Topics in complex and real geometry

March 15 - May 15, 2003

The goal of the special research period **Topics in Complex and Real Geometry** March 15, 2003 - May 15, 2003 was to bring together experts of different Universities in order to achiev strong synergetic effects in training young Ph.D and PostDoc, focusing on fundamental problems in Complex and Real Geometry. The activity of the period was structured in minicourses, survey and advanced lectures offered by leading experts, and talks. In total there were about 50 one-hour talks and 10 minicourses for about 40 hours. Furthemore we had 5 one-hour courses and 27 half hour communications. During the bimester a special session also took place, dedicated to the annual meeting of the *European group of Real Algebraic and Analytic Geometry;* in this week the most recent results on the topics of the RAAG were presented in a 5 days series of talks and communications.

Short description of the main contributions.

Subanalytic geometry: Starting from the main questions, that is extension, division and composition of differentiable functions, already clear in the works of Whitney, Thom and Lojasiewicz, in his 3-hours minicourse, **E. Bierstone**, after a short historical introduction, presented recent results on the extension problem. An important tool was the notion of bundle of differential operators on a singular space and also the characterization of those subanalytic sets that admit a stratified coherent structure. This class, which includes all semianalytic sets, seems to be the right tame class from the point of wiew of geometry, analysis, algebraic invariants. In this framing, in the talk of **A. M. Chollet** conditions to get Weierstrass division by a hyperbolic polinomial for Denjoy-Carlemann classes were illustrated.

Global semianalytic sets: The techniques of Real Algebra are not easily translated from the algebraic setting to the analytic one. One possibility is to consider the sets that are definable in the ring of analytic function on an analytic manifold. The main difficulty is the fact that there are too many orderings in the real spectrum of the field of meromorphic functions on M. This and many other problems were discussed in the (4-hours) course of **C. Andradas**, e.g. the 17-th Hilbert problem, Artin-Lang property, Null and Positivstellensatz, connected components, closure and finiteness property. Most of these problems are still open, some work has been done mainly for dimension 1 and 2 and for compact manifolds. In the case of dimension 1 one has a good description of the real spectrum of the field of meromorphic functions on M and this allows us to get Artin-Lang property (a translation from the constructible sets of the real spectrum and the sets defined by global analytic functions) and hence for instance the fact that connected components and closures of global semianalytic sets are still global. A few results in general dimension regarding Lojasiewicz inequality and finiteness property were recently obtained; here finiteness means that an open (closed) global semianalytic set is a finite union of open (closed) basic sets. An Artin-Lang-like property was presented in the talk by **L.Pernazza.** The so called semi-pfaffian geometry is somewhat related to this context: in his talk **A. Gabrielov** introduced a spectral sequence associated with a surjective continuous map that provides upper bounds on the Betti numbers of a set defined by an expression with quantifiers, in terms of the Betti numbers of some auxiliary sets defined by quantifier-free formulas. This applies to semi-pfaffian sets.

Nash functions and manifolds: The ring of Nash functions (that is, analytic functions that are algebraic over the polynomial ring) has nice local properties that have been known for a long time. Concerning global properties, the main problem is the fact that there is no good vanishing property of the cohomolgy groups. A problem raising in this context is whether a Nash function on a Nash subset of a Nash manifold can be extended to the whole manifold. In a 4 hour course **M. Coste** presented the solution to this problem using hard commutative algebra as Neron-Severi desingularization.

