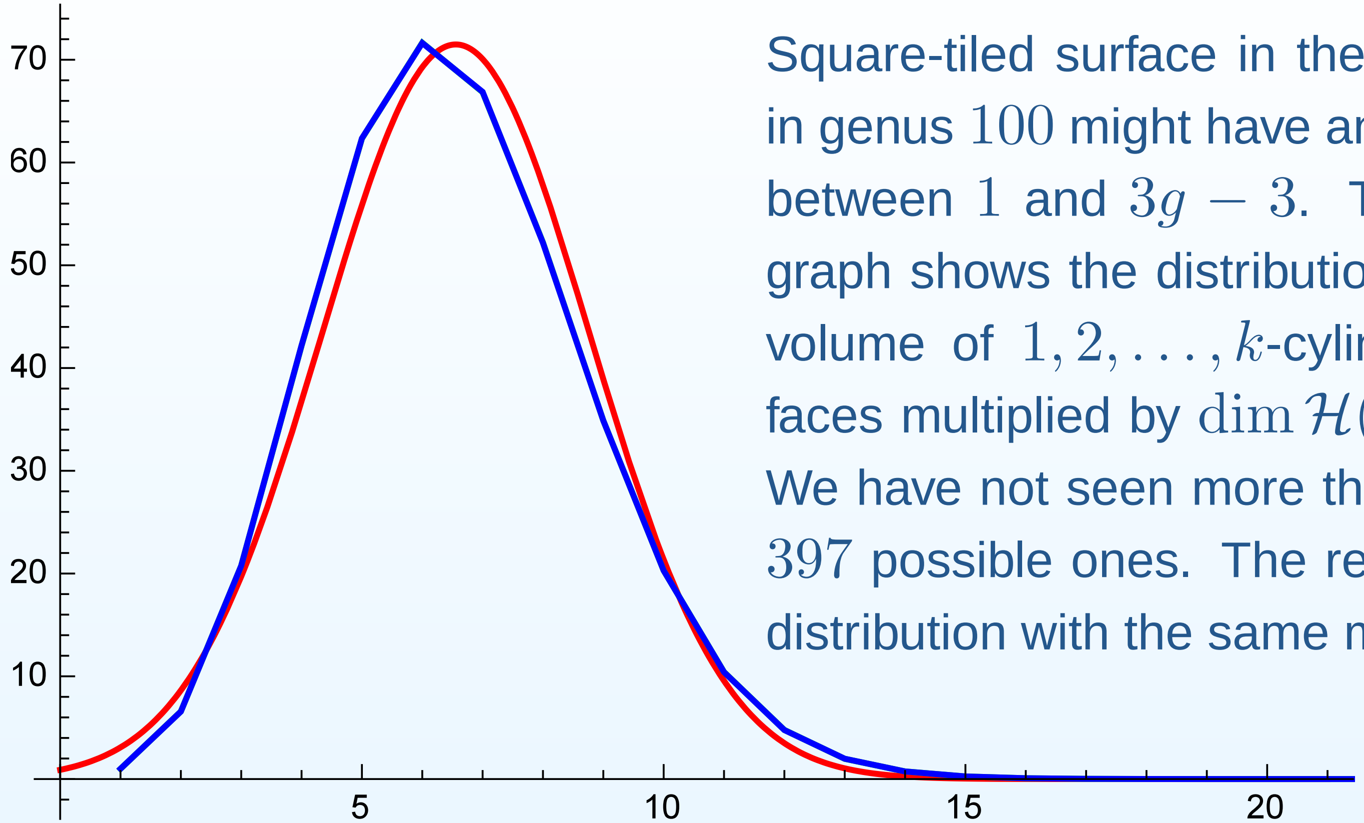

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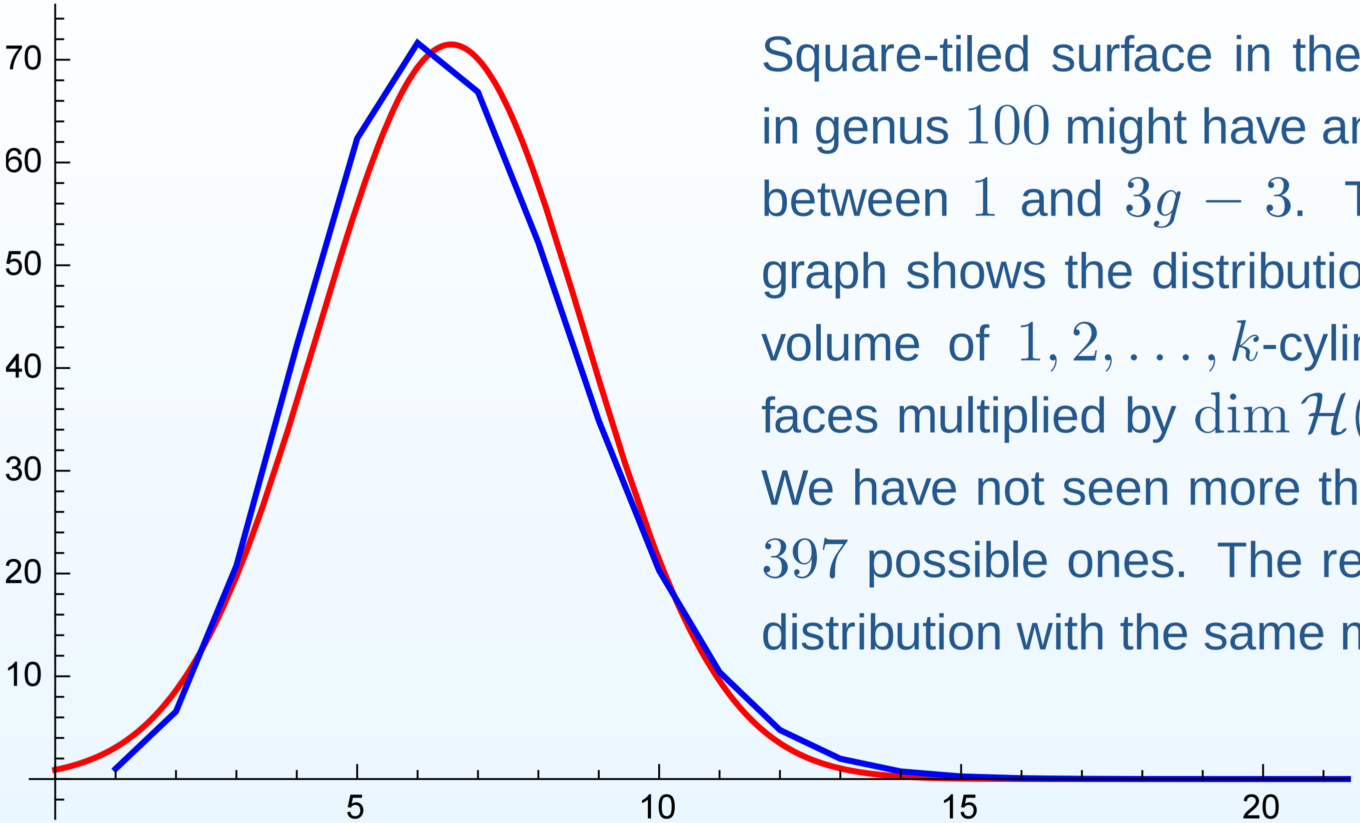

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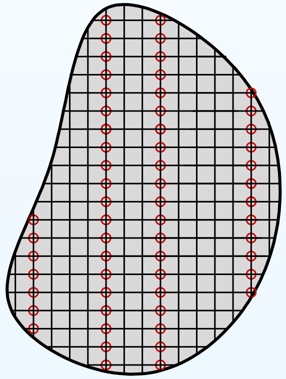
Experimental evaluation of volumes

The Equidistribution Theorem allows to compute approximate values of volumes experimentally. Choose some ball B (or some box) in the stratum. Consider a sufficiently small grid in it and collect statistics of frequency $p_1(B)$ of 1-cylinder square-tiled surfaces (pillow-case covers) in our grid in B.



Now compute the **absolute** contribution $c_1(\mathcal{H})$ of all 1-cylinder square-tiled surfaces to $\operatorname{Vol}\mathcal{H}_1$; it is easier than for k-cylinder ones with $k > 2$. By the