Phase Space Analysis of Partial Differential Equations
February 15 - May 15, 2004

The goal of the trimester was:

to provide a broad, accessible and updated presentation of Microlocal Analysis and of some of its most recent applications to Partial Differential Equations (PDE), both linear and nonlinear, and to related fields of analysis and geometry;

to introduce open problems and exciting directions of new research;

to offer a wide outline of advanced topics to the interested young mathematicians;

to provide specialists from various Universities the opportunity to meet and exchange ideas.

Research Directions

The last half century has seen impressive developments in the analytic tools and in the ensuing deepening of our understanding of PDE theory. Such a progress has been made possible in particular by the greatly expanded reach of the Fourier transform in suitably adapted versions (e.g., beyond groups to manifolds as Fourier integrals, or to the analytic category as the FBI transform), leading to:

> pseudodifferential calculus (in its Weyl formulation) and its refinements based on Littlewood-Paley decompositions;
> paradifferential operators, extending pseudodifferential operators to nonlinear equations and allowing a deep analysis of the singularities of their solutions;
> Fourier integral operators and the WKB methods of linear, as well as nonlinear, geometrical optics and their connections with symplectic geometry.

Intimately linked to Fourier analysis are the Strichartz inequalities which have been shown, in the last decades, to play a crucial role in:

> the iterative methods leading to the solution of nonlinear PDE from that of related linear PDE;
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> the study of the local L^p regularity of solutions - mainly of hyperbolic and Schrödinger-type equations.

Lectures, conferences and seminars have been devoted to all these aspects of modern analysis. Prominence has also been given to some of the most important applications: compressible and incompressible flows, Euler and Navier-Stokes equations, dispersive and bilinear estimates for equations of relativity theory and quantum mechanics, etc.

Pseudodifferential operators and oscillatory integrals (Fourier integral operators) are also closely related to harmonic analysis. Since the last month of this trimester overlapped with the trimester in Harmonic Analysis, the audience and the lecturers of both trimesters were allowed to investigate areas of common interest.

The activity consisted mainly in 3 in-depth courses (duration of 12 hours), in about 20 minicourses (duration of 4 - 8 hours), in about 10 survey lectures (2 - 3 hours), and finally in some research lectures (1 hour). Courses, minicourses and lectures were offered by leading experts.

A total of 43 junior participants from 13 countries attended the various activities of the trimester. Among them 25 received a grant (partially) covering travel and/or living expenses. The participants were from Algeria, Belgium, Brasil, Bulgaria, France, Germany, India, Italy, Japan, Poland, Sweden, Tunisia, and Ukraine. The costs of the invited speakers and the reimboursements to junior participants were mostly covered by a grant from the Istituto Nazionale di Alta Matematica, in the framework of the program "Bimestri e trimestri intensivi di ricerca".

Jean-Michel Bony, Ferruccio Colombini, Sergiu Klainerman, Nicolas Lerner, and François Treves formed the Scientific Committee, while the members of the Local Organizing Committee were Ferruccio Colombini and Ludovico Pernazza.

A volume comprising the texts of many of the courses given is in press.

Here is a short description of the contributions, formally divided among these 5 different fields - although many of them belong to two or more of them: **Microlocal analysis, Fluid mechanics, Hyperbolic equations, Strichartz estimates, Other related fields** (Uniqueness, Schrödinger operators, hypoellipticity).