Real algebraic varieties and maps: Under this title we summarize several topics developed in the bimester concerning the topology of real algebraic varieties. The 4 hours course of **G. Mikhalkin** was dedicated to an emergent topic, namely the so called tropical geometry. There is an interesting relation between amoebas, that are images of algebraic varieties under the moment map, and combinatorial constructions of real algebraic varieties. Amoebas are closed unbounded regions of peculiar shape which carry information on the initial algebraic variety. There exists a deformation (known in different areas of mathematics by different names: patchworking, dequantization, passing to the large complex limit) which turns the amoebas into piecewise-linear polyhedral complexes. It turns out that the resulting objects can be considered as algebraic varieties over the so-called tropical (or (max,+)) semiring. As an application of tropical geometry, G. Mikhalkin showed how to compute the Gromov-Witten invariants of the projective plane (and other toric surfaces) by lattice paths in polygons. Combinatorial construction of real algebraic varieties was the subject of the talk of **I. Itenberg,** too: he presented Viro method and discussed the relation with tropical algebraic varieties. **K. Kurdyka** presented his positive answer to an old problem dating back to J. Ax and E. Borel: are injective endomorphisms of real algebraic sets surjective? The proof, in a somewhat more general setting, uses Borel-Moore homology for semialgebraic sets. The essential ingredient is the notion of arcwise symmetric set.
The Betti numbers of compact nonsingular real algebraic varieties have a unique extension to "virtual Betti numbers" defined for all real algebraic varieties, if we ask that they keep a special additive property. Even though these numbers are not topological invariants, they have interesting applications to the topology of algebraic varieties. This subject was discussed in the talk of **C.McCrory.**
A survey lecture on constructive desingularization was given by **O. Villamajor** , in particular some recent progress over fields of characteristic zero were discussed, **J. Bochnak** spoke about approximation of algebraic morphisms. Other aspects of the topology of real algebraic varieties were presented by **J. Huisman**, in particular surgery of real varieties, and by **V. Kharlamov** who dealt with the rate of growth of the number of topologically different plane real algebraic curves of given degree d.

Analytic vector fields: The subject of the 3 hours course of **A. Parusinski** was the proof of the gradient conjecture of R. Thom recently given by Kurdyka, Mostowski, Parusinski. The conjecture states that the integral curves of the gradient of a real analytic function f on an open subset of the euclidean n-space admit tangent lines at their limit points. The proof of the conjecture is based on the theory of singularities. The existence of analytic trajectories and the o-minimal case were also discussed.

Three manifolds, symplectic and hyperbolic geometry: During the trimester three talks were devoted to this topic. **R. Benedetti** described some (mostly geometric and topological) aspects of the construction of invariants for 3-manifolds endowed with principal flat PSL(2,C)-bundles based, in the "classical" case on Rogers dilogarithm, in the "quantum" one on Faddeev-Kashaev's matrix dilogarithms derived from the cyclic representations of a suitable Borel quantum algebra. These invariants are explicitely expressed as state sums over the hyperbolic ideal tetrahedra of any fixed so-called I-triangulation which encodes the manifold and the bundle. **P. Lisca** talked about his recent work that generalizes earlier results of Eliashberg and McDuff on the classification of symplectic fillings of the contact three-manifold (L,C). In particular, he showed that the set known to parametrize the deformations of the singularity also parametrizes the minimal symplectic fillings of (L,C). Along the way all the symplectic fillings of (L,C) were described. **C. Petronio**'s talk sketched the complete classification of the manifolds arising as exceptional Dehn fillings of the smallest known triply cusped hyperbolic 3-manifold (a link complement in the 3-sphere). **J. Duval** gave a proof of the hyperbolicity of the complement of five lines in general position in the projective plane which extends to the almost complex case as well.

During the bimester we also had a survey lecture by **A. Prestel** on the state of the art about positive polynomials and Schmudgen theorem.

Complex analysis in infinite dimensions: This subject was treated by **L. Lempert** in a 4-hours course. After giving of a general account, he turned to the problem of the generalization to Stein manifolds of cohomological result in infinite dimension.

Noncritical holomorphic functions on Stein manifolds This is the title of a 4-hours course given by **F. Forstneric** on a very recent result, proved by Forstneric thimself, concerning the number of holomorphic functions on a Stein manifold with independent differential. This is directly related to the outstanding conjecture: a Stein manifold whose complex tangent bundle is trivial is a covering of an open subset of the n-dimensional complex space.

Constructive Hyperbolicity: Around 1970, S. Kobayashi suggested that a general hypersurface of high enough degree is hyperbolic (i.e., every holomorphic map from the complex plane to it is constant) and has hyperbolic complement. Since then, a significant progress was done in a work by Green and Griffiths, Siu and Yeung, Demailly and El Goul, McQuillan; related to this the following topics were treated in a 4-hour course by **M. Zaidenberg**: Kobayashi hyperbolicity and Brody entire curves; absorbing stratifications and stability of hyperbolicity; algebraic hyperbolicity of projective hypersurfaces and their complements; smooth quintics with hyperbolic complements;
M. Green's value distribution theorems; hyperbolic hypersurfaces of Fermat-Waring type; hyperbolic hypersurfaces provided by symmetric products of curves; hyperbolic non-percolation; examples of degree 8 hyperbolic surfaces.