Equdistribution Theorem, the volume of the ambient stratum is $\operatorname{Vol}\mathcal{H}_1 = \frac{c_1}{p_1}$.

The statistics $p_1(\mathcal{H})$ can be, actually, collected using interval exchanges, which simplifies the experiment. Approximate values of volumes were extremely useful in debugging numerous normalization factors in rigorous answers in the implementation by E. Goujard of the method of Eskin–Okounkov.

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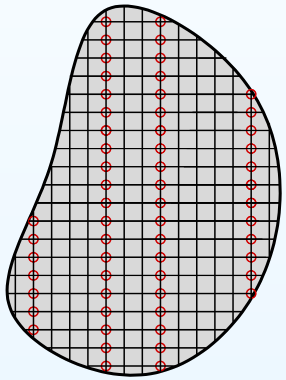

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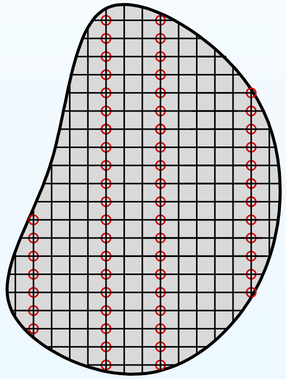

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Artistic image of a billiard in a polygon

Varvara Stepanova. Joueurs de billard. Thyssen Museum, Madrid