Microlocal analysis. In his 12-hours course ***Jean-Michel Bony*** (École Polytechnique, Paris) gave a deep introduction to pseudodifferential and paradifferential calculus. He discussed Littlewood-Paley theory, characterization of Sobolev and Hölder spaces, decomposition of classical symbols, boundedness in L^2; oscillatory integrals, classical pseudodifferential operators, symbolic calculus, wave front sets; hypoelliptic operators of constant strength; Weyl-Hörmander calculus, paraproduct and paradifferential operators, symbolic calculus; paralinearization of nonlinear differential equations, microlocal ellipticity, and propagation of singularities. Some courses were focused on the **local solvability** problem and its links with microlocal analysis. ***Paulo Domingo Cordaro*** (Universidade de S. Paulo) presented a general picture of the problem of (local) solvability for the differential complex associated to a locally integrable structure. The emphasis was on structures of corank one and on the question of solvability in top degree, two cases where some complete answers can be provided. At the end of the lectures some open questions were also presented. ***Nils Dencker*** (Lunds Universitet) presented a proof of the Nirenberg-Treves conjecture: Nirenberg and Treves formulated in 1970 the conjecture that condition Psi is necessary and sufficient for solvability of (pseudo-)differential operators of principal type. This condition determines the sign changes of the imaginary part of the principal symbol, along the bicharacteristics of the real part. He obtains local solvability by proving a localizable estimate of the adjoint operator with a loss of two derivatives (compared with the elliptic case). ***Ludovico Pernazza*** (Università di Pavia) gave an overview of the known results on local solvability for operators of principal type and showed some recent results obtained for operators not of principal type. After an exposition of Hörmander's theorem on the necessity of condition Psi and of Lerner's theorem on the sufficiency of this condition in dimension 2, he discussed a new condition Psi for operators not of principal type leading to new positive and negative results. Two courses were devoted to **systems of differential operators** and microlocal analysis. ***Alberto Parmeggiani*** (Università di Bologna), having revised classical inequalities by Gårding, Melin, Hörmander and Fefferman-Phong in the scalar case, exposed methods to detect necessary conditions (both in scalar and system case) that lead to the concept of localized operators. He then exhibited some classes of systems for which the Fefferman-Phong inequality does not hold, and some other classes (systems with symplectic double characteristics) for which the Fefferman-Phong inequality does hold. ***François Treves*** (Rutgers University) gave a short review of some of the landmark discoveries of I.M.Gel'fand and L.A.Dickey, linking the algebra of classical pseudodifferential operators (in a single variable), or equivalently the algebra Symb of their symbols, to the hierarchy based on the Korteweg-de Vries equation. He then outlined some of his recent results: his approach aims at freeing the algebraic aspects of the theory (such as the so-called Poisson structures) from the choice of a specific algebra A of possible solutions. Finally, as an important example of how microlocal analysis can be applied to the study of Strichartz estimates, ***Claude Zuily*** (Université de Paris-Sud) in his course introduced the classical Fourier-Bros-Iagoniltzer transform and investigated its simplest properties. Then he showed how this theory can be used to prove Strichartz estimates for asymptotically flat and non-trapped perturbations of the usual Schrödinger equation.

Fluid mechanics. The 12-hours course of ***Jean-Yves Chemin*** (École Polytechnique, Paris) focused on the problem of localisation in frequence space and fluidomecanics. The goal of these lectures was to study the incompressible Navier-Stokes system in $\mathbf{R}^{\mathbf{d}}$ with d=2 or d=3 and to show what could be the impact of techniques of localization in frequency space in the study of this system. After the proof of classical wellposedness theorem, he introduced Littlewood-Paley theory and translated some smallness condition given previously in terms of Besov spaces. As an illustration, he studied the problem of the existence and uniqueness of trajectories for scaling invariant solutions of Navier-Stokes equations. This was an opportunity to revisit the Cauchy-Lipschitz theorem. In the last chapter, he presented an anisotropic model of the incompressible Navier-Stokes system coming from the study of geophysical fluids; in this three dimensional model, the three dimensional laplacian becomes a bidimensional laplacian. Another course related to fluidodynamics has been given by ***Isabelle Gallagher*** (École Polytechnique, Paris). She observed that it is usual for evolution PDEs arising from a physical problem to be associated with some "energy" conservation, and also to have a "scaling" property. To both aspects one can naturally associate some invariant functional spaces. The question she addressed was to try to understand what can happen when one of the spaces is "embedded" in the other: for instance, can energy-type techniques improve the local-in-time results in scale invariant spaces, if the energy space is itself scale invariant? She discussed three different systems, accounting for the three different possibilities: when the energy space is scale invariant (the Navier-Stokes equation in two space dimensions), when the energy space is "below" scaling (the Navier-Stokes equation in three space dimensions) and when the energy space is "above" scaling (the cubic wave equation in three space dimensions).