Complex geometry of orbits: The following topics were covered in the 4-hours course given by **Alan T. Huckleberry** in various level of detail: Basic facts on compact complex homogeneous spaces of complex semi-simple groups; complex geometry of orbits; precise descriptions of cycle spaces of open orbits; examples of (Andreotti-Norguet) transfer from the cohomology of orbits to function theory level of the associated cycle domains. Numerous open problems were suggested.

On the geometry of Positive Cones in compact Kaehler manifolds: The goal of this 4-hours course given by **J. P. Demally** was to describe the geometry of the Kaehler cone of an arbitrary compact Kaehler manifold, as well as of the cone of pseudo-effective divisors. It turns out that both cones are entirely computable in terms of analytic cycles and Hodge structures. As a consequence, we show for instance that the Kaehler cone is generically invariant with respect to the Gauss-Manin connection in any deformation, and we prove also a long-standing conjecture asserting that a projective variety is unruled if and only if its canonical line bundle is not pseudo-effective. The methods rely on the theory of positive currents and Monge-Ampere equations.

CR geometry, extension of CR functions, envelope of holomorphy and analytic discs, pluripolar sets: S. Dragomir gave an overview on subelliptic harmonic maps in the context of CR geometry. The class of the real-analytic strictly pseudoconvex manifolds which admit embeddings into a hyperquadric in low codimension was discussed by **P. Ebenfeld. B. Joericke** presented his joint work with N. Shcherbina where they provide geometric conditions on compact subsets of real manifolds contained in a sphere which ensure that the envelope of holomorphy of suitably small one-sided neighbourhoods of its complement (in the sphere) is single-sheeted or, respectively multisheeted. **N. Shcherbina** proved that the graph of a continuous function is pluripolar if and only if the function is holomorphic. **C. Laurent** discussed extensions of CR continuos functions and **D. Hill** and **M. Nacinovich** dealt with weak pseudoconcavity, maximum modulus principle and weak unique continuation. **A. Perotti** presented some results on the characterization of traces of pluriharmonic functions on the boundary of a domain. Such a problem can be studied by means of a local or a global approach. The local approach, in which one finds tangential differential conditions satisfied by the traces, requires the non-vanishing of the Levi form of the boundary. The global approach, first proposed by Fichera in the 1980's, consists in finding the orthogonal subspace of the traces of pluriharmonic functions w.r.t. the L^2-norm and it has greater generality. Very recently G. Henkin and V. Michel proved that a local Hartogs-Bochner- Principle on real-analytic CR-manifold M holds true if and only if M is nowhere stricly pseudoconvex. A positive result was presented by **E. Porten**, valid for hypersurfaces which are given as C^2-graphs. As a corollary a global Hartogs- Bochner-Principle on weakly 2-concave hypersurfaces was derived.

Approximate solutions of d-bar, and Poletsky theory of discs: Poletsky's construction of plurisubharmonic functions was generalized to the case of manifolds by J-P. Rosay. The problem for one-sided holomorphic extendibility of CR functions on a hypersurface was discussed by **Anna Siano**. A positive answer was shown if the lowest degree nontrivial homogeneous polynomial in the Taylor expansion of r satisfies sector property. A new proof of that theorem is based on attaching a singular disc.
Finally, let f be a continuous function in the strip Im} z| <1 in the complex plane. Suppose that for every real r the restriction of f to the circle |z-r|=1 extends holomorphically inside the circle. Does this imply that f is holomorphic in the strip? This question has been open for over a decade. **A. Tumanov** answers the question in the affirmative using methods of several complex variables.

Functional spaces of holomorphic functions, asymptotic expansions, hyperbolic polynomials:
Let S be a finite sequence of points in the unit ball B of C^n. An analytic disc through S is a holomorphic map from the unit disc D to B s.t. f(s)=S. One can easily see that if S is an interpolating sequence (in the Carleson sense) for the Hardy space H^\infty(B) of constant less than C, then for all discs (s,f) through S, s is still an interpolating sequence for H^\infty(D). The reciprocal is still unknown. **E. Amar** studied the same problem for the

Hardy space $H^2(B)$ instead of $H^\infty(B)$ and proved that the analogous reciprocal has a negative answer. **C. de Fabritiis** discussed the following problem. Consider a family of spaces of holomorphic maps on the punctured plane which are square integrable with respect to a given weight. Give a complete classification of the ones which are finite dimensional or invariant for the group of rotations, and study composition operators on these spaces. **D. Barlet** discussed singularities of analytic germs in the context of the asymptotic expansions of the oscillating integrals associated to a complex Milnor fibration and of monodromy. **A-M. Chollet** discussed the Weierstrass Division by Hyperbolic polynomials.