Hyperbolic equations. Many courses and lectures have been dedicated to this argument. **Quasilinear hyperbolic equations:** in his course ***Serge Alinhac*** (Université de Paris-Sud) gave a deep introduction to the geometric methods in the study of quasilinear hyperbolic systems. He discussed perturbation techniques for quasilinear wave equations (Klainerman's method, Klainerman's inequality) and gave some elements of Riemanian geometry. He then introduced energy inequalities for the wave equation, the geometry associated to an optical function, energy inequalities for Maxwell and Bianchi equations, the commutation formula for the scalar case, the tensorial case and the quasiradial point of view. Finally, he gave some applications to the study of the behaviour of free solutions to the wave equation on a curved background and to the study of the "blowup at infinity". The problem of global existence of small solutions for nonlinear symmetric hyperbolic systems has been tackled by ***Kunihiko Kajitani*** (Tsukuba University): he investigated the decay of solutions of the Cauchy problem for first order symmetric hyperbolic systems with variable coefficients and obtained the existence of global small amplitude solutions of quasilinear symmetric hyperbolic systems. In order to derive decay estimates of solutions he made use of the spectral theory for first order symmetric hyperbolic systems and, in particular, of the integral representation of the wave operator. Finally, ***Hajer Bahouri*** (Université de Tunis) in her lecture gave an outline of some recent results of local wellposedness for quasilinear wave equations for initial data that are less regular than what is required by the energy method. To go below the regularity prescribed by the classical theory of strictly hyperbolic equations, one has to use the particular properties of the wave equation. The purpose of her talk was to emphasize the importance of ideas coming from microlocal analysis to prove such results. **Nonlinear hyperbolic equations**: ***Sergiu Klainerman*** (Princeton University) treated nonlinear hyperbolic equations; his course gave an overview of some of the recent developments in the mathematical theory of general relativity, in particular those in connection with the problem of evolution. This course was concentrated in particular on recent results on the stability of the Minkowski space as well as some recent progress on the bounded L^2 curvature conjecture obtained in collaboration with I. Rodnianski. They recently established a lower bound estimate on the radius of injectivity of null hypersurfaces imbedded in Einstein spacetimes, which depends only on the flux of the spacetime curvature along the hypersurface. ***Thomas Sideris*** (University of California, Santa Barbara) presented recent results on the global existence of solutions to the initial value problem for 3-dimensional nonlinear elastodynamics close to the reference configuration, in both the compressible and incompressible case. The equations of motion were presented using a field-theoretic approach with the least action principle, allowing for a natural introduction to energy methods based on the Galilean invariance. Dispersive estimates were derived via the combination of generalized Sobolev inequalities and a new series of weighted L^2 estimates. A detailed analysis of the nonlinear structure of the equations uncovers a null condition which is necessary to cancel the resonances of each wave family. The results were explained within the context of nonlinear hyperbolic PDE with an emphasis on the evolution of techniques used therein. ***Guy Métivier*** (Université de Bordeaux) in his course dealt with the stability analysis of nonlinear multidimensional boundary layers and shock waves, emphasizing the construction of symmetrizers used to prove maximal energy estimates. The first part was devoted to boundary value problems for hyperbolic systems, reviewing classical materials with a few new improvements: plane wave analysis, Lopatinski determinant, the method of symmetrizers, Kreiss' construction of symmetrizers, the block structure condition. In the second part, he gave an extension of the previous notions to hyperbolic-parabolic systems, that is, viscous perturbations of hyperbolic equations. The third part was devoted to a short introduction to applications to boundary layers: profiles equations, the plane wave analysis, an introduction of the Evans function, the conjugation lemma, symmetrizers and adapted paradifferential calculi. ***Jeffrey Rauch*** (Michigan University, Ann Arbor) gave an introduction to the methods of nonlinear geometric optics. The goal was to start with the simplest linear examples to motivate the various ansatz which serve to construct approximate solutions with error tending to zero as wavelength tends to zero. The "Phase Space " aspect comes from the fact that the solutions are rapidly oscillating: the link is created through the pair “position, direction of oscillation in the cotangent bundle”. Propagation of this pair along null bicharacteristics corresponds to the laws of geometric optics going back to Fermat's principle of least time. Maxwell’s equations have been treated by ***Yann Brenier*** (Université de Nice): more precisely, he studied the Born-Infeld system, a nonlinear version of Maxwell's equations. He showed that, using the energy density and the Poynting vector as additional independent variables, the BI system can be augmented as a 10x10 system of hyperbolic conservation laws. The resulting augmented system has some similarity with MHD equations and enjoys remarkable properties (existence of a convex entropy, galilean invariance, full linear degeneracy). Then, he investigated several limit regimes of the augmented BI equations, using a relative entropy method going back to Dafermos, and recovered the Maxwell equations for low fields, some pressureless MHD equations (describing string motion) for high fields, and pressureless gas equations for very high fields. Some courses dealt with **wave maps** problems: wave maps are an exact analogue to harmonic maps, if one replaces the background metric with a Lorentzian metric; of course, the system of quasilinear PDE thus arising is of hyperbolic type. ***Piero D'Ancona*** (Università di Roma 1) in the first part of his course showed some quite strong results of ill-posedness (namely, non-uniqueness) in the "subcritical" case, i.e., in Sobolev spaces of order lower than n/2. In the second part of the course he discussed the following result: in the critical Sobolev space of index n/2, the Cauchy problem is ill-posed in the Hadamard (or Bourgain-Kenig-Ponce-Vega) sense, i.e. the solution flow is not uniformly continuous in the corresponding spaces. This result can be interpreted as a form of instability of the wave map system in the critical case. ***Joachim Krieger*** (Princeton University) outlined the techniques that led to his recent result, joint with D. Tataru, on global regularity of wave maps originating on $R^{\mathbf{d}}$, d3, provided the initial data are smooth and small in the critical Sobolev space. These techniques are both analytic (null-frame spaces, extensive use of microlocalization, multilinear null-form estimates) as well as geometric (technique of moving frames, use of the Coulomb Gauge, Hodge-type decompositions, vector fields method). Finally the case of **linear hyperbolic equations** was treated by ***Tatsuo Nishitani*** (Osaka University), who lectured on effectively hyperbolic Cauchy problems. He treatedmultiple characteristics, necessary conditions of Ivrii-Petkov, effective hyperbolicity, generalized flows, generalized characteristic curves, generalized bicharacteristics, as well as microlocal and local hyperbolic a priori estimates, uniqueness and existence results, metrics with large parameters associated to a surface, symbols and weights, specializing symbols and the regularization of time function. Finally, he studiedhyperbolic polynomials, the semi-continuity of hyperbolic cones, and applications of semi-continuity.