Holomorphic vector fields, holomorphic dynamics, families of holomorphic maps: C. Bisi characterized the commuting polynomial automorphisms C^2, up their meromorphic extension to P^2 and looking at their dynamics on the line at infinity. A polynomial refinement of a theorem of M. Suzuki concerning the meromorphicity of periods of complete vector fields was proved by **M. Brunella**. **Tien Cuong Dinh** gave a new approach for the problem of value distribution of meromorphic mappings and used the method to study the dynamics of correspondences and the problem of distribution of holomorphic sectionsof positive line bundles (joint work with N. Sibony). Holomorphic dynamics of a bidisc was discussed by **C. Frosini.** In particular Wolff points of a holomorphic self-map f of the bidisc are characterized in terms of the properties of the components of the map f itself. **J. P. Vigue** discussed the iterates of an analytic family of holomorphic mappings and fixed points on a product. In particular results on the existence of fixed points of holomorphic mappings on the product of two bounded strictly convex domains were obtained.

Group action on complex manifolds: This topic was discussed by **N. Sibony** and **S. Trapani.**

Geometric aspects of holomorphic convexity: Several talks were devoted to various aspects of holomorphic convexity. **B. Berndtsson** gave a sufficient condition for a hermitian holomorphic vector bundle over the disk to be quasi-isometric to the trivial bundle. One consequence is a version of Cartan's lemma on the factorization of matrices with uniform bounds. (Joint work with J-P Rosay.) **J. Leiterer** gave a talk on some results on the relative Oka-Grauert principle (in the setting of Oka-pairs introduced by Forster and Ramspott) obtained in a joint paper with V. Vajaitu. **G. Marinescu** gave a talk on a stronger version with new proofs of the compactification theorem of Siu-Yau and extending Nadel's theorems to dimension 2. **T. Oshawa** discussed pseudoconvexity, L^2 extension and vector bundles. **V. Vajaitu** gave some criteria for holomorphic convexity. He characterized holomorphic convexity of complex spaces in terms of 1-convexity of its principal hypersurfaces and $H^1(X,O_X)$.

Levi flat hypersurfaces, boundary of holomorphic chains with parameters and applications to the boundary problem for Levi flat hypersurfaces: J. Brinkschulte proved nonexistence of higher codimensional Levi-flat CR manifolds in compact symmetric spaces. **P. Dolbeault** discussed boundaries of holomorphic 1-chains with C^∞ parameter. **D. Zaitsev** gave a report on a joint work with P. Dolbeault and G. Tomassini on boundaries of Levi-flat hypersurfaces, where conditions guaranteeing that a compact real submanifold bounds a Levi-flat hypersurface in C^3 are provided.

Symplectic, Kaehler and algebraic geometry: These topics were treated in many talks. **V.**

Ancona discussed Fano threefolds which are P}^1-bundles on smooth projective surfaces with the aim to describe the small quantum cohomology ring in these explicit cases. **C. Arezzo** gave a talk on some recent results obtained in a joint paper (with A. Gigi, G. P. Pirola) on the existence result a KE metric on deformation types of Fano threefolds. **S. Borghesi** gave a short introduction to the so called "degree formulae" for smooth, projective algebraic varieties over a field. **P. de Bartolomeis** considered several deformation theories for symplectic structures and relations among them, with a special emphasis on symplectic deformations of Kaehler manifolds. **S. Kosarew** explained some approaches to define new notions of moduli spaces, having better properties than the classical ones. He discussed modified Hilbert and cycle spaces, product structures, and homotopy invariance properties of local and global moduli spaces. An interesting feature in this context is a non linear version of Grauert's coherence theorem for direct images. **L. Migliorini** reported on joint works with M.A.de Cataldo. The main result says that if L is a globally generated line bundle on a projective manifold X, one can define a filtration on the cohomology groups of X such that the cup product operation on the graded piece of degree zero of this filtration satisfies the Hard Lefscehtz theorem and the Hodge Riemann bilinear relations.