Strichartz estimates. ***Daniel Tataru*** (University of California, Berkeley) gave a 12-hours course on phase space analysis and nonlinear dispersive equations. The aim of the course was to introduce a phase space approach to the study of linear and nonlinear dispersive equations. He first introduced the Bargman transform and a corresponding approach to the study of pseudodifferential and Fourier integral operators. Then he considered Schrödinger-like equations and obtained phase space representations for the Fourier integral operators governing the corresponding evolution. In other words, he obtained representations of solutions as superpositions of wave packets, i.e. special solutions which are highly localized in the phase space and move essentially along the Hamilton flow. A more delicate application of this was to study long time outgoing parametrices for Schrödinger equations with variable coefficients principal part. The second part of the course was devoted to a related approach for the wave equation, fro which the spirit is similar, but the geometry is different. The main application that was discussed regarded the study of quasilinear wave equations. In his course ***Jacob Sterbenz*** (Princeton University) gave an introduction with some details to a recent joint work with M. Machedon on the local and global regularity properties of certain field equations which arise in mathematical physics, particularly those of Yang-Mills type. The focus of this work is the so-called "low regularity" approach to dispersive PDE, which is an attempt to understand the long term behaviour of equations by proving local wellposedness of the Cauchy problem in scale invariant Sobolev (or Besov) spaces. In this way, one is led to discover the regularity properties of the equations in question in spaces which are much rougher than those that can be treated by the “classical” methods of energy estimates and Sobolev embeddings. ***Hart Smith*** (Washington University, Seattle) in his course dealt with wave packet methods and boundary value problems. His lectures concerned the L^p norm of eigenfunctions for metrics of limited differentiability. The results also apply to functions with spectrum contained in a unit range of frequencies. The question of interest was to bound the L^p norm in terms of a power of the frequency, given that the L^2 norm is bounded by 1. C. D. Sogge established the sharp exponent for compact Riemannian manifolds with smooth metrics, and the first result discussed in these lectures was that the same exponent is valid for metrics which are only twice continuously differentiable. Then the case of metrics in Hölder classes was discussed, giving examples showing that the exponents can be strictly larger and establishing the best possible exponent for a range of p near 2. Some courses were focused on **Strichartz estimates for Schrödinger equation**. The course of ***Patrick Gérard*** (Université de Paris-Sud) was devoted to some phase space methods used in the study of the Cauchy problem for the nonlinear Schrödinger equation on Riemannian compact manifolds of dimension 2 or 3. In the first part the link between WKB methods, dispersion effects and Strichartz estimates was discussed, with applications to global existence for NLS on general compact manifolds. In the second part these results were improved on specific manifolds such as spheres; the main tools in this case were multilinear estimates of eigenfunctions, combined with analysis in conormal Bourgain spaces. The subject of the course of ***Vladimir Georgiev*** (Università di Pisa) was the study of dispersive and Strichartz-type estimates for Schrödinger equation with potential and the stability of solitary solutions. He considered the Schrödinger equation in 3-dimensional space with small potential in the Lorentz space and proved Strichartz-type estimates. Finally, he applied these estimates to show the stability of solitary waves for the Maxwell- Schrödinger system in presence of an external Coulomb potential of attractive type. In his lecture, ***Atanas Stefanov*** (Kansas University) presented a version of the classical Strichartz estimates in mixed Lebesgue norms for the solutions of the Schrödinger equation with magnetic potential. He also discussed applications to concrete geometric PDE.

Other related fields. The **uniqueness problem** was the subject of a course and a lecture. The course of ***Daniele Del Santo*** (Università di Trieste) was devoted to present some results on uniqueness in the Cauchy problem for singular principally normal operators. The first part of the course concerned some classical topics, mainly the technique of Carleman estimates and Hörmander's theorem on uniqueness in the Cauchy problem for principally normal operators when the initial surface is strongly pseudoconvex. The proof of the latter result was given stressing the fact that the crucial point is the Fefferman-Phong inequality. In the second part of the course the attention was turned on some possible generalizations of the recalled Hörmander's result. In particular the use of singular weights in Carleman estimates leads to the definition of singular principally normal operators. These operators have the compact uniqueness in the Cauchy problem when the characteristics roots are simple. The main point in the proof was again a Fefferman-Phong type estimate which holds in locally tempered Weyl Calculus. ***Jorge Hounie*** **(**Universidade de S. Carlos) gave a strong uniqueness theorem for planar vector fields.Let L be a locally integrable planar vector field. The strong uniqueness property implies that if u is a solution of $Lu = 0$ for $y > 0$ and its boundary value u(x,0) vanishes on a set of positive measure or has a zero of exponential order it must vanish identically in a neighborhood of $\{y=0\}$. The main example of a vector field with this uniqueness property is the Cauchy-Riemann vector field. In his lecture he characterized geometrically the locally integrable planar vector fields that possess the strong uniqueness property. A result on pointwise convergence to the boundary value was also given for bounded solutions. Two courses were given on the **Schrödinger equation**. ***David Jerison*** (M.I.T.) in his course proved that certain Schrödinger operators with short-range but rough potentials do not have embedded eigenvalues. The first step was to show that this theorem follows from a Carleman inequality. On its own, the proof of the Carleman inequality involved an explicit construction of an inverse for the operator, using separation of variables, Bessel functions and spherical harmonics. An important tool were symbol-type estimates for the Bessel and Legendre functions across the entire range of values of the order and argument. This eventually reduces matters to more typical oscillatory integral bounds. Another tool developed were optimal operator bounds, that bounds take advantage not only of the size of the coefficients, but also of their cancellation properties. ***Vesselin Petkov*** (Université de Bordeaux) analysed trace formulae and resonances for Schrödinger operators. The trace formulae are well known for operators with point spectrum. On the other hand, for Schrödinger-type operators with continuous spectrum one expects to define the resonances and to obtain trace formulae connecting the trace of some operators with the resonances. In this direction the spectral shift function (SSF) plays the role of the function counting eigenvalues. In his lectures he presented an approach for the trace formulae working for short and long range perturbations. The main result was that the derivative of the SSF modulo regular terms is always a sum of harmonic measures related to the complex resonances and Dirac measures related to the real eigenvalues. Many phisical applications concerning the distribution of the resonances, Breit-Wigner approximation etc. are closely related to the representation of the derivative of SSF as a sum of measures. ***Waichiro Matsumoto*** (Ryukoku University) investigated a problem posed by M. Kac: ”**Can one hear the shape of a drum?”** There exist in literature some positive results of uniqueness for smooth convex domains, and some counterexamples for nonconvex domains. Natural questions are: 1) Does there exist a nonconvex domain on which the answer is yes? 2) Does there exist a convex counterexample? 3) Does there exist a smooth counterexample? 4) What is the necessary and sufficient condition for uniqueness? In his lectures he gave positive answer to the first question and showed the shapes of some nonconvex drums with uniqueness. The question of **Gevrey hypoellipticity** of a sum of squares of vector fields was the subject of the lecture given by ***Antonio Bove*** (Università di Bologna). He studied the analytic singularities of viscosity solutions of equations of eikonal type and obtained that the analytic singular support of these functions has an analytic stratification. The singular support can be identified with the cut-locus of the distance from the boundary of an open set, when the interior is equipped with a degenerate Riemannian metric. He applied the result to elliptic equations as well as to model operators of Grusin